A NEW RESULT FOR WEIGHTED ARITHMETIC MEAN SUMMABILITY FACTORS OF INFINITE SERIES INVOLVING ALMOST INCREASING SEQUENCES

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ABSTRACT. In this paper, a known theorem dealing with weighted mean summability methods of non-decreasing sequences has been generalized for $|A, p_n; \delta|_k$ summability factors of almost increasing sequences. Also, some new results have been obtained concerning $|N, p_n|_k$, $|N, p_n; \delta|_k$ and $|C, 1; \delta|_k$ summability factors.

1. INTRODUCTION

Let $\sum a_n$ be a given infinite series with the partial sums $(s_n)$. We denote $u_n^\alpha$ the $n$th Cesàro mean of order $\alpha$, with $\alpha > -1$, of the sequence $(s_n)$, that is (see [9]),

$$u_n^\alpha = \frac{1}{A_n^\alpha} \sum_{v=0}^{n} A_n^{\alpha-1} s_v$$  \hspace{1cm} (1)

where

$$A_n^\alpha = \frac{(\alpha + 1)(\alpha + 2)\ldots(\alpha + n)}{n!} = O(n^\alpha), \quad A_n^{\alpha-n} = 0 \quad \text{for} \quad n > 0. \hspace{1cm} (2)$$

A series $\sum a_n$ is said to be summable $|C, \alpha; \delta|_k$, $k \geq 1$ and $\delta \geq 0$, if (see [10]),

$$\sum_{n=1}^{\infty} n^{\delta k + k-1} |u_n^\alpha - u_{n-1}^\alpha|^k < \infty.$$  \hspace{1cm} (3)

If we take $\delta = 0$, then we have $|C, \alpha|_k$ summability (see [12]).

Let $(p_n)$ be a sequence of positive numbers such that

$$P_n = \sum_{v=0}^{n} p_v \to \infty \quad \text{as} \quad n \to \infty, \quad (P_{-i} = p_{-i} = 0, \quad i \geq 1).$$  \hspace{1cm} (4)

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The sequence-to-sequence transformation

\[ w_n = \frac{1}{P_n} \sum_{v=0}^{n} p_v s_v \]  

defines the sequence \((w_n)\) of the weighted arithmetic mean or simply the \((\tilde{N}, p_n)\) mean of the sequence \((s_n)\) generated by the sequence of coefficients \((p_n)\) (see [11]).

The \((\tilde{N}, p_n)\) mean of \((s_n)\) reduces to the Cesàro mean \((C, 1)\) when \((p_n) = \frac{1}{n+1}\) [17]. \((\tilde{N}, p_n)\) means were used in many applications of summability theory such as Tauberian and Korovkin type-theorems (see e.g. [18], [19] and [2]).

The series \(\sum a_n\) is said to be summable \(|\tilde{N}, p_n; \delta|k, k \geq 1\) and \(\delta \geq 0\), if (see [5]),

\[ \sum_{n=1}^{\infty} \left( \frac{P_n}{p_n} \right)^{\delta k + k - 1} |\Delta w_{n-1}| < \infty. \]  

where

\[ \Delta w_{n-1} = - \frac{p_n}{P_n P_{n-1}} \sum_{v=1}^{n} P_{v-1} a_v, \quad n \geq 1. \]  

In the special case if we take \(\delta = 0\), we have \(|\tilde{N}, p_n|k\) summability (see [3]). When \(p_n = 1\) for all values of \(n\), \(|\tilde{N}, p_n; \delta|k\) summability is the same as \(|C, 1; \delta|k\) summability. Also if we take \(\delta = 0\) and \(k = 1\), then we have \(|\tilde{N}, p_n|\) summability.

