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On Some Bivariate Gauss-Weierstrass Operators

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ABSTRACT. The aim of the paper is to investigate the approximation properties of bivariate generalization of Gauss-Weierstrass operators associated with the Riemann-Liouville operator. In particular, the approximation error will be estimated by these operators in the space of functions defined and continuous in the half-plane $(0, \infty) \times \mathbb{R}$, and bounded by certain exponential functions.

Keywords: Gauss-Weierstrass operator, Linear operators, Approximation order

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1. INTRODUCTION

Numerous issues related to positive linear integral operators were and still are the subject of research. The reason lays with their numerous applications in different domains of mathematics and physics. The classical Gauss-Weierstrass singular integral

$$W(f;x,t) = \frac{1}{2\sqrt{\pi t}} \int_{\mathbb{R}} \exp\left(-\frac{(x-y)^2}{4t}\right) f(y)dy, \tag{1.1}$$

has been studied systematically in the past. The integral W is a solution of the heat equation. The details can be found, for example, in [13]. Approximation properties of the operator W were given in many papers and monographs (see, for example, [13, 14, 18]). In [4], Anastassiou and Mezei investigated the smooth Gauss-Weierstrass singular integral operators (not in general positive) over the real line regarding their simultaneous global smoothness preservation property with respect to the L^p norm, by involving higher order moduli of smoothness. Some Lipschitz type results for the smooth Gauss-Weierstrass type singular integral operators were established in [17]. Approximation properties of the classical Gauss-Weierstrass integrals for functions of two variables in exponential weighted space were presented in [11] and a certain modification of these integrals which has a better order of approximation than the classical integrals was investigated in [19]. Khan and Umar (see [16]) gave a generalization of the Gauss-Weierstrass integrals and obtained the rate of convergence of the integral operator. In [5], Aral proposed a definition of the λ-Gauss Weierstrass singular integral with the kernel depending on a nonisotropic distance, its generalization, and gave some approximation properties of these integrals in certain function spaces. In [3], Anastassiou and Duman studied statistical L_p-approximation properties of the double Gauss-Weierstrass singular integral operators which do not need to be positive. Similar issues were also examined in the complex case in note [2]. Recently, various q-generalizations of Gauss-Weierstrass singular integral operators based on q-calculus (see [15]) and their approximation properties were investigated intensively (see, for example, [1, 6, 7, 8]).

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The aim of this paper is to study approximation properties of the family of bivariate Gauss-Weierstrass operators associated with the Riemann-Liouville operator (see [10]). This family is of the form

$$V_\alpha^t(f)(r,x) = V_\alpha(f;r,x,t) = \int_{\mathbb{R}} \int_0^\infty K_\alpha^t(r,x,s,y) f(s,y) ds dy,$$

where the kernel is defined by

$$K_\alpha^t(r,x,s,y) = \frac{(2t)^{-(\alpha+3/2)}}{\sqrt{2\pi}} e^{-\frac{r^2+s^2+(x-y)^2}{4t}} \left(\frac{rs}{2t}\right)^{-\alpha} I_\alpha\left(\frac{rs}{2t}\right) s^{2\alpha+1},$$

for $\alpha \geq -\frac{1}{2}$, $r>0$, $x \in \mathbb{R}$, $t>0$, and I_α is a modified Bessel function

$$I_\alpha(z) = \sum_{k=0}^\infty \frac{z^{\alpha+2k}}{2^{\alpha+2k} k! \Gamma(\alpha+k+1)}.$$

