On generalized weakly symmetric $(LCS)_n$-manifolds

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Abstract

The object of the present paper is to study generalized weakly symmetric and weakly Ricci symmetric $(LCS)_n$-manifolds. Our aim is to bring out different type of curvature restrictions for which $(LCS)_n$-manifolds are sometimes Einstein and some other time remain $\eta$-Einstein. Finally, the existence of such manifold is ensured by a non-trivial example.

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1. Introduction

The notion of Lorentzian concircular structure manifolds (briefly $(LCS)_n$-manifolds) has been initiated by Shaikh [25]. Thereafter, a lot of study has been carried out. For details we refer [5, 12, 19, 27–30, 33] and the references therein.

The notion of weakly symmetric Riemannian manifold have been introduced by Tamássy and Binh [34]. Thereafter, a lot research has been carried out in this topic. For details, we refer to see [1, 2, 10, 13, 14, 21–24, 26, 31, 32] and the references there in.

In the spirit of Tamássy and Binh [34], a Riemannian manifold $(M^n, g)(n > 2)$, is said to be a weakly symmetric manifold, if its curvature tensor $\bar{R}$ of type $(0, 4)$ is not identically zero and admits the identity

\[
(\nabla_X \bar{R})(Y, U, V, W) = A_1(X) \bar{R}(Y, U, V, W) + B_1(Y) \bar{R}(X, U, V, W) + B_1(U) \bar{R}(Y, X, V, W) + D_1(V) \bar{R}(Y, U, X, W) + D_1(W) \bar{R}(Y, U, V, X)
\]

where $A_1, B_1$ & $D_1$ are non-zero 1-forms defined by $A_1(X) = g(X, \sigma_1), B_1(X) = g(X, g_1)$ and $D_1(X) = g(X, \pi_1)$, for all $X$ and $\bar{R}(Y, U, V, W) = g(\bar{R}(Y, U)V, W)$, $\nabla$ being the operator of the covariant differentiation with respect to the metric tensor $g$. An $n$-dimensional Riemannian manifold of this kind is denoted by $(WS)_n$-manifold.
Keeping in tune with Dubey [11], the author have introduced the notion of a generalized weakly symmetric Riemannian manifold (which is abbreviated hereafter as (GWS)$_n$-manifold). An $n$-dimensional Riemannian manifold is said to be generalized weakly symmetric if it admits the equation

\[(\nabla_X \bar{R})(Y, U, V, W) = A_1(X) \bar{R}(Y, U, V, W) + B_1(Y) \bar{R}(X, U, V, W) + B_1(W) \bar{R}(Y, X, V, W) + D_1(V) \bar{R}(X, Y, U, W) + D_1(W) \bar{R}(Y, X, U, V) + D_2(V) \bar{G}(Y, U, V, W) + D_2(W) \bar{G}(Y, U, V, X)\]  \hspace{0.5cm} (1.2)

where

\[\bar{G}(Y, U, V, W) = [g(U, V)g(Y, W) - g(Y, V)g(U, W)]\]  \hspace{0.5cm} (1.3)

and $A_i$, $B_i$ & $D_i$ are non-zero 1-forms defined by $A_i(X) = g(X, \sigma_i)$, $B_i(X) = g(X, g_i)$, and $D_i(X) = g(X, \pi_i)$, for $i = 1, 2$. The beauty of such $(GWS)_n$-manifold is that it has the flavour of