Let \(A = (a_{nv})\) be a normal matrix, i.e., a lower triangular matrix of nonzero diagonal entries. Given a normal matrix \(A = (a_{nv})\), we associate two lower semimatrices \(\tilde{A} = (\tilde{a}_{nv})\) and \(\hat{A} = (\hat{a}_{nv})\) as follows:

\[ \tilde{a}_{nv} = \sum_{i=v}^{n} a_{ni}, \quad n, v = 0, 1, \ldots \]  

and

\[ \hat{a}_{00} = a_{00}, \quad \hat{a}_{nv} = \tilde{a}_{nv} - \tilde{a}_{n-1,v}, \quad n = 1, 2, \ldots \]  

Then \(A\) defines the sequence-to-sequence transformation, mapping the sequence \(s = (s_n)\) to \(As = (A_n(s))\), where

\[ A_n(s) = \sum_{v=0}^{n} a_{nv} s_v, \quad n = 0, 1, \ldots \]  

It may be noted that \(\tilde{A}\) and \(\hat{A}\) are the well-known matrices of series-to-sequence and series-to-series transformations, respectively. Then, we have

\[ A_n(s) = \sum_{v=0}^{n} a_{nv} s_v = \sum_{v=0}^{n} a_{nv} \sum_{i=0}^{v} a_i = \sum_{i=0}^{n} a_i \sum_{v=i}^{n} a_{nv} \]
\[ \sum_{i=0}^{n} a_i \bar{\alpha}_{ni} = \sum_{v=0}^{n} \bar{\alpha}_{nv} a_v. \]  

(11)

Since \( \bar{\alpha}_{n-1,n} = \sum_{i=n}^{n-1} a_{n-1,i} = 0, \)

\[ \bar{\Delta} A_n(s) = A_n(s) - A_{n-1}(s) = \sum_{v=0}^{n} \bar{\alpha}_{nv} a_v - \sum_{v=0}^{n-1} \bar{\alpha}_{n-1,v} a_v \]

\[ = \sum_{v=0}^{n} (\bar{\alpha}_{nv} - \bar{\alpha}_{n-1,v}) a_v + \bar{\alpha}_{n-1,n} a_n = \sum_{v=0}^{n} \bar{\alpha}_{nv} a_v. \]  

(12)

The series \( \sum a_n \) is said to be summable \( |A,p_n;\delta|_k, k \geq 1 \) and \( \delta \geq 0 \), if (see [16])

\[ \sum_{n=1}^{\infty} \left( \frac{P_n}{p_n} \right)^{\delta k + k-1} |\bar{\Delta} A_n(s)|^k < \infty \]  

(13)

where

\( \Delta A_n(s) = A_n(s) - A_{n+1}(s), \) and \( \bar{\Delta} A_n(s) = A_n(s) - A_{n-1}(s). \)

By a weighted mean matrix we state

\[ a_{nv} = \begin{cases} \frac{P_v}{p_n}, & 0 \leq v \leq n \\ 0, & v > n, \end{cases} \]

where \( (p_n) \) is a sequence of positive numbers with \( P_n = p_0 + p_1 + p_2 + \ldots + p_n \to \infty \) as \( n \to \infty. \)

If we take \( \delta = 0 \), then \( |A,p_n;\delta|_k \) summability is the same as \( |A,p_n|_k \) summability (see [20]) and if we take \( \delta = 0 \) and \( a_{nv} = \frac{P_v}{p_n} \), then \( |A,p_n;\delta|_k \) summability is the same as \( |\tilde{N},p_n|_k \) summability. Also, if we take \( \delta = 0 \), \( a_{nv} = \frac{P_v}{p_n} \) and \( p_n = 1 \) for all \( n, \) then \( |A,p_n;\delta|_k \) summability is the same as \( |C,1|_k \) summability.

2. The Known Results

Quite recently, Bor has proved the following theorems concerning on weighted arithmetic mean summability factors of infinite series.

**Theorem 1.** Let \( (X_n) \) be a positive non-decreasing sequence and suppose that there exists sequences \( (\beta_n) \) and \( (\lambda_n) \) such that

\[ |\Delta \lambda| \leq \beta_n, \]

\[ \beta_n \to 0 \quad \text{as} \quad n \to \infty \]

(14)

(15)

\[ \sum_{n=1}^{\infty} n |\Delta \beta| X_n < \infty, \]

\[ |\lambda| X_n = O(1). \]  

(16)

(17)
If
\[
\sum_{n=1}^{m} \frac{|s_n|^k}{n} = O(X_m) \quad \text{as} \quad m \to \infty, \tag{18}
\]
and \((p_n)\) is a sequence that
\[
P_n = O(np_n), \tag{19}
\]
\[
P_n \Delta p_n = O(p_n p_{n+1}), \tag{20}
\]
then the series \(\sum a_n \frac{P_n \lambda_n}{np_n}\) is summable \(|N, p_n|k, k \geq 1\).

