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Research Article

Study of velocity and shear stress for unsteady flow of incompressible Oldroyd-B fluid between two concentric rotating circular cylinders

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Abstract

This investigation deals with the study of unsteady flow of incompressible Oldroyd-B fluid between two rotating circular cylinders, both cylinders are rotating around their common axis ($r = 0$). The governing differential equations are formulated with appropriate boundary conditions and then solved by means of Laplace and Hankel transforms to obtain velocity and shear stress for unsteady flow of Oldroyd-B fluid between two infinite concentric rotating circular cylinders. The obtained solutions can easily be reduced to equivalent solutions for Maxwell and classical Newtonian fluids. Finally, the influence of different physical parameters on the fluid velocity and shear stress is graphically underlined and discussed.

Mathematics Subject Classification (2010). 76A05, 76A10

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1. Introduction

Over the years, non-Newtonian fluids have been considered due to their practical importance and huge-unfold applications in various branches of engineering, science and technology: particularly in drilling operations, material processing, oil exploitation, polymer chemical industries, and bioengineering. A number of industrially important fluids including exotic lubricants, extrusions of polymers, food stuffs, drilling mud, slurry type fuels, suspension and colloidal mixtures display non-Newtonian characteristics. In literature, for non-Newtonian fluids, a wide range of models are offered to explore their behaviors and properties [1, 2, 23, 25], because a particular model cannot define all the multifaceted properties of non-Newtonian fluids. Amongst these, Oldroyd-B fluid model is an important non-Newtonian viscoelastic model, which has attained much attention of the researchers [4, 5, 12, 16, 19, 24] because of its wide spread industrial applications. With the recent advances of complex and viscoelastic materials; applications of Oldroyd-B fluid have increased. Both theoretically and practically, the flow analysis of such fluids is very vital. Hayat et al. presented a detailed analysis of some simple flows of Oldroyd-B fluid [7]. Fetecau and Fetecau investigated the flow characteristics of Oldroyd-B fluids that flow

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unsteadily in a rectangular channel [3]. Tanveer et al. discussed magneto-hydrodynamic flow of generalized Oldroyd-B fluid over an infinite oscillating plate with slip condition using Fox H-function [22].

Motion of the fluid under translating or rotating cylinders is of great importance to both practical and theoretical fields. The study of viscoelastic fluid flows in the region of rotating circular cylinders is of vital significance as this types of fluid flows have many uses in several industries, like food and petroleum industries, chemical engineering, medicines, and bioengineering. Moreover, such flows have wide coverage on the development of energy generation and in astrophysical and geophysical fluid dynamics. The academic workers and engineers are very much interested in the geometry of such types of flows [6,17]. The literature about motion under translating or rotating cylinders for non-Newtonian fluids is not so well organized, but some interesting studies of such types of fluid flows are given by Jamil et al. [8,9], Kamran et al. [10,11], and Mahmood et al. [14,15].

The motivation of this study is to examine the flow of Oldroyd-B fluid between two coaxially rotating cylinders. At time $t = 0$, the fluid is at rest. Due to rotational shear stress which is time-dependent, the inner cylinder starts rotation about its own axis and the outer cylinder is rotating around its axis at time $t = 0^+$ through the angular velocity $R_2\,\omega\,t$. The flow of Oldroyd-B fluid is then generated by the rotation of two cylinders which at time $t = 0^+$ begin to rotate around their common axis. Closed form solutions for velocity and shear stress for the flow of Oldroyd-B fluid between two rotating cylinders are derived under series form in terms of generalized G functions with the help of Laplace and Hankel transforms. These solutions, which are new in the literature, give the complete pattern of flow field and have widespread applications in many industrial fields. Moreover, the derived expressions for velocity and shear stress are in the most simplified form, and the point worth mentioning is that these expressions are free from convolution product and integral of the product of generalized G function. Furthermore, the effects of various physical parameters on velocity field and shear stress are examined and illustrated graphically.

2. Basic equations

We write down the basic equations governing the motion of an incompressible non-Newtonian fluid. These are the equation of continuity

$$\operatorname{div}\mathbf{u} = 0\,, \tag{2.1}$$

and the linear momentum equation (in absence of body forces)

$$\operatorname{div}\mathbf{T} = \rho\,\frac{d\,\mathbf{u}}{d\,t}\,, \tag{2.2}$$

where $\mathbf{u}$ is the velocity field, $\mathbf{T}$ is the Cauchy stress tensor, ρ is the constant density, and $d/dt = \partial_t + \mathbf{u}\cdot\nabla$ is the material time derivative.

The constitutive equations of an incompressible Oldroyd-B fluid are given by

$$\mathbf{T} = -p\,\mathbf{I} + \mathbf{S}\;; \qquad (\lambda_1\,\partial_t + 1)\,\mathbf{S} = \mu\,(\lambda_2\,\partial_t + 1)\,\mathbf{A}\,, \tag{2.3}$$

where $-\,p\,\mathbf{I}$ is the spherical stress due to the constraint of incompressibility, $\mathbf{S}$ is the extra stress tensor, μ is the dynamic viscosity, λ_1 is the relaxation time, λ_2 is the retardation time, and $\mathbf{A}$ is the first Rivlin-Ericksen tensor defined as [20]

$$\mathbf{A} = \mathbf{L} + \mathbf{L}^{\top},$$

where $\mathbf{L}$ is the velocity gradient and the superscript $\top$ denotes the transpose operator.