(i) locally symmetric space [7] (for $A_i = B_i = D_i = 0$),
(ii) locally recurrent space [36] (for $A_1 \neq 0$, $A_2 = B_1 = D_1 = 0$),
(iii) generalized recurrent space [11] (for $A_1 \neq 0$. $B_1 = D_1 = 0$),
(iv) pseudo symmetric space [8] \hspace{0.5cm} (for $\frac{D_1}{2} = B_1 = D_1 = H_1 \neq 0$, $A_2 = B_2 = D_2 = 0$),
(v) generalized pseudo symmetric space [3] (for $\frac{D_1}{2} = B_1 = D_1 = H_1 \neq 0$),
(vi) semi-pseudo symmetric space [35] (for $A_i = B_2 = D_2 = 0$, $B_1 = D_1 \neq 0$),
(vii) generalized semi-pseudo symmetric space [4] (for $A_1 = 0$, $B_1 = D_1 \neq 0$),
(viii) almost pseudo symmetric space [9] (for $A_1 = H_1 + K_1$, $B_1 = D_1 = H_1 \neq 0$ and $A_2 = B_2 = D_2 = 0$),
(ix) almost generalized pseudo symmetric space [6] (for $A_1 = H_i + K_i$, $B_1 = D_1 = H_i \neq 0$),
(x) weakly symmetric space [34] (for $A_1$, $B_1$, $D_1 \neq 0$, $A_2 = B_2 = D_2 = 0$).

An $n$-dimensional Riemannian manifold is said to be generalized weakly Ricci symmetric if it admits the equation

\[(\nabla_X S)(Y, Z) = A_1(X)S(Y, Z) + B_1(Y)S(X, Z) + D_1(Z)S(Y, X) + A_2(X)\bar{g}(Y, Z) + B_2(Y)\bar{g}(X, Z) + D_2(Z)\bar{g}(Y, X)\]  \hspace{0.5cm} (1.4)

where and $A_i$, $B_i$ & $D_i$ are non-zero 1-forms defined by $A_i(X) = g(X, \sigma_i)$, $B_i(X) = g(X, g_i)$, and $D_i(X) = g(X, \pi_i)$, for $i = 1, 2$. The beauty of generalized weakly Ricci symmetric manifold is that it has the flavour of Ricci symmetric, Ricci recurrent, generalized Ricci recurrent, pseudo Ricci symmetric, generalized pseudo Ricci symmetric, semi-pseudo Ricci symmetric, generalized semi-pseudo Ricci symmetric, almost pseudo Ricci symmetric, almost generalized pseudo Ricci symmetric and weakly Ricci symmetric space as special cases.

Now, if the vectors associated to the 1-forms $A_1$, $B_1$ & $D_1$ are respectively co-directional with that of $A_2$, $B_2$ & $D_2$ that is $A_1(X) = \phi A_2(X)$, $B_1(X) = \phi B_2(X)$ & $D_1(X) = \phi D_2(X) \forall X$, where $\phi$ being a non-zero constant function, then the relation (1.4) turns into

\[(\nabla_X Z)(Y, U) = A_1(X)Z(Y, U) + B_1(Y)Z(X, U) + D_1(U)Z(X, U)\]

where $Z(X, Y) = S(X, Y) + \phi g(X, Y)$ is well known $Z$-tensor introduced in ([15, 18]).

This leads to the following

**Proposition 1.1.** Every generalized weakly Ricci symmetric manifold is a weakly $Z$-symmetric manifold provided the vector fields associated to the 1-forms $A_1$, $B_1$ \& $D_1$ are co-directional with that of $A_2$, $B_2$ \& $D_2$ respectively.
Our work is structured as follows. Section 2 is concerned with \((LCS)_n\)-manifolds and some known results. In section 3, we have investigated a generalized weakly symmetric \((LCS)_n\)-manifold and it is observed that such a space is an \(\eta\)-Einstein manifold provided
\(B^*(\xi) \neq -\alpha\). We also tabulated different type of curvature restrictions for which \((LCS)_n\)-manifolds are sometimes Einstein and some other time remain \(\eta\)-Einstein. Section 4, is concerned with a generalized weakly Ricci-symmetric \((LCS)_n\)-manifold which is found to be an \(\eta\)-Einstein space. Finally, we have constructed an example of a generalized weakly symmetric \((LCS)_n\)-manifold.