In paper [9], the operator V_α is considered for functions belonging to L^p, $1 \leq p \leq \infty$ and S, which is a space of infinitely differentiable functions, rapidly decreasing together with all their derivatives, even with respect to the first variable.
It is known (see [9, Proposition 3.4]) that the operator V_α is a positive linear operator from L^p into itself and for every $f \in L^p$, $1 \leq p \leq \infty$, we have

$$||V_\alpha^t(f)||_{L^p} \leq ||f||_{L^p}.$$

Moreover, for every $1 \leq p < \infty$, the family $(V_\alpha^t)_{t>0}$ is strongly continuous semigroup of operators on L^p and it is called Gauss semigroup associated with the Riemann-Liouville operator.
Armi and Rachdi proved that if $f \in S$, then V_α is a function of the class C^∞ on $(0,\infty)\times\mathbb{R}\times(0,\infty)$ and satisfies the following equations (see [9]):

$$\text{(1.2)} \quad \begin{aligned} &\frac{\partial u(r,x,t)}{\partial t} = \frac{\partial^2 u(r,x,t)}{\partial x^2} + \frac{2\alpha+1}{r}\frac{\partial u(r,x,t)}{\partial r} + \frac{\partial^2 u(r,x,t)}{\partial r^2}, \\ &\lim_{t\to 0^+} V_\alpha(f;r,x,t) = f(r,x) \quad \text{uniformly on} \quad (0,\infty)\times\mathbb{R}. \end{aligned}$$

An interesting fact related to the study of the operator V_α is the following remark. If $f(r,x) = f_1(r)f_2(x)$, then

$$\text{(1.3)} \quad V_\alpha(f;r,x,t) = W_\alpha(f_1;r,t) W(f_2;x,t),$$

where

$$W_\alpha(f_1;r,t) = \frac{1}{2t}\int_0^\infty r^{-\alpha} s^{\alpha+1} \exp\left(-\frac{r^2+s^2}{4t}\right) I_\alpha\left(\frac{rs}{2t}\right) f_1(s)\, ds$$

and W is defined by (1.1). Note that $W_{-\frac{1}{2}}$ is the classical Gauss-Weierstrass integral (1.1) and

$$W_{-\frac{1}{2}}(f_1;r,t) = \frac{1}{2\sqrt{\pi t}} \int_{\mathbb{R}} \exp\left(-\frac{(r-s)^2}{4t}\right) \tilde{f}_1(s) ds,$$

where

$$\tilde{f}_1(s) = \begin{cases} f_1(s) & \text{if } s \geq 0, \\ f_1(-s) & \text{if } s < 0. \end{cases}$$

It is worth mentioning that for $f(s) = s^{2k}$, $k \in \mathbb{N}$, the function $W_\alpha(f)$ is a polynomial called radial heat polynomial [12].
Some properties of the operator W_α, in particular, an estimation of the rate of convergence, were studied in [20].

In this work, we will investigate approximation properties of V_α in the space E_K, $K \geq 0$, consisting of all continuous functions f defined on the half-plane $(0,\infty)\times\mathbb{R}$, and such that

$$|f(r,x)| \leq Me^{K(r^2+x^2)}$$

for some $M>0$. The norm in E_K is given by

$$||f||_{E_K} = \sup_{(r,x)\in D} e^{-K(r^2+x^2)}|f(r,x)|,$$

where $D = \{(r,x) : r > 0, x \in \mathbb{R}\}$. Observe that if $0 \leq K_1 \leq K_2$, then $E_{K_1} \subset E_{K_2}$ and $\|f\|_{K_2} \leq \|f\|_{K_1}$.
We shall prove that the operator V_α is bounded and maps E_K into $E_{K+\delta}$, where $\delta > 0$. Moreover, we shall estimate an order of approximation by this operator.

2. Approximation properties

Applying the method used in [20], we can prove the following theorem.

Theorem 2.1. *Let $f \in E_K$.*

(a) *The function $V_\alpha(f)$ is of the class C^∞ in the set*

$$\Omega = \left\{(r,x,t); r>0, x\in\mathbb{R}, 0<t<\frac{1}{4K}\right\}$$

(if $K=0$, then $0<t<\infty$).

(b) *The function $V_\alpha(f)$ is a solution of the equation (1.2) in Ω and*

$$\lim_{(r,x,t)\to(r_0,x_0,0^+)} V_\alpha(f;r,x,t) = f(r_0,x_0)$$

for every $(r_0,x_0)\in\Omega$. Moreover, we have

$$\lim_{t\to 0^+} V_\alpha(f;r,x,t) = f(r,x)$$

in every closed subset in Ω.