For the problem under consideration, let us take velocity field and extra-stress of the following form [4,21]

$$\mathbf{u} = \mathbf{u}\,(\,r\,,\,t\,) = \,q\,(\,r\,,\,t\,)\,\mathbf{e}_\theta\;; \qquad \mathbf{S} = \mathbf{S}\,(\,r\,,\,t\,)\,, \tag{2.4}$$

where $\mathbf{e}_\theta$ is the transverse unit vector of cylindrical coordinates. Moreover, initial conditions, when the fluid is at rest, are

$$\mathbf{u}\,(r\,,\,0\,) = \mathbf{0}\;; \qquad\qquad \mathbf{S}\,(r\,,\,0\,) = \mathbf{0}\,. \tag{2.5}$$

The governing equations related to such type of flow of Oldroyd-B fluid in the absence of pressure gradient in axial direction are given by [4, 19]

$$(\lambda_1\,\partial_t + 1)\,\partial_t\,q\,(r\,,\,t) = \nu\,(\lambda_2\,\partial_t\, + 1)\left[\partial_r^2\, + \frac{1}{r}\,\partial_r\, - \frac{1}{r^2}\right]q\,(r\,,\,t)\,; \tag{2.6}$$

$$(\lambda_1\,\partial_t + 1)\,\sigma(r\,,\,t) = \mu\,(\lambda_2\,\partial_t + 1)\left[\partial_r - \frac{1}{r}\right]q\,(r\,,\,t)\,, \tag{2.7}$$

where ν represents the kinematic viscosity and $\sigma\,(r\,,\,t) = S_{r\theta}\,(r\,,\,t)$ is the shear stress which is different from zero.

3. Formulation and solutions of the problem

We consider an annular region between two straight infinite circular cylinders of radii R_1 and $R_2(> R_1)$, filled with incompressible Oldroyd-B fluid under the assumption to be at rest initially, as shown in Fig.1. At time $t = 0^+$, both cylinders begin to rotate about their common axis. The inner cylinder starts rotation because a shear stress given in equation (3.1) is applied on its boundary [11], and the outer cylinder is rotating around its axis through the angular velocity $R_2\,\omega\,t$.

$$\sigma\,(R_1,\,t) = g\,\lambda_1^{-1}\left[\frac{R_1}{r}\right]^2 M_{1,\,-1}(\,-\,\lambda_1^{-1},\,t\,)\,, \tag{3.1}$$

where g is a constant, and M represents the generalized functions defined by [13]

$$M_{x,\,y}(b\,,\,t) = \pounds^{-1}\left\{\frac{s^y}{s^{\,x} - b}\right\} = \sum_{j=0}^{\infty}\frac{(b)^j\,t^{(j+1)\,x-y-1}}{\Gamma[(j+1)\,x - y]}\;; \tag{3.2}$$

$$\mathrm{Re}\,(x-y) > 0\;; \qquad \mathrm{Re}\,(s) > 0\;; \qquad \left|\frac{b}{s^{\,x}}\right| < 1\,,$$

where $\Gamma(\bullet)$ is the Gamma function.

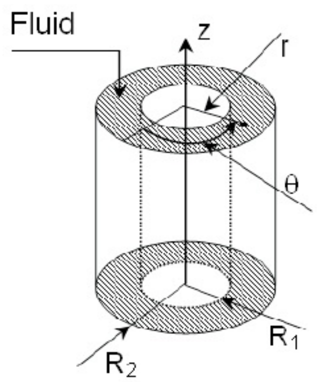

Figure 1. Geometry of the problem

Owing to the shear, the fluid between two rotating cylinders gradually starts moving and its velocity in cylindrical coordinates $(r,\,\theta,\,z)$ is given in equation $(2.4)_1$.

Based on the above suppositions, the governing equations of incompressible Oldroyd-B fluid, corresponding to this motion are given by equations (2.6), (2.7). The corresponding initial and boundary conditions are

$$q\,(r\,,\,0) = \partial_t\, q\,(r\,,\,0) = 0\ ; \qquad \sigma(r\,,\,0) = 0\ ; \qquad r \in [R_1\,,\,R_2]\,, \tag{3.3}$$

and

$$\begin{aligned} &(\lambda_1\,\partial_t\,+1)\,\sigma\,(r\,,\,t)\Big|_{r\,=\,R_1} = \mu\,(\lambda_2\,\partial_t\,+1)\left[\partial_r - \frac{1}{r}\right] q\,(r\,,\,t)\Big|_{r\,=\,R_1} = g\ ; \\ &q\,(r\,,\,t)\Big|_{r\,=\,R_2} = R_2\,\omega\,t\ ; \qquad t > 0\,, \end{aligned} \tag{3.4}$$

where ω represents the angular acceleration of outer cylinder.

3.1. Velocity field

Application of Laplace transform to equation (2.6), taking into account the initial and boundary conditions given in equations (3.3), (3.4) gives

$$\overline{q}\,(r\,,\,s)\left[(\lambda_2\,s+1)\left\{\partial_r^2 + \frac{1}{r}\,\partial_r - \frac{1}{r^2}\right\} - \frac{s\,(\lambda_1\,s+1)}{\nu}\right] = 0\,, \tag{3.5}$$

where $\overline{q}\,(r\,,\,s) = \pounds\{q(r\,,\,t)\}$, $\pounds$ denotes the Laplace transform operator, and

$$\left[\partial_r\,-\frac{1}{r}\right]\overline{q}\,(r\,,\,s)\Big|_{r\,=\,R_1} = \frac{g}{\mu\,s\,(\lambda_2\,s+1)}\ ; \qquad \overline{q}\,(r\,,\,s)\Big|_{r\,=\,R_2} = \frac{R_2\,\omega}{s^2}\,, \tag{3.6}$$

where s is the transform parameter.