2. \((LCS)_n\)-manifolds and some known results

An \(n\)-dimensional Lorentzian manifold \(M\) is a smooth connected paracompact Hausdorff manifold with a Lorentzian metric \(g\), that is, \(M\) admits a smooth symmetric tensor field \(g\) of type (0, 2) such that for each point \(p \in M\), the tensor \(g_p : T_pM \times T_pM \rightarrow R\) is a non-degenerate inner product of signature \((-\ldots,+,\ldots+)\), where \(T_pM\) denotes the tangent vector space of \(M\) at \(p\) and \(R\) is the real number space. A non-zero vector \(v \in T_pM\) is said to be timelike (resp., non-spacelike, null, spacelike) if it satisfies \(g_p(U,U) < 0\) (resp, \(\leq 0\), \(= 0, > 0\), [20]. The category to which a given vector falls is called its causal character.

Let \(M^n\) be a Lorentzian manifold admitting a unit timelike concircular vector field \(\xi\), called the characteristic vector field of the manifold. Then we have
\[g(\xi, \xi) = -1.\] (2.1)

Since \(\xi\) is a unit concircular vector field, there exists a non-zero 1-form \(\eta\) such that for
\[g(X, \xi) = \eta(X)\] (2.2)

the equation of the following form holds
\[(\nabla_X \eta)(Y) = \alpha \{g(X,Y) + \eta(X)\eta(Y)\}\] (\(\alpha \neq 0\)) (2.3)

for all vector fields \(X, Y\) where \(\nabla\) denotes the operator of covariant differentiation with respect to the Lorentzian metric \(g\) and \(\alpha\) is a non-zero scalar function satisfies
\[\nabla_X \alpha = (X\alpha) = \alpha(X) = \rho \eta(X),\] (2.4)

\(\rho\) being a certain scalar function. If we put
\[\phi X = \frac{1}{\alpha} \nabla_X \xi,\] (2.5)

then from (2.3) and (2.5), we have
\[\phi X = X + \eta(X)\xi,\] (2.6)

from which it follows that \(\phi\) is a symmetric (1, 1) tensor. Thus the Lorentzian manifold \(M^n\) together with the unit timelike concircular vector field \(\xi\), its associated 1-form \(\eta\) and (1,1) tensor field \(\phi\) is said to be a Lorentzian concircular structure manifold (briefly \((LCS)_n\)-manifold) [5]. In a \((LCS)_n\)-manifold, the following relations hold [25]:
\[
\begin{align*}
\eta(\xi) &= -1, & \phi \circ \xi &= 0, \quad (2.7) \\
\eta(\phi X) &= 0, & g(\phi X, \phi Y) &= g(X, Y) + \eta(X)\eta(Y), \quad (2.8) \\
\eta(R(X, Y)Z) &= (\alpha^2 - \rho)[g(Y, Z)\eta(X) - g(X, Z)\eta(Y)], \quad (2.9) \\
R(X, Y)\xi &= (\alpha^2 - \rho)[\eta(Y)X - \eta(X)Y], \quad (2.10) \\
S(X, \xi) &= (n - 1)(\alpha^2 - \rho)\eta(X) \quad (2.11)
\end{align*}
\]
for any vector fields $X, Y, Z$.

**Lemma 2.1.** Let $(M^n, g)$ be a $(LCS)_n$-manifold. Then for any $X; Y; Z$ the following relation holds:

\[
(\nabla_W S)(X, \xi) = (n - 1)[\alpha(\alpha^2 - \rho)g(X, W) + (2\alpha\rho - \beta)\eta(W)\eta(X)] - \alpha S(X, W)
\]

(2.12)