In what follows, it will be useful to consider the functions:

$$\psi_{0,0}(r,x) = e^{K(r^2+x^2)}, \quad \psi_{0,i}(r,x) = x^i e^{K(r^2+x^2)},$$
$$\psi_{i,0}(r,x) = r^{2i}e^{K(r^2+x^2)} \quad \text{for} \quad i=1,2.$$

Using (see [20])

$$\int_0^\infty s^{\alpha+2b+1}\exp\left(-as^2\right) I_\alpha(\beta s)\,ds = \sum_{k=0}^\infty \frac{\beta^{\alpha+2k}\Gamma(\alpha+k+b+1)}{k!\Gamma(\alpha+k+1)a^{\alpha+k+b+1}2^{\alpha+2k+1}},$$

$\alpha \geq -\frac{1}{2}$, $b\geq 0$, $a>0$, $\beta>0$ and the equation (1.3), we have the following lemma.

Lemma 2.1. *Let $I = \left(0,\frac{1}{4K}\right)$ for $K>0$ and $I=(0,\infty)$ for $K=0$. For $t\in I$, we have*

$$\begin{aligned}
V_\alpha(\psi_{0,0};r,x,t) &= A,\\
V_\alpha(\psi_{0,1};r,x,t) &= Ax(1-4Kt)^{-1},\\
V_\alpha(\psi_{0,2};r,x,t) &= A\left[2x^2(1-4Kt)^{-2}+2t(1-4Kt)^{-1}\right],\\
V_\alpha(\psi_{1,0};r,x,t) &= A\left[r^2(1-4Kt)^{-2}+4t(\alpha+1)(1-4Kt)^{-1}\right],\\
V_\alpha(\psi_{2,0};r,x,t) &= A\left[r^4(1-4Kt)^{-4}+8tr^2(\alpha+2)(1-4Kt)^{-3}\right.\\
&\quad \left.+16t^2(\alpha+2)(\alpha+1)(1-4Kt)^{-2}\right],
\end{aligned}$$

where $A = (1-4Kt)^{-(\alpha+\frac{3}{2})} e^{\frac{K(r^2+x^2)}{1-4Kt}}$.

Theorem 2.2. *Let* $f \in E_K$. *If* $K > 0$, *then for every* $\delta > 0$ *and* $t \in (0, \frac{\delta}{4K(K+\delta)})$, *the operator* V_α *maps the space* E_K *in* $E_{K+\delta}$ *and*

$$\left\|V_\alpha^t(f)\right\|_{K+\delta} \leq \left(1+\frac{\delta}{K}\right)^{\alpha+\frac{3}{2}} \|f\|_K. \tag{2.4}$$

If $K = 0$, *then* V_α *maps the space* E_0 *into itself and*

$$\left\|V_\alpha^t(f)\right\|_0 \leq \|f\|_0. \tag{2.5}$$

Proof. By the positivity and linearity of V_α, we get

$$|V_\alpha(f;r,x,t)| \leq V_\alpha(|f|;r,x,t) \leq \|f\|_K V_\alpha(\psi_{0,0};r,x,t) = A\|f\|_K.$$

From above we have (2.5) for $K = 0$.
Let $K > 0$. If $\delta > 0$ and $t \in (0, \frac{\delta}{4K(K+\delta)})$, then $\frac{K}{1-4Kt} < K+\delta$. Hence

$$\begin{aligned} \|V_\alpha^t\|_{K+\delta} &= \sup_{(r,x)\in D} e^{-(K+\delta)(r^2+x^2)} |V_\alpha(f;r,x,t)| \\ &\leq \sup_{(r,x)\in D} e^{-\frac{K}{1-4Kt}(r^2+x^2)} |V_\alpha(f;r,x,t)| \\ &\leq (1-4K)^{-(\alpha+\frac{3}{2})} \|f\|_K \leq \left(1+\frac{\delta}{K}\right)^{\alpha+\frac{3}{2}} \|f\|_K, \end{aligned}$$

which gives (2.4). □

3. Rate of convergence

In this section, we shall state an estimate of the rate of convergence of the integral V_α in terms of the modulus of continuity.
Let $\delta > 0$ and

$$\omega(f;E_K,\delta) = \sup_{\sqrt{(s-r)^2+(y-x)^2}\leq\delta} |f(s,y)-f(r,x)| e^{-K(s^2+y^2)}, \quad K \geq 0.$$