Let us denote finite Hankel transform of the function $\overline{q}\,(r\,,\,s)$ by [18, 23]

$$H\{\overline{q}\,(r\,,\,s)\} = \overline{q}_{_H}\,(r\,,\,s) = \int_{R_1}^{R_2} r\,B_1(r\,b_n)\,\overline{q}\,(r\,,\,s)\,dr\ ; \qquad n = 1, 2, 3, \ldots\,, \tag{3.7}$$

where

$$B_1(r\,b_n) = J_1(r\,b_n)\,Y_1(R_2 b_n) - J_1(R_2 b_n)\,Y_1(r\,b_n)\,, \tag{3.8}$$

where $J_k(\bullet)$ and $Y_k(\bullet)$ are Bessel functions of order k of the first and second kind respectively, and b_n are the positive roots of $B_1(r\,b_n) = 0$.

Applying Hankel transform to equation (3.5), taking into account the conditions given in equation (3.6) and using the following relation

$$\begin{aligned} &\int_{R_1}^{R_2} r\,B_1(r\,b_n)\left\{\partial_r^2 + \frac{1}{r}\,\partial_r - \frac{1}{r^2}\right\}\overline{q}\,(r\,,\,s)\,dr = -\,b_n^2\,\overline{q}_{_H}\,(r\,,\,s) \\ &+\frac{2\,g}{\pi\,\mu\,b_n s\,(\lambda_2\,s+1)} + \frac{b_n R_2^2\,\omega}{s^2}\,B_2(R_2 b_n)\,, \end{aligned} \tag{3.9}$$

where

$$B_2\,(R_2 b_n) = J_2(R_2 b_n)\,Y_2(R_1 b_n) - J_2(R_1 b_n)\,Y_2(R_2 b_n)\,,$$

we have

$$\begin{aligned} \overline{q}_{_H}\,(r\,,\,s) = &\frac{2\,g}{\pi\,\mu\,b_n^3}\left[\frac{1}{s} - \frac{\lambda_2\,\nu\,b_n^2 + \lambda_1\,s+1}{s\,(\lambda_2\,\nu\,b_n^2 + \lambda_1\,s+1) + \nu\,b_n^2}\right] + \frac{R_2^2\,\omega\,B_2(R_2\,b_n)}{b_n} \\ &\times\left[\frac{1}{s}\left\{\frac{1}{s} - \frac{1}{\nu\,b_n^2}\right\} - \frac{1}{s\,(\lambda_2\,\nu\,b_n^2 + \lambda_1\,s+1) + \nu\,b_n^2}\left\{\lambda_1 - \frac{\lambda_2\,\nu\,b_n^2 + \lambda_1\,s+1}{\nu\,b_n^2}\right\}\right] \end{aligned} \tag{3.10}$$

Now applying inverse Hankel transform formula of $\bar{q}_H\,(r\,,\,s)$ defined as [18, 23]

$$\bar{q}\,(r,\,s) = \frac{\pi^2}{2}\sum_{n=1}^{\infty}\frac{b_n^2 J_1^2\,(R_2 b_n)\,B_1(r\,b_n)}{J_1^2\,(R_1 b_n) - J_1^2(R_2 b_n)}\;\bar{q}_H\,(r\,,\,s)\,, \tag{3.11}$$

to equation (3.10), and using the following relation [11]

$$\int_{R_1}^{R_2}(r^2 - R^2)\,B_1(r\,b_n)\,dr = \frac{4}{\pi\,b_n^3}\left[\frac{R_2}{R_1}\right]^2, \tag{3.12}$$

we arrive at

$$\begin{aligned}\bar{q}\,(r\,,\,s) &= \frac{g}{2\,\mu\,s}\left[\frac{R_1}{R_2}\right]^2\left\{\frac{r^2 - R_2^2}{r}\right\} - \frac{\pi\,g}{\mu}\sum_{n=1}^{\infty}\frac{J_1^2\,(R_2\,b_n)\,B_1(r\,b_n)}{b_n[J_1^2\,(R_1\,b_n) - J_1^2\,(R_2\,b_n)]} \\ &\times\frac{(\lambda_2\,\nu\,b_n^2 + \lambda_1\,s + 1)}{s\,(\lambda_2\,\nu\,b_n^2 + \lambda_1\,s + 1) + \nu\,b_n^2} + \frac{\pi^2}{2}\sum_{n=1}^{\infty}\frac{J_1^2(R_2\,b_n)\,B_1(r\,b_n)}{J_1^2(R_1\,b_n) - J_1^2(R_2\,b_n)}R_2^2\,\omega\,B_2\,(R_2\,b_n) \\ &\times\left[\frac{1}{s}\left\{\frac{b_n}{s} - \frac{1}{\nu\,b_n}\right\} - \frac{1}{s\,(\lambda_2\,\nu\,b_n^2 + \lambda_1\,s + 1) + \nu\,b_n^2}\left\{\lambda_1\,b_n + \frac{\lambda_2\,\nu\,b_n^2 + \lambda_1\,s + 1}{\nu\,b_n}\right\}\right]\end{aligned} \tag{3.13}$$

Using the following relation

$$\frac{\lambda_1}{s\,(\lambda_2\,\nu\,b_n^2 + \lambda_1 s + 1) + \nu\,b_n^2} = \sum_{h=0}^{\infty}\sum_{l=0}^{h}\frac{h\,!\,\lambda_2^l}{l\,!\,(h-l)\,!}\,\frac{s^{(l-h-1)}}{[s + \frac{1}{\lambda_1}]^{h+1}}\left[\frac{-\nu\,b_n^2}{\lambda_1}\right]^h \tag{3.14}$$