In this connection we would like to mention that equation (2.3) is the defining property of concircular or unit time-like torse-forming vector field. In ([16], Theorem 2.1), the authors proved that a Lorentzian manifold is twisted, i.e. the metric is written in the form

\[
ds^2 = -dt^2 + f(t, x^\gamma)^2 \tilde{g}_{\alpha\beta} dx^\alpha dx^\beta,
\]

if and only if it admits a unit time-like torse-forming vector field. Moreover eq (2.4) and the consequent integrability relations (2.10) and (2.11) in [16] ensure that the unit time-like vector is an eigen vector of the Ricci tensor. Also, Proposition 3.7 of [17] ensures that the space-time is a generalized Robertson-Walker space-time, i.e. the metric is written in the form

\[
ds^2 = -dt^2 + f(t)^2 \tilde{g}_{\alpha\beta} dx^\alpha dx^\beta,
\]

$\tilde{g}$ being the metric tensor of a $n - 1$ dimensional Riemannian manifold.

### 3. Generalized weakly symmetric $(LCS)_n$-manifold

A non-flat $n$-dimensional $(LCS)_n$-manifold $(M^n; g)$ $(n > 2)$, is termed as generalized weakly symmetric manifold, if its Riemannian curvature tensor $\tilde{R}$ of type $(0; 4)$ is not identically zero and admits the identity

\[
(\nabla_X \tilde{R})(Y, U, V, W) = A^\ast(X)\tilde{R}(Y, U, V, W) + B^\ast(Y)\tilde{R}(X, U, V, W)
\]

\[+ D^\ast(W)\tilde{R}(Y, U, V, X) + \alpha^\ast(X)G(Y, U, V, W)
\]

\[+ \beta^\ast(Y)G(X, U, V, W) + \beta^\ast(U)G(Y, X, V, W)
\]

\[+ \gamma^\ast(V)G(Y, U, X, W) + \gamma^\ast(W)G(Y, U, V, X)
\]

(3.1)

where

\[
G(Y, U, V, W) = [g(U, V)g(Y, W) - g(Y, V)g(U, W)]
\]

(3.2)

and $A^\ast$, $B^\ast$, $D^\ast$, $\alpha^\ast$, $\beta^\ast$ & $\gamma^\ast$ are non-zero 1-forms which are defined as $A^\ast(X) = g(X, \theta_1), B^\ast(X) = g(X, \phi_1), D^\ast(X) = g(X, \pi_1), \alpha^\ast(X) = g(X, \theta_2), \beta^\ast(X) = g(X, \phi_2)$ and $\gamma^\ast(X) = g(X, \pi_2)$.

Now, contracting $U$ over $V$ in both sides of (3.1) we find

\[
(\nabla_X S)(Y, W) = A^\ast(X)S(Y, W) + B^\ast(Y)S(X, W) + D^\ast(W)S(Y, X)
\]

\[- B^\ast(R(Y, X)W) + D^\ast(R(X, W)Y) + (n - 1)[\alpha^\ast(X)
\]

\[g(Y, W) + \beta^\ast(Y)g(X, W) + \gamma^\ast(W)g(Y, X) - \beta^\ast(Y)g(X, W)
\]

\[+ [\beta^\ast(X) + \gamma^\ast(X)]g(Y, W) - \gamma^\ast(W)g(X, Y)
\]

(3.3)

which yields

\[(n - 1)[\alpha(\alpha^2 - \rho)g(X, W) + (2\alpha\rho - \beta)\eta(W)\eta(X)] - \alpha S(X, W)
\]

\[= (\alpha^2 - \rho)[(n - 1)\{A^\ast(X)\eta(W) + D^\ast(W)\eta(X)\} + \eta(W)B^\ast(X)
\]

\[- g(X, W)B^\ast(\xi) + \eta(W)D^\ast(X) - \eta(X)D^\ast(W)] + B^\ast(\xi)S(X, W)
\]

\[+ (n - 1)[\alpha^\ast(X)\eta(W) + \beta^\ast(\xi)g(X, W) + \gamma^\ast(W)\eta(X)]
\]

\[= \beta^\ast(\xi)g(X, W) + [\beta^\ast(X) + \gamma^\ast(X)]\eta(W) - \gamma^\ast(W)\eta(X)
\]