**Theorem 2.** [6] Let \((X_n)\) be a positive non-decreasing sequence. If the sequences \((X_n), (\beta_n), (\lambda_n), (p_n)\) satisfy the conditions (14)-(17), (19)-(20) of Theorem 1, and
\[
\sum_{n=1}^{m} \left( \frac{P_n}{p_n} \right) \delta^k \frac{|s_n|^k}{n} = O(X_m) \quad \text{as} \quad m \to \infty, \tag{21}
\]
\[
\sum_{n=v+1}^{m+1} \left( \frac{P_n}{p_n} \right) \delta^{k-1} \frac{1}{p_{n-1}} = O \left( \frac{P_v}{p_v} \delta^k \frac{1}{P_v} \right) \quad \text{as} \quad m \to \infty, \tag{22}
\]
then the series \(\sum a_n \frac{P_n \lambda_n}{np_n}\) is summable \(|N, p_n; \delta|k, k \geq 1\) and \(0 < \delta < 1/k\).

**Theorem 3.** [7] Let \((X_n)\) be a positive non-decreasing sequence. If the sequences \((X_n), (\beta_n), (\lambda_n), (p_n)\) satisfy the conditions (14)-(17), (19)-(20) of Theorem 1, and
\[
\sum_{n=1}^{m} \frac{|s_n|^k}{n^{k-1} X_n} = O(X_m) \quad \text{as} \quad m \to \infty, \tag{23}
\]
then the series \(\sum a_n \frac{P_n \lambda_n}{np_n}\) is summable \(|N, p_n; \delta|k, k \geq 1\) and \(0 \leq \delta < 1/k\).

**Theorem 4.** [7] Let \((X_n)\) be a positive non-decreasing sequence. If the sequences \((X_n), (\beta_n), (\lambda_n), (p_n)\) satisfy the conditions (14)-(17), (19)-(20) of Theorem 1, and
\[
\sum_{n=1}^{m} \frac{|s_n|^k}{n^{k-1} X_n^k} = O(X_m) \quad \text{as} \quad m \to \infty, \tag{24}
\]
then the series \(\sum a_n \frac{P_n \lambda_n}{np_n}\) is summable \(|N, p_n|k, k \geq 1\).

**Theorem 5.** [8] Let \((X_n)\) be a positive non-decreasing sequence. If the sequences \((X_n), (\beta_n), (\lambda_n), (p_n)\) satisfy the conditions (14)-(17), (19)-(20) of Theorem 1, condition (22) of Theorem 2, and
\[
\sum_{n=1}^{m} \left( \frac{P_n}{p_n} \right) \delta^k \frac{|s_n|^k}{n X_n^{k-1}} = O(X_m) \quad \text{as} \quad m \to \infty, \tag{25}
\]
then the series \( \sum a_n \frac{P_n}{np_n} \) is summable \( |\mathcal{N}, p_n; \delta|_k \), \( k \geq 1 \), \( 0 \leq \delta < 1/k \).

We need the following lemmas.

**Lemma 6.** [13] Under the conditions on \((X_n), (\beta_n), \) and \((\lambda_n)\) as expressed in the statement of Theorem 1, we have the following:

\[
\begin{align*}
nX_n\beta_n &= O(1), \quad (26) \\
\sum_{n=1}^{\infty} \beta_n X_n &< \infty. \quad (27)
\end{align*}
\]

**Lemma 7.** [15] If the conditions (19) and (20) of Theorem 1 are satisfied, then

\[
\Delta \left( \frac{P_n}{np_n} \right) = O \left( \frac{1}{n} \right).
\]

**Remark 8.** Under the conditions on the sequence \((\lambda_n)\) of Theorem 1, we have that \((\lambda_n)\) is bounded and \(\Delta \lambda_n = O(1/n)\) (see [4]).