in equation (3.13), we get

$$\begin{aligned}\bar{q}(r\,,\,s) &= \frac{g}{2\,\mu\,s}\left[\frac{R_1}{R_2}\right]^2\left\{\frac{r^2 - R_2^2}{r}\right\} - \frac{\pi\,g}{\mu\,\lambda_1}\sum_{n=1}^{\infty}\frac{J_1^2(R_2\,b_n)B_1(r\,b_n)}{b_n[J_1^2(R_1\,b_n) - J_1^2(R_2\,b_n)]} \\ &\times\sum_{h=0}^{\infty}\sum_{l=0}^{h}\frac{h\,!\,\lambda_2^l}{l\,!\,(h-l)\,!}\,\frac{s^{(l-h-1)}\,(\lambda_2\,\nu\,b_n^2 + \lambda_1\,s + 1)}{[s + \frac{1}{\lambda_1}]^{h+1}}\left[\frac{-\nu\,b_n^2}{\lambda_1}\right]^h \\ &+ \frac{\pi^2\,R_2^2\,\omega}{2}\sum_{n=1}^{\infty}\frac{J_1^2(R_2\,b_n)\,B_1(r\,b_n)\,B_2\,(R_2\,b_n)}{J_1^2(R_1\,b_n) - J_1^2(R_2\,b_n)}\left[\frac{1}{s}\left\{\frac{b_n}{s} - \frac{1}{\nu\,b_n}\right\}\right. \\ &\left.+\sum_{h=0}^{\infty}\sum_{l=0}^{h}\frac{h\,!\,\lambda_2^l}{l\,!\,(h-l)\,!}\,\frac{s^{(l-h-1)}\,\{(\lambda_2 - \lambda_1)\,\nu\,b_n^2 + \lambda_1\,s + 1\}}{\lambda_1\nu\,b_n\,[s + \frac{1}{\lambda_1}]^{h+1}}\left[\frac{-\nu\,b_n^2}{\lambda_1}\right]^h\right]\end{aligned} \tag{3.15}$$

In order to avoid exhausting and lengthy computations of residues and contour integrals, the discrete inverse Laplace transform is utilized in equation (3.15), taking into account the following relation [13]

$$G_{x,\,y,\,z}\,(b\,,\,t) = \pounds^{-1}\left\{\frac{s^y}{(s^x - b)^z}\right\} = \sum_{j=0}^{\infty}\frac{(b)^j\,\Gamma(j+z)\,t^{(j+z)\,x-y-1}}{\Gamma(z)\,\Gamma(j+1)\,\Gamma[(j+z)\,x - y]}\;; \tag{3.16}$$

$$\mathrm{Re}\,(xz - y) > 0\;; \qquad \left|\frac{b}{s^x}\right| < 1\,,$$

to get the velocity field as

$$q(r\,,\,t)=\frac{g}{2\,\mu}\left[\frac{R_1}{R_2}\right]^2\left\{\frac{r^2-R_2^2}{r}\right\}-\frac{\pi\,g}{\mu\,\lambda_1}\sum_{n=1}^{\infty}\frac{J_1^2(R_2\,b_n)\,B_1(r\,b_n)}{b_n\left[J_1^2(R_1\,b_n)-J_1^2(R_2\,b_n)\right]}\tag{3.17}$$

$$\times\sum_{h=0}^{\infty}\sum_{l=0}^{h}\frac{h\,!\,\lambda_2^l}{l\,!\,(h-l)\,!}\left[\frac{-\nu\,b_n^2}{\lambda_1}\right]^h\Bigg[\lambda_1\,G_{1,\,l-h,\,h+1}\,(-\lambda_1^{-1},\,t)+\{1+\lambda_2\,\nu\,b_n^2\}$$

$$\times\,G_{1,\,l-h-1,\,h+1}\,(-\lambda_1^{-1},\,t)\Bigg]+\frac{\pi^2R_2^2\,\omega}{2}\sum_{n=1}^{\infty}\frac{J_1^2(R_2\,b_n)\,B_1(r\,b_n)\,B_2\,(R_2\,b_n)}{J_1^2(R_1\,b_n)-J_1^2(R_2\,b_n)}$$

$$\times\Bigg[b_nt-\frac{1}{\nu\,b_n}+\frac{1}{\lambda_1}\sum_{h=0}^{\infty}\sum_{l=0}^{h}\frac{h\,!\,\lambda_2^l}{l\,!\,(h-l)\,!}\left[\frac{-\nu b_n^2}{\lambda_1}\right]^h\Bigg\{\frac{\lambda_1}{\nu b_n}\,G_{1,\,l-h,\,h+1}(-\lambda_1^{-1},t)$$

$$+\Big[\frac{1}{\nu\,b_n}+(\lambda_2-\lambda_1)\,b_n\Big]G_{1,\,l-h-1,\,h+1}\,(-\lambda_1^{-1},\,t)\Bigg\}\Bigg]$$

3.2. Shear stress

By implementing Laplace transform to equation (2.7), we get

$$(\lambda_1\,s+1)\,\overline{\sigma}(r\,,\,s)=\mu\,(\lambda_2\,s+1)\left[\partial_r-\frac{1}{r}\right]\overline{q}\,(r\,,\,s)\tag{3.18}$$

For finding shear stress $\sigma(r\,,\,t)$, we write equation (3.10) in the following form

$$\begin{aligned}\overline{q}_H\,(r\,,\,s)\;&=\;\frac{1}{(\lambda_2\,s+1)}\left[\frac{2\,g}{\pi\,\mu\,b_n^3}\left\{\frac{1}{s}-\frac{\lambda_1\,s+1}{s\,(\lambda_2\,\nu\,b_n^2+\lambda_1\,s+1)+\nu\,b_n^2}\right\}\right.\\&+\;\left.\frac{R_2^2\,\omega\,B_2(R_2\,b_n)}{b_ns}\left\{\frac{1}{s}-\frac{\lambda_2\,\nu\,b_n^2\,(\lambda_2+1)-\lambda_1\,s-1}{s\,(\lambda_2\,\nu\,b_n^2+\lambda_1\,s+1)+\nu\,b_n^2}\right\}\right]\end{aligned}\tag{3.19}$$