(3.4)
for $Y = \xi$. Setting $X = W = \xi$ in the foregoing equation, we obtain
\[
-(2\alpha \rho - \beta) = (\alpha^2 - \rho)[A^*(\xi) + B^*(\xi)] + [\alpha^*(\xi) + \beta^*(\xi) + \gamma^*(\xi)].
\tag{3.5}
\]
In a weakly symmetric $(LCS)_n$-manifold we have the relation (3.4). Setting $X = \xi$ in (3.4) we get
\[
(n - 2)[(\alpha^2 - \rho)D^*(W) + \gamma^*(W)] = [(n - 1)\{(2\alpha \rho - \beta) + (\alpha^2 - \rho)\{A^*(\xi) + B^*(\xi)\}\}
+ (\alpha^2 - \rho)D^*(\xi)]\eta(W) + [(n - 1)\{\alpha^*(\xi) + \beta^*(\xi)\}
+ \gamma^*(\xi)]\eta(W).
\tag{3.6}
\]
In view of (3.5), the relation (3.6) reduces to
\[
[(\alpha^2 - \rho)D^*(W) + \gamma^*(W)] = -[(\alpha^2 - \rho)D^*(\xi) + \gamma^*(\xi)]\eta(W).
\tag{3.7}
\]
Again, contracting over $Y$ and $W$ in (3.1) we get
\[
(\nabla_X S)(U, V) = A^*(X)S(U, V) + B^*(R(X, U)V) + B^*(U)S(X, V)
+ D^*(V)S(U, X) + D^*(R(X, V)U) + (n - 1)[\{\alpha^*(X)g(U, V)
+ \beta^*(U)g(X, V) + \gamma^*(V)g(X, U)\}] + [\gamma^*(X)g(U, V)
- \gamma^*(V)g(U, X) + \beta^*(X)g(U, V) - \beta^*(U)g(X, V).
\tag{3.8}
\]
Setting $V = \xi$ in (3.8) and using (2.12), (2.11), we get
\[
(n - 1)[\alpha(\alpha^2 - \rho)g(X, U) + (2\alpha \rho - \beta)\eta(U)\eta(X)] - \alpha S(X, U)
= (\alpha^2 - \rho)[(n - 1)\{A^*(X)\eta(U) + B^*(U)\eta(X)\} + B^*(X)\eta(U) - B^*(U)\eta(X)
+ D^*(X)\eta(U) - D^*(\xi)g(X, U)] + D^*(\xi)S(U, X) + (n - 1)[\{\alpha^*(X)\eta(U)
+ \beta^*(U)\eta(X) + \gamma^*(\xi)g(X, U)\}] + [\gamma^*(X)\eta(U)
- \gamma^*(\xi)g(U, X) + \beta^*(X)\eta(U) - \beta^*(U)\eta(X),
\tag{3.9}
\]
which turns into
\[
[(\alpha^2 - \rho)B(U) + \beta(U)] = -[(\alpha^2 - \rho)B(\xi) + \beta(\xi)]\eta(U)
\tag{3.10}
\]
for $X = \xi$ and
\[
[(\alpha^2 - \rho)A^*(X) + \alpha^*(X)] = -[(\alpha^2 - \rho)A^*(\xi) + \alpha^*(\xi)]\eta(X)
\tag{3.11}
\]
for $U = \xi$. In view of (3.5), (3.7), (3.10) and (3.11) we have
\[
(2\alpha \rho - \beta)\eta(X) = (\alpha^2 - \rho)[A^*(X) + B^*(X) + D^*(X)]
+ [\alpha^*(X) + \beta^*(X) + \gamma^*(X)].
\tag{3.12}
\]
Now, making use of (3.10)-(3.12) in (3.4), we find that
\[
-\{\alpha + B^*(\xi)\}S(X, W) = [(n - 2)\beta^*(\xi) - (\alpha^2 - \rho)\{(n - 1)\alpha + B^*(\xi)\}]g(X, W)
- (n - 2)[(2\alpha \rho - \beta)\eta(W)\eta(X) + (\alpha^2 - \rho)\{A^*(X)\eta(W)
+ D^*(W)\eta(X)\}] + \{\alpha^*(X)\eta(W) + \gamma^*(W)\eta(X )\}
\tag{3.13}
\]
which leaves
\[
S(X, W) = \left[\frac{(n - 2)(\alpha^2 - \rho)\{\alpha - B^*(\xi)\}}{\alpha + B^*(\xi)}\right]g(X, W)
- \frac{(n - 2)[\{\alpha^2 - \rho\}B^*(\xi) + \gamma^*(\xi)]}{\alpha + B^*(\xi)}\eta(W)\eta(X).
\tag{3.14}
\]
after a straight forward calculation. Approaching in a different manner, we can also have