3. The Main Results

A positive sequence \((b_n)\) is said to be almost increasing if there exists a positive increasing sequence \((z_n)\) and two positive constants \(C\) and \(B\) such that \(Cz_n \leq b_n \leq Bz_n\) (see [1]). It is known that every increasing sequences is an almost increasing sequence but the converse need not be true. In this paper we generalize Theorem 5 to \(|A, p_n; \delta|_k\) summability method using almost increasing sequences and normal matrix instead of non-decreasing sequences and weighted mean matrix, respectively. The following our main theorem is generalized the above results concerning \(|\mathcal{N}, p_n|_k\) and \(|\mathcal{N}, p_n; \delta|_k\) summability methods.

**Theorem 9.** [22] Let \(k \geq 1\) and \(0 \leq \delta < 1/k\). Let \(A = (a_{nv})\) be a positive normal matrix such that

\[
\begin{align*}
p_{n0} &= 1, \quad n = 0, 1, \ldots, \quad (28) \\
a_{n-1,v} &\geq a_{nv}, \quad \text{for } n \geq v + 1, \quad (29) \\
a_{nn} &= O \left( \frac{p_n}{n} \right), \quad (30) \\
\sum_{v=1}^{n-1} a_{nv} \hat{a}_{n,v+1} &= O(a_{nn}), \quad (31)
\end{align*}
\]

\[
\begin{align*}
\sum_{n=v+1}^{m+1} \left( \frac{P_n}{p_n} \right)^{\delta k} \left| \Delta_v (\hat{a}_{nv}) \right| &= O \left\{ \left( \frac{P_v}{p_v} \right)^{\delta k-1} \right\} \quad \text{as } m \to \infty, \quad (32) \\
\sum_{n=v+1}^{m+1} \left( \frac{P_n}{p_n} \right)^{\delta k} \left| \hat{a}_{n,v+1} \right| &= O \left\{ \left( \frac{P_v}{p_v} \right)^{\delta k} \right\} \quad \text{as } m \to \infty. \quad (33)
\end{align*}
\]
Let \((X_n)\) be an almost increasing sequence. If the sequences \((X_n)\), \((\beta_n)\), \((\lambda_n)\), and \((p_n)\) satisfy all the conditions of Theorem 5, then the series \(\sum a_n \frac{P_n \lambda_n}{np_n}\) is summable \(|A, p_n, \delta|_k, k \geq 1, 0 \leq \delta < 1/k\).

4. PROOF OF THEOREM 9

Proof. Let \((V_n)\) denotes the A-transform of the series \(\sum a_n \frac{P_n \lambda_n}{np_n}\). Then, by the definition, we have that

\[
\tilde{\Delta} V_n = \sum_{v=1}^{n} \tilde{a}_{nv} a_v \frac{P_v \lambda_v}{vp_v}.
\]

Applying Abel’s transformation to this sum, we have that

\[
\tilde{\Delta} V_n = \sum_{v=1}^{n-1} \Delta_v \left( \frac{\tilde{a}_{nv} P_v \lambda_v}{vp_v} \right) \sum_{r=1}^{v} a_r + \frac{\tilde{a}_{nn} P_n \lambda_n}{np_n} \sum_{r=1}^{n} a_r
\]

\[
\tilde{\Delta} V_n = \sum_{v=1}^{n-1} \Delta_v \left( \frac{\tilde{a}_{nv} P_v \lambda_v}{vp_v} \right) s_v + \frac{\tilde{a}_{nn} P_n \lambda_n}{np_n} s_n,
\]

\[
\tilde{\Delta} V_n = \frac{a_{nn} P_n \lambda_n}{np_n} s_n + \sum_{v=1}^{n-1} \frac{P_v \lambda_v}{vp_v} \Delta_v (\tilde{a}_{nv}) s_v + \sum_{v=1}^{n-1} \tilde{a}_{n,v+1} \lambda_v \Delta \left( \frac{P_v}{vp_v} \right) s_v
\]

\[
+ \sum_{v=1}^{n-1} \tilde{a}_{n,v+1} \frac{P_{v+1}}{(v+1)p_{v+1}} \Delta \lambda_v s_v
\]

\[
\tilde{\Delta} V_n = V_{n,1} + V_{n,2} + V_{n,3} + V_{n,4}.
\]