Application of inverse Hankel transform to equation (3.19) and utilizing relation (3.12), we have

$$\begin{aligned}\overline{q}\,(r\,,\,s)=&\frac{g}{\mu\,(\lambda_2s+1)}\left[\frac{R_1^2}{2\,s\,R_2^2}\left\{\frac{r^2-R_2^2}{r}\right\}-\pi\sum_{n=1}^{\infty}\frac{J_1^2(R_2\,b_n)\,B_1(r\,b_n)}{b_n[J_1^2(R_1\,b_n)-J_1^2(R_2\,b_n)]}\right.\\&\left.\times\frac{\lambda_1s+1}{s\,(\lambda_2\nu\,b_n^2+\lambda_1s+1)+\nu\,b_n^2}\right]+\frac{R_2^2\,\omega\,\pi^2}{2}\sum_{n=1}^{\infty}\frac{b_nJ_1^2(R_2\,b_n)\,B_1(r\,b_n)\,B_2(R_2\,b_n)}{J_1^2(R_1\,b_n)-J_1^2(R_2\,b_n)}\\&\times\left[\frac{1}{s\,(\lambda_2s+1)}\left\{\frac{1}{s}+\frac{\lambda_2\nu\,b_n^2(\lambda_2+1)-\lambda_1s-1}{s\,(\lambda_2\nu\,b_n^2+\lambda_1s+1)+\nu\,b_n^2}\right\}\right],\end{aligned}\tag{3.20}$$

where

$$\begin{aligned}\left[\partial_r-\frac{1}{r}\right]\overline{q}\,(r\,,\,s)=&\frac{g}{\mu\,(\lambda_2s+1)}\left[\frac{R_1^2}{s\,r^2}+\pi\sum_{n=1}^{\infty}\frac{J_1^2(R_2\,b_n)\widetilde{B}_1(r\,b_n)}{J_1^2(R_1\,b_n)-J_1^2(R_2\,b_n)}\right.\\&\left.\times\frac{\lambda_1s+1}{s\,(\lambda_2\nu\,b_n^2+\lambda_1s+1)+\nu\,b_n^2}\right]-\frac{R_2^2\,\omega\,\pi^2}{2}\sum_{n=1}^{\infty}\frac{b_n^2J_1^2(R_2\,b_n)\widetilde{B}_1(r\,b_n)B_2(R_2\,b_n)}{J_1^2(R_1\,b_n)-J_1^2(R_2\,b_n)}\\&\times\left[\frac{1}{s\,(\lambda_2s+1)}\left\{\frac{1}{s}+\frac{\lambda_2\nu\,b_n^2(\lambda_2+1)-\lambda_1s-1}{s\,(\lambda_2\nu\,b_n^2+\lambda_1s+1)+\nu\,b_n^2}\right\}\right]\end{aligned}\tag{3.21}$$

Substituting equation (3.21) into equation (3.18), yields

$$\begin{aligned}\overline{\sigma}(r\,,\,s) &= \frac{g}{s\,(\lambda_1 s+1)}\left[\frac{R_1}{r}\right]^2 + \pi g \sum_{n=1}^{\infty} \frac{J_1^2(R_2\,b_n)\widetilde{B}_1(r\,b_n)}{J_1^2(R_1\,b_n)-J_1^2(R_2\,b_n)} \\ &\times \frac{1}{s\,(\lambda_2 \nu\, b_n^2+\lambda_1 s+1)+\nu\, b_n^2} - \frac{\mu R_2^2\,\omega\,\pi^2}{2}\sum_{n=1}^{\infty}\frac{b_n^2 J_1^2(R_2\,b_n)\widetilde{B}_1(r\,b_n)B_2(R_2\,b_n)}{J_1^2(R_1\,b_n)-J_1^2(R_2\,b_n)} \\ &\times\left[\frac{1}{s\,(\lambda_1 s+1)}\left\{\frac{1}{s}+\frac{\lambda_2\nu\, b_n^2(\lambda_2+1)-\lambda_1 s-1}{s\,(\lambda_2\nu\, b_n^2+\lambda_1 s+1)+\nu\, b_n^2}\right\}\right]\end{aligned} \tag{3.22}$$

Utilizing equation (3.14) into equation (3.22), we have

$$\begin{aligned}\overline{\sigma}(r\,,\,s) &= \frac{g}{s\,(\lambda_1 s+1)}\left[\frac{R_1}{r}\right]^2 + \frac{\pi g}{\lambda_1}\sum_{n=1}^{\infty}\frac{J_1^2(R_2\,b_n)\widetilde{B}_1(r\,b_n)}{J_1^2(R_1\,b_n)-J_1^2(R_2\,b_n)} \\ &\times\sum_{h=0}^{\infty}\sum_{l=0}^{h}\frac{h\,!\,\lambda_2^l}{l\,!\,(h-l)\,!}\frac{s^{(l-h-1)}}{\left[s+\frac{1}{\lambda_1}\right]^{h+1}}\left[\frac{-\nu\, b_n^2}{\lambda_1}\right]^h - \frac{\mu R_2^2\,\omega\,\pi^2}{2} \\ &\times\sum_{n=1}^{\infty}\frac{b_n^2 J_1^2(R_2\,b_n)\widetilde{B}_1(r\,b_n)B_2(R_2\,b_n)}{J_1^2(R_1\,b_n)-J_1^2(R_2\,b_n)}\left[\frac{1}{s\,(\lambda_1 s+1)}\left\{\frac{1}{s}+\frac{1}{\lambda_1}\right.\right. \\ &\times\left.\left.\sum_{h=0}^{\infty}\sum_{l=0}^{h}\frac{h\,!\,\lambda_2^l}{l\,!\,(h-l)\,!}\frac{s^{(l-h-1)}\{\lambda_2\nu b_n^2(\lambda_2+1)-\lambda_1 s-1\}}{\left[s+\frac{1}{\lambda_1}\right]^{h+1}}\left[\frac{-\nu\, b_n^2}{\lambda_1}\right]^h\right\}\right]\end{aligned} \tag{3.23}$$