\[ S(X, W) = \left[(\alpha^2 - \rho) + (n - 2) \left( \frac{\alpha - \gamma^*}{\alpha + D^*(\xi)} \right) \right] g(X, W) \]

\[ - \frac{(n - 2)\left[ (\alpha^2 - \rho)D^*(\xi) + \beta^*(\xi) \right]}{\alpha + D^*(\xi)} \eta(W)\eta(X). \] (3.15)

This leads to the followings.

**Theorem 3.1.** A generalized weakly symmetric \((LCS)_n\)-manifold \(M^n(\phi, \xi, \eta, g)(n > 2)\) is an \(\eta\)-Einstein provided that \(B^*(\xi) \neq -\alpha\).

**Theorem 3.2.** In an \((LCS)_n\)-manifold the following table hold good

<table>
<thead>
<tr>
<th>Type of curvature restriction</th>
<th>Nature of the space corresponding to curvature restriction</th>
</tr>
</thead>
<tbody>
<tr>
<td>locally symmetric space</td>
<td>Einstein space</td>
</tr>
<tr>
<td>locally recurrent space</td>
<td>Einstein space</td>
</tr>
<tr>
<td>generalized recurrent space</td>
<td>Einstein space</td>
</tr>
<tr>
<td>pseudo symmetric space</td>
<td>(\eta)-Einstein space</td>
</tr>
<tr>
<td>generalized pseudo symmetric space</td>
<td>(\eta)-Einstein space</td>
</tr>
<tr>
<td>semi-pseudo symmetric space</td>
<td>(\eta)-Einstein space</td>
</tr>
<tr>
<td>almost pseudo symmetric space</td>
<td>(\eta)-Einstein space</td>
</tr>
<tr>
<td>almost generalized pseudo symmetric space</td>
<td>(\eta)-Einstein space</td>
</tr>
<tr>
<td>weakly symmetric space</td>
<td>(\eta)-Einstein space</td>
</tr>
</tbody>
</table>

Note that if a manifold is locally recurrent, then it is Ricci recurrent, i.e. \(\nabla_k R_{jl} = \beta_k R_{jl}\), for a non-null one form \(\beta_i\) which leaves after transvection \(\nabla_k R = \beta_k R\). Consequently, the manifold is Ricci flat as it is known that the scalar curvature of an Einstein manifold is constant. Thus we can state the following corollary.