Observe that

$$\omega(f;E_K,\delta_1) \leq \omega(f;E_K,\delta_2) \quad \text{for} \quad 0 < \delta_1 \leq \delta_2,$$

$$\omega(f;E_K,\lambda\delta) \leq (1+\lambda)\omega(f;E_K,\delta) \quad \text{for} \quad \lambda > 0.$$

Theorem 3.3. *Let* $f \in E_K$, $K \geq 0$ *and* $A = (1-4Kt)^{-(\alpha+\frac{3}{2})} e^{\frac{K(r^2+x^2)}{1-4Kt}}$. *We have*

$$|V_\alpha(f;r,x,t) - f(r,x)| \leq 2A\omega(f;E_K,\delta),$$

where

$$\begin{aligned} \delta &= \Big\{x^2 - 2x^2(1-4Kt)^{-1} + x^2(1-4Kt)^{-2} + 2t(1-4Kt)^{-1} \\ &+ \big[r^4 - 2r^4(1-4Kt)^{-2} + r^4(1-4Kt)^{-4} - 8tr^2(\alpha+1)(1-4Kt)^{-1} \\ &+ 8tr^2(\alpha+2)(1-4Kt)^{-3} + 16t^2(\alpha+2)(\alpha+1)(1-4Kt)^{-2}\big]^{1/2}\Big\}^{1/2} \end{aligned}$$

for $r > 0$, $x \in \mathbb{R}$, $0 < t < \frac{1}{4K}$ *and* $K > 0$.

If $K = 0$, *we have*

$$|V_\alpha(f;r,x,t) - f(r,x)| \leq 2\omega\left(f; E_0, \sqrt{2t + \sqrt{8tr^2 + 16t^2(\alpha+2)(\alpha+1)}}\right)$$

for $r > 0$, $x \in \mathbb{R}$, $t > 0$.

Proof. Let $\delta > 0$. Using the property of the modulus of continuity, we obtain

$$|f(s,y) - f(r,x)| \leq e^{K(s^2+y^2)}\omega\left(f; E_K, \sqrt{(s-r)^2 + (y-x)^2}\right)$$

for $f \in E_K$. From this, we get

$$\begin{aligned} &|f(s,y) - f(r,x)| \\ &\quad\leq e^{K(s^2+y^2)}\left(1 + \frac{\sqrt{(s-r)^2+(y-x)^2}}{\delta}\right)\omega\left(f; E_K, \delta\right) \\ &\quad\leq e^{K(s^2+y^2)}\left(1 + \frac{(s-r)^2+(y-x)^2}{\delta^2}\right)\omega\left(f; E_K, \delta\right). \end{aligned}$$

In view of $(s-r)^2 \leq |s^2 - r^2|$, we can write

$$|f(s,y) - f(r,x)| \leq e^{K(s^2+y^2)}\left(1 + \frac{|s^2-r^2| + (y-x)^2}{\delta^2}\right)\omega\left(f; E_K, \delta\right).$$

The operator V_α is positive and linear (see also [9]), so

$$\begin{aligned} &|V_\alpha(f;r,x,t) - f(r,x)| \\ &\quad\leq V_\alpha(|f - f(r,x)|; r,x,t) \\ &\quad\leq \omega\left(f; E_K, \delta\right) V_\alpha\left(\psi_{0,0} + \frac{x^2\psi_{0,0} - 2x\psi_{0,1} + \psi_{0,2} + \phi\psi_{0,0}}{\delta^2}; r,x,t\right), \end{aligned}$$

where $\phi(s,y) = |s^2 - r^2|$. Observe that

$$\begin{aligned} V_\alpha\left(\phi\psi_{0,0}; r,x,t\right) &\leq \left\{V_\alpha(\psi_{0,0};r,x,t)V_\alpha(\phi^2\psi_{0,0};r,x,t)\right\}^{1/2} \\ &= \left\{V_\alpha(\psi_{0,0};r,x,t)\left[r^4V_\alpha(\psi_{0,0};r,x,t)\right.\right. \\ &\quad \left.\left. - 2r^2V_\alpha(\psi_{1,0};r,x,t) + V_\alpha(\psi_{2,0};r,x,t)\right]\right\}^{1/2}. \end{aligned}$$