By taking inverse Laplace transform and utilizing equations (3.2), (3.16), the shear stress can be acquired as

$$\begin{aligned}\sigma(r\,,\,t) &= \frac{g}{\lambda_1}\left[\frac{R_1}{r}\right]^2 M_{1,\,-1}(-\lambda_1^{-1},\,t) + \frac{\pi g}{\lambda_1}\sum_{n=1}^{\infty}\frac{J_1^2(R_2\,b_n)\widetilde{B}_1(r\,b_n)}{J_1^2(R_1\,b_n)-J_1^2(R_2\,b_n)} \\ &\times\sum_{h=0}^{\infty}\sum_{l=0}^{h}\frac{h\,!\,\lambda_2^l}{l\,!\,(h-l)\,!}\left[\frac{-\nu\, b_n^2}{\lambda_1}\right]^h G_{1,\,l-h-1,\,h+1}(-\lambda_1^{-1},\,t) - \frac{\mu\, R_2^2\,\omega\,\pi^2}{2\,\lambda_1} \\ &\times\sum_{n=1}^{\infty}\frac{b_n^2 J_1^2(R_2\,b_n)\widetilde{B}_1(r\,b_n)B_2(R_2\,b_n)}{J_1^2(R_1\,b_n)-J_1^2(R_2\,b_n)}\left[M_{1,\,-2}(-\lambda_1^{-1},\,t)+\frac{1}{\lambda_1}\right. \\ &\times\sum_{h=0}^{\infty}\sum_{l=0}^{h}\frac{h\,!\,\lambda_2^l}{l\,!\,(h-l)\,!}\left[\frac{-\nu\, b_n^2}{\lambda_1}\right]^h\left\{(\lambda_2\nu b_n^2-1)\,G_{1,\,l-h-2,\,h+2}(-\lambda_1^{-1},\,t)\right. \\ &\left.\left.+(\lambda_2^2\nu b_n^2-\lambda_1)\,G_{1,\,l-h-1,\,h+2}(-\lambda_1^{-1},\,t)\right\}\right]\end{aligned} \tag{3.24}$$

Now taking into account the following results

$$\frac{1}{\lambda_1}\,M_{1,\,-1}(-\lambda_1^{-1},\,t) = 1-e^{-t/\lambda_1}\ ; \qquad \frac{1}{\lambda_1}\,M_{1,\,-2}(-\lambda_1^{-1},\,t) = t+\lambda_1+\lambda_1 e^{-t/\lambda_1}\,,$$

equation (3.24) yields

$$\sigma(r\,,\,t)=g\left[\frac{R_1}{r}\right]^2\left\{1-e^{-t/\lambda_1}\right\}+\frac{\pi g}{\lambda_1}\sum_{n=1}^{\infty}\frac{J_1^2(R_2\,b_n)\widetilde{B}_1(r\,b_n)}{J_1^2(R_1\,b_n)-J_1^2(R_2\,b_n)} \tag{3.25}$$
$$\times\sum_{h=0}^{\infty}\sum_{l=0}^{h}\frac{h\,!\,\lambda_2^l}{l\,!\,(h-l)\,!}\left[\frac{-\nu\,b_n^2}{\lambda_1}\right]^h G_{1,\,l-h-1,\,h+1}(-\lambda_1^{-1},\,t)-\frac{\mu R_2^2\,\omega\pi^2}{2}$$
$$\times\sum_{n=1}^{\infty}\frac{b_n^2J_1^2(R_2\,b_n)\widetilde{B}_1(r\,b_n)B_2(R_2\,b_n)}{J_1^2(R_1\,b_n)-J_1^2(R_2\,b_n)}\Bigg[t+\lambda_1+\lambda_1e^{-t/\lambda_1}+\frac{1}{\lambda_1^2}$$
$$\times\sum_{h=0}^{\infty}\sum_{l=0}^{h}\frac{h\,!\,\lambda_2^l}{l\,!\,(h-l)\,!}\left[\frac{-\nu\,b_n^2}{\lambda_1}\right]^h\Big\{(\lambda_2\nu b_n^2-1)\,G_{1,\,l-h-2,\,h+2}(-\lambda_1^{-1},\,t)$$
$$+(\lambda_2^2\nu b_n^2-\lambda_1)\,G_{1,\,l-h-1,\,h+2}(-\lambda_1^{-1},\,t)\Big\}\Bigg]$$

4. Limiting cases

Solutions for Maxwell and classical Newtonian fluids, executing the same flow, can be obtained as limiting cases of our general solutions.

4.1. Maxwell fluid

By setting $\lambda_2\to 0$ in equations (3.17) and (3.25), the expressions for velocity and shear stress associated to Maxwell fluid can be recovered.

4.2. Classical Newtonian fluid

By taking $\lambda_1,\,\lambda_2\to 0$ in equations (3.17) and (3.25), the velocity field and related shear stress for classical Newtonian fluid can be obtained.

5. Graphical representation and discussion

In this section, we illustrate the obtained results graphically and discuss the effects of various substantial parameters on velocity field and shear stress.