To complete the proof of Theorem 9, by Minkowski inequality, it is sufficient to show that

\[
\sum_{n=1}^{\infty} \frac{P_n}{p_n} \delta^{k+1} \left| V_{n,r} \right|^k < \infty, \text{ for } r = 1, 2, 3, 4.
\]

First, by applying Hölder’s inequality with indices \(k\) and \(k'\), where \(k > 1\) and \(\frac{1}{k} + \frac{1}{k'} = 1\), we have that

\[
\sum_{n=1}^{m} \left( \frac{P_n}{p_n} \right)^{\delta k+1} \left| V_{n,1} \right| \leq \sum_{n=1}^{m} \left( \frac{P_n}{p_n} \right)^{\delta k+1} \frac{k}{n^{k}} a_{nn} \left( \frac{P_n}{p_n} \right)^{k} |\lambda_n|^{k} \left| s_n \right|^{k} = O(1) \sum_{n=1}^{m} \left( \frac{P_n}{p_n} \right)^{\delta k} \frac{1}{n^{k}} |\lambda_n|^{k} \left| s_n \right|^{k}
\]

\[
= O(1) \sum_{n=1}^{m} \left( \frac{P_n}{p_n} \right)^{\delta k} \frac{1}{n^{k-1}} |\lambda_n| \left| s_n \right|^{k}
\]

\[
= O(1) \sum_{n=1}^{m-1} \Delta |\lambda_n| \sum_{v=1}^{n} \left( \frac{P_v}{p_v} \right)^{\delta k} \frac{1}{v^{k-1}} |\lambda_n| \left| s_n \right|^{k} + O(1) |\lambda_m| \sum_{n=1}^{m} \left( \frac{P_n}{p_n} \right)^{\delta k} \frac{1}{n^{k-1}}
\]

\[
= O(1) \sum_{n=1}^{m-1} \Delta |\lambda_n| \sum_{v=1}^{n} \left( \frac{P_v}{p_v} \right)^{\delta k} \frac{1}{v^{k-1}} |\lambda_n| \left| s_n \right|^{k} + O(1) |\lambda_m| \sum_{n=1}^{m} \left( \frac{P_n}{p_n} \right)^{\delta k} \frac{1}{n^{k-1}}
\]
as $m \to \infty$. By applying Hölder’s inequality with indices $k$ and $k'$, where $k > 1$ and $\frac{1}{k} + \frac{1}{k'} = 1$ and as in $V_{n,1}$, we have that

$$
\sum_{n=2}^{m+1} \left( \frac{P_n}{p_n} \right)^{\delta k + k - 1} |V_{n,2}|^k = \sum_{n=2}^{m+1} \left( \frac{P_n}{p_n} \right)^{\delta k + k - 1} \left| \sum_{v=1}^{n-1} P_v \lambda_v \Delta_v(\hat{a}_{nv}) s_v \right|^k
$$

$$
\leq \sum_{n=2}^{m+1} \left( \frac{P_n}{p_n} \right)^{\delta k + k - 1} \left\{ \sum_{v=1}^{n-1} |\Delta_v(\hat{a}_{nv})||\lambda_v|^k|s_v|^k \frac{1}{v^k} \left( \frac{P_v}{p_v} \right)^k \right\} \left( \frac{P_n}{p_n} \right)^{\delta k + k - 1}
$$

$$
= O(1) \sum_{n=2}^{m+1} \left( \frac{P_n}{p_n} \right)^{\delta k + k - 1} a_{nn}^{-1} \left| \sum_{v=1}^{n-1} |\Delta_v(\hat{a}_{nv})||\lambda_v|^k|s_v|^k \frac{1}{v^k} \left( \frac{P_v}{p_v} \right)^k \right|
$$

$$
= O(1) \sum_{n=2}^{m+1} \left( \frac{P_n}{p_n} \right)^{\delta k} \frac{1}{X_v^{k-1}} |s_v|^k |\lambda_v| \frac{1}{v^k} \left( \frac{P_v}{p_v} \right)^{k-1}
$$

$$
= O(1) \sum_{n=2}^{m+1} \left( \frac{P_n}{p_n} \right)^{\delta k} \frac{1}{X_v^{k-1}} |s_v|^k |\lambda_v| \frac{1}{v^k} v^{-k-1}
$$

$$
= O(1) \sum_{v=1}^{m} \left( \frac{P_v}{p_v} \right)^{\delta k} \frac{1}{X_v^{k-1}} |s_v|^k |\lambda_v| \frac{1}{v^k} v^{-k}
$$