Hence

$$\begin{aligned} &|V_\alpha(f;r,x,t) - f(r,x)| \\ &\leq \omega\left(f; E_K, \delta\right)\Big\{V_\alpha\left(\psi_{0,0};r,x,t\right) \\ &\quad + \frac{1}{\delta^2}\left[x^2V_\alpha\left(\psi_{0,0};r,x,t\right) - 2xV_\alpha\left(\psi_{0,1};r,x,t\right) + V_\alpha\left(\psi_{0,2};r,x,t\right)\right] \\ &\quad + \frac{1}{\delta^2}\left[V_\alpha(\psi_{0,0};r,x,t)\left(r^4V_\alpha(\psi_{0,0};r,x,t) - 2r^2V_\alpha(\psi_{1,0};r,x,t)\right.\right. \\ &\quad \left.\left. + V_\alpha(\psi_{2,0};r,x,t)\right)\right]^{1/2}\Big\}. \end{aligned}$$

If $K = 0$, then from Lemma 2.1, we have

$$\begin{aligned} V_\alpha\left(\psi_{0,0}; r, x, t\right) &= 1, \\ V_\alpha\left(\psi_{0,1}; r, x, t\right) &= x, \\ V_\alpha\left(\psi_{0,2}; r, x, t\right) &= 2x^2 + 2t, \\ V_\alpha\left(\psi_{1,0}; r, x, t\right) &= r^2 + 4t(\alpha+1), \\ V_\alpha\left(\psi_{2,0}; r, x, t\right) &= r^4 + 8tr^2(\alpha+2) + 16t^2(\alpha+2)(\alpha+1). \end{aligned}$$

Hence, we conclude

$$|V_\alpha(f; r, x, t) - f(r, x)| \leq 2\omega\left(f; E_0, \sqrt{2t + \sqrt{8tr^2 + 16t^2(\alpha+2)(\alpha+1)}}\right)$$

for $r > 0$, $x \in \mathbb{R}$, $t > 0$.
For $K > 0$, we obtain from Lemma 2.1 the following estimation

$$\begin{aligned} &|V_\alpha(f; r, x, t) - f(r, x)| \\ &\leq A\omega\left(f; E_K, \delta\right) \\ &\quad \times \left\{1 + \frac{1}{\delta^2}\left[x^2 - 2x^2(1-4Kt)^{-1} + x^2(1-4Kt)^{-2} + 2t(1-4Kt)^{-1}\right]\right. \\ &\quad + \frac{1}{\delta^2}\left[r^4 - 2r^4(1-4Kt)^{-2} + r^4(1-4Kt)^{-4} - 8tr^2(\alpha+1)(1-4Kt)^{-1}\right. \\ &\quad \left.\left. + 8tr^2(\alpha+2)(1-4Kt)^{-3} + 16t^2(\alpha+2)(\alpha+1)(1-4Kt)^{-2}\right]^{1/2}\right\}. \end{aligned}$$

Setting

$$\begin{aligned} \delta &= \left\{x^2 - 2x^2(1-4Kt)^{-1} + x^2(1-4Kt)^{-2} + 2t(1-4Kt)^{-1}\right. \\ &\quad + \left[r^4 - 2r^4(1-4Kt)^{-2} + r^4(1-4Kt)^{-4} - 8tr^2(\alpha+1)(1-4Kt)^{-1}\right. \\ &\quad \left.\left. + 8tr^2(\alpha+2)(1-4Kt)^{-3} + 16t^2(\alpha+2)(\alpha+1)(1-4Kt)^{-2}\right]^{1/2}\right\}^{1/2}, \end{aligned}$$

we get the assertion. □

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