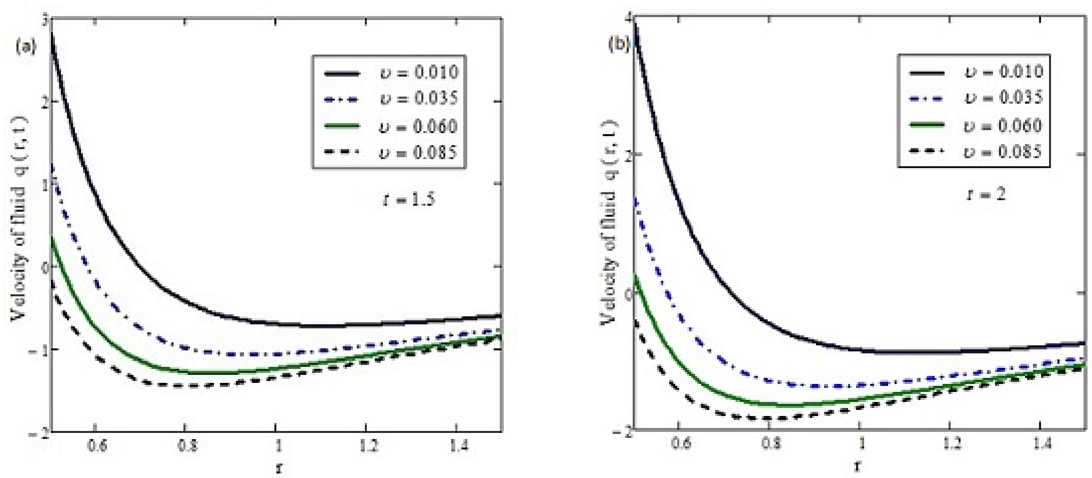

Figure 2. Effect of kinematic viscosity on velocity.

Figures 2 and 3 demonstrate the changes in velocity and shear stress related to kinematic viscosity ν. From these figures, it can be observed that the larger value of ν decrease both velocity and shear stress (in absolute value).

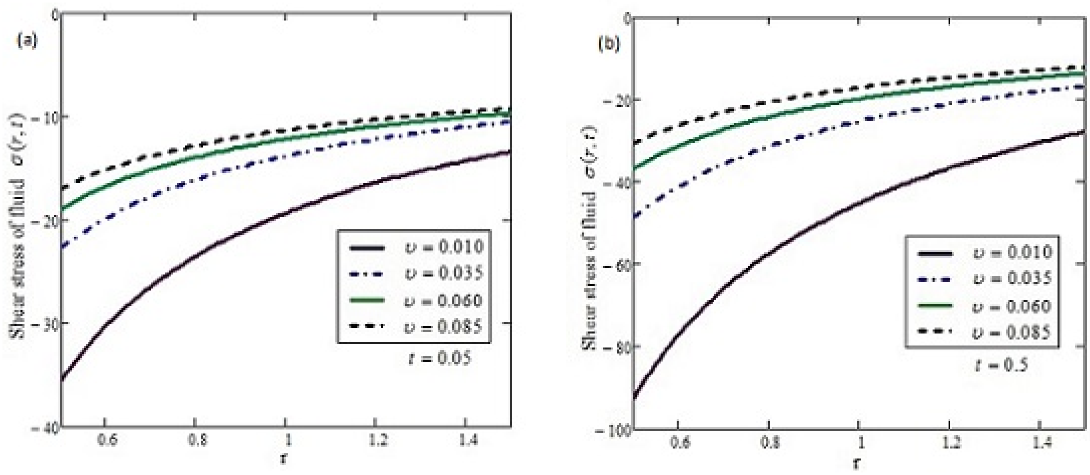

Figure 3. Effect of kinematic viscosity on shear stress.

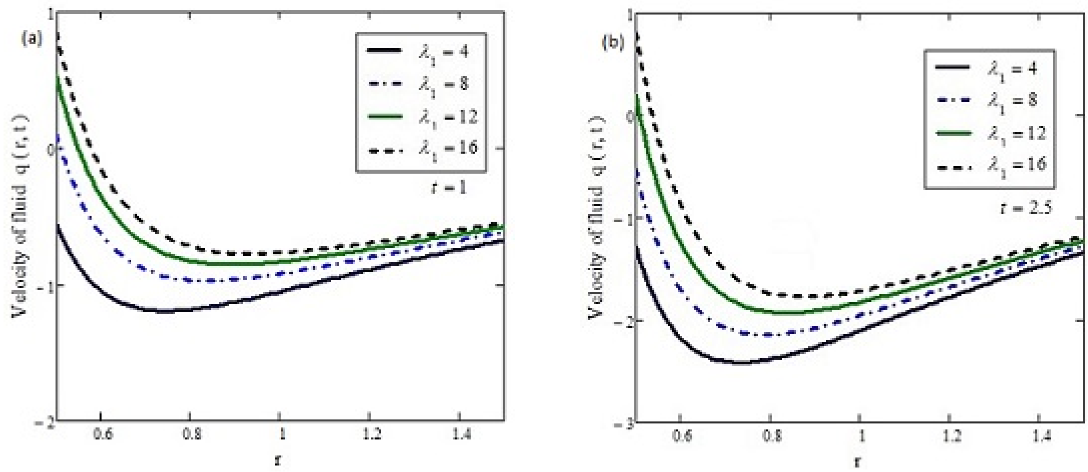

Figure 4. Effect of relaxation parameter on velocity.

Figures 4 and 5 elaborate the effects of relaxation parameter λ_1 on velocity and shear stress. It can be seen that with the increase of λ_1, the velocity increases while shear stress (in absolute value) varies inversely with this parameter.