Also, by using conditions of Theorem 9, we obtain that

$$
\sum_{n=2}^{m+1} \left( \frac{P_n}{p_n} \right)^{\delta k + k - 1} |V_{n,3}|^k = \sum_{n=2}^{m+1} \left( \frac{P_n}{p_n} \right)^{\delta k + k - 1} \left| \sum_{v=1}^{n-1} a_{nv}^k \Delta_v(\hat{a}_{nv+1}) \lambda_v s_v \right|^k
$$

$$
= O(1) \sum_{n=2}^{m+1} \left( \frac{P_n}{p_n} \right)^{\delta k + k - 1} \left\{ \sum_{v=1}^{n-1} a_{nv}^k \Delta_v(\hat{a}_{nv+1}) |\lambda_v|^k |s_v|^k \frac{1}{v^k} \right\} \left( \frac{P_n}{p_n} \right)^{\delta k + k - 1}
$$

$$
= O(1) \sum_{n=2}^{m+1} \left( \frac{P_n}{p_n} \right)^{\delta k + k - 1} a_{nn}^{-1} \left( \frac{P_v}{p_v} \right)^{k-1} \Delta_v(\hat{a}_{nv+1}) |\lambda_v|^k |s_v|^k \frac{1}{v^k}
$$

$$
= O(1) \sum_{v=1}^{m} \left( \frac{P_v}{p_v} \right)^{\delta k} |\lambda_v|^k |\lambda_v|^k |s_v|^k \frac{1}{v^k} \sum_{n=v+1}^{m+1} \hat{a}_{nv+1}
$$
Finally, by virtue of the hypotheses of Theorem 9, by Lemma 6, we have $v \beta_v = O(1)$, then

\begin{align*}
&= O(1) \sum_{v=1}^{m} \left( \frac{P_v}{p_v} \right)^{\delta k} |\lambda_v|^{k-1} |\lambda_v| s_v \left| \frac{v}{p_v} \right| \\
&= O(1) \sum_{v=1}^{m} \left( \frac{P_v}{p_v} \right)^{\delta k} \frac{1}{X_v^{k-1}} |\lambda_v| s_v \left| \frac{v}{p_v} \right| = O(1) \quad \text{as} \quad m \to \infty.
\end{align*}

This completes the proof of Theorem 9.

**Conclusion 10.** If we take $\delta = 0$ in Theorem 9, then Theorem 9 reduces to $|A, p_m|$ summability theorem (see [21]).

Let $(X_n)$ be a positive non-decreasing sequence. The following results have been obtained.

1. If we take $a_{nv} = \frac{P_{nv}}{p_{nv}}$ in Theorem 9, then Theorem 9 reduces to Theorem 5.

2. If we take $\delta = 0$ and $a_{nv} = \frac{P_{nv}}{p_{nv}}$ in Theorem 9, then we obtain Theorem 4 and
if we put $\delta = 0$ and $k = 1$ in Theorem 5, we have a known result of Mishra and Srivastava dealing with $[N, p_n]$ summability factors of infinite series (see [15]).

3. If we take $a_{nv} = \frac{p_v}{p_n}$ and $p_n = 1$ for all values of $n$ in Theorem 9, then we obtain a known result of Mishra and Srivastava concerning the $[C, 1; \delta]_k$ summability factors of infinite series.

4. If we take $\delta = 0$, $a_{nv} = \frac{p_v}{p_n}$ and $p_n = 1$ for all values of $n$ in Theorem 9, then we obtain a known result of Mishra and Srivastava concerning the $[C, 1]_k$ summability factors of infinite series (see [14]).

References


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