**Corollary 3.3.** Every locally recurrent \((LCS)_n\)-manifold is Ricci flat.

4. Generalized weakly Ricci symmetric \((LCS)_n\)-manifold

A non-flat \(n\)-dimensional \((LCS)_n\)-manifold \((M^n; g)(n > 2)\), is said to be a generalized weakly Ricci symmetric manifold, if its Ricci tensor \(S\) of type \((0, 2)\) is not identically zero and admits the identity

\[ (\nabla_X S)(Y, Z) = A^*_1(X)S(Y, Z) + B^*_1(Y)S(X, Z) + D^*_1(Z)S(Y, X) \]

\[ + A^*_2(X)g(Y, Z) + B^*_2(Y)g(X, Z) + D^*_2(Z)g(Y, X) \]  (4.1)

where \(A^*_i, B^*_i & D^*_i\) are non-zero 1-forms which are defined as \(A^*_i(X) = g(X, \theta_i)\), \(B^*_i(X) = g(X, \phi_i)\), \(D^*_1(X) = g(X, \pi_i)\) for \(i = 1, 2\). Setting, \(Y = \xi\) in (4.1) and then making use of (2.12), we have

\[ (n - 1)[\alpha(\alpha^2 - \rho)g(X, Z) + (2\alpha \rho - \beta)\eta(Z)\eta(X)] - \alpha S(X, Z) \]

\[ = (\alpha^2 - \rho)(n - 1)[A^*_1(X)\eta(Z) + D^*_1(Z)\eta(X)] + B^*_1(\xi)S(X, Z) \]

\[ + A^*_2(X)\eta(Z) + B^*_2(\xi)g(X, Z) + D^*_2(Z)\eta(X) \]  (4.2)
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which yields

\[
(\alpha^2 - \rho)(n - 1)[A_1^*(\xi) + B_1^*(\xi) + D_1^*(\xi)] + [A_2^*(\xi) + B_2^*(\xi) + D_2^*(\xi)] = -(n - 1)(2\alpha\rho - \beta),
\]

for \(X = Z = \xi\).

Setting \(Z = \xi\) in (4.2) we obtain

\[
(n - 1)(\alpha^2 - \rho)[A_1^*(X) + A_2^*(\xi)] = -[A_2^*(X) + A_2^2(\xi)\eta(X)].
\]

Proceeding in a similar manner we can find

\[
(\alpha^2 - \rho)(n - 1)[B_1^*(X) + B_2^*(\xi)] = -[B_2^*(X) + B_2^2(\xi)\eta(X)],
\]

(4.5)

\[
(\alpha^2 - \rho)(n - 1)[D_1^*(X) + D_2^*(\xi)] = -[D_2^*(\xi) + D_2^2(\xi)\eta(X)].
\]

(4.6)

\textbf{Theorem 4.1.} In a generalized weakly Ricci symmetric \((LCS)_{\eta}\)-manifold \(M^n(\phi, \xi, \eta, g)(n > 2)\) the 1-forms are related by

\[
(\alpha^2 - \rho)(n - 1)[A_1^*(X) + B_1^*(X) + D_1^*(X)] + [A_2^*(X) + B_2^*(X) + D_2^*(X)]
= (n - 1)(2\alpha\rho - \beta)\eta(X).
\]

(4.7)

\textbf{Proof.} Adding (4.4), (4.5) & (4.6) and then making use of (4.3) in the resultant, one can easily obtain (4.7). \(\square\)

Now, making use of (4.3)-(4.7) in (4.2), we find that

\[
S(X, Z) = \left[\frac{(n - 1)\alpha(\alpha^2 - \rho) - B_2^*(\xi)}{\alpha + B_1^*(\xi)}\right]g(X, Z)
- \left[\frac{(\alpha^2 - \rho)(n - 1)B_1^*(\xi) + B_2^*(\xi)}{\alpha + B_1^*(\xi)}\right]\eta(X)\eta(Z)
\]

(4.8)

This leads to the followings

\textbf{Theorem 4.2.} A generalized weakly Ricci symmetric \((LCS)_{\eta}\)-manifold \(M^n(\phi, \xi, \eta, g)\) is an \(\eta\)-Einstein provided that \(B_1^*(\xi) \neq -\alpha\).

\textbf{Theorem 4.3.} In an \((LCS)_{\eta}\)-manifold the following table holds good

<table>
<thead>
<tr>
<th>Type of curvature restriction</th>
<th>Nature of