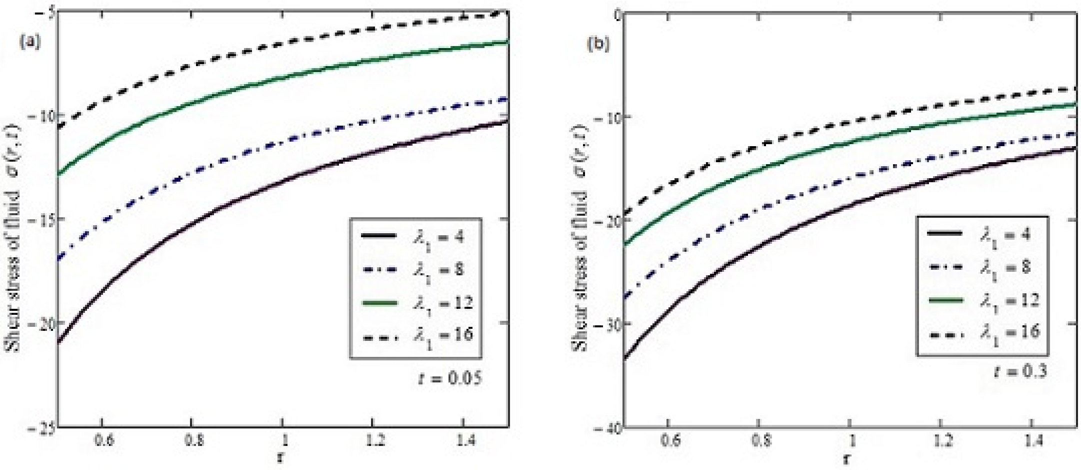

Figure 5. Effect of relaxation parameter on shear stress.

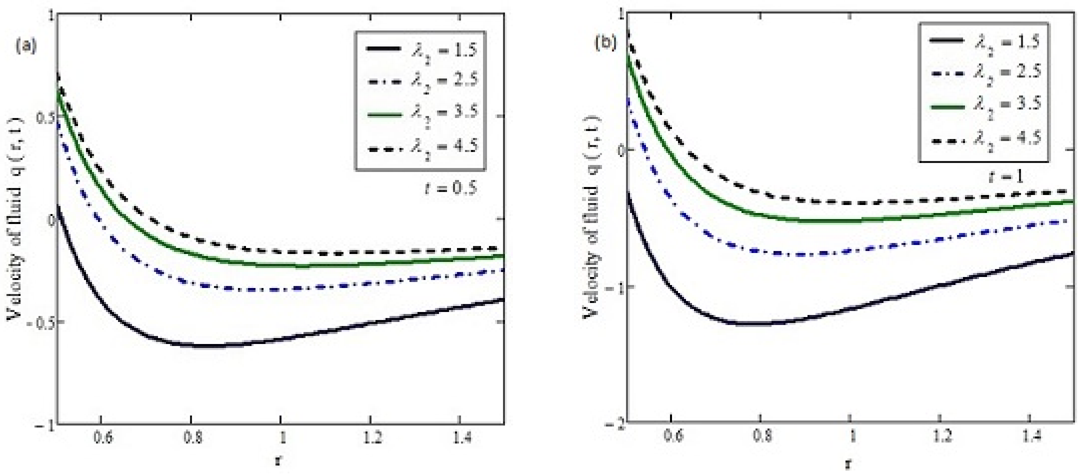

Figure 6. Effect of retardation parameter on velocity.

Figures 6 and 7 depict variations in velocity and shear stress due to retardation parameter λ_2. From here, it can be clearly observed that the larger value of λ_2 increase both velocity and shear stress (in absolute value).

6. Conclusions

The present study is focused on the derivation of velocity and shear stress for unsteady flow of incompressible Oldroyd-B fluid between two infinite concentric rotating circular cylinders. The motion of the fluid is produced by two cylinders which at time $t = 0^+$ begin to rotate around their common axis. Series solutions of governing differential equations have been derived by using Laplace and Hankel transforms which is most effective method for the proposed problem. For $\lambda_2 \to 0$ or $\lambda_1 \to 0$ and $\lambda_2 \to 0$, similar solutions for Maxwell fluids, respectively, classical Newtonian fluids can be recovered as limiting cases of our general results. Moreover, the acquired results are sketched graphically, and the effects of pertinent parameters on velocity and shear stress are discussed thoroughly.

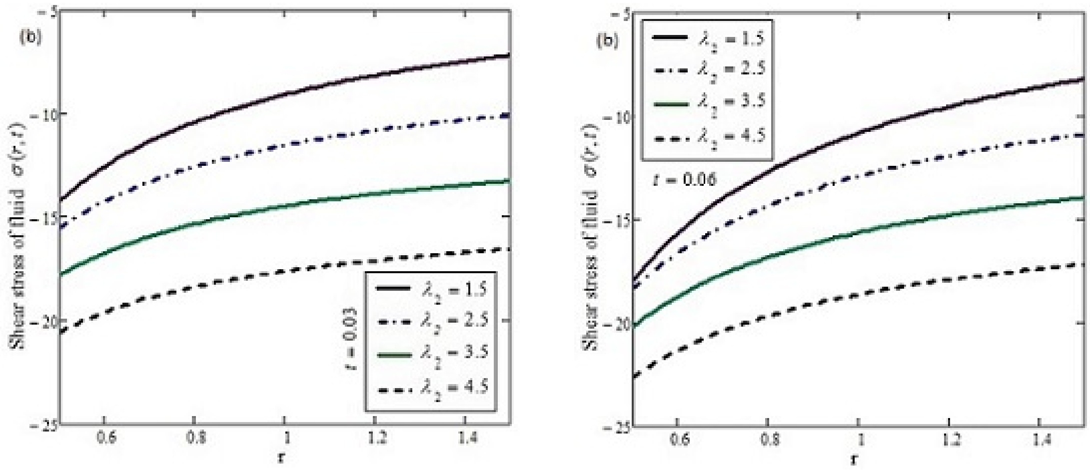

Figure 7. Effect of retardation parameter on shear stress.

The obtained results have many engineering applications, e.g., lubrication oil between two rotating cylinders/shafts which appears in many engineering designs especially in Mechanical Machinery. The derived results categorically indicate the following findings:

- The fluid velocity decreases as we increase the value of kinematic viscosity ν, while velocity of fluid increases with increasing values of both relaxation λ_1 and retardation λ_2 parameters.
- The shear stress (in absolute value) decreases with increasing values of kinematic viscosity ν and relaxation parameter λ_1, while the influence of retardation parameter λ_2 on shear stress is contrary to that of ν and λ_1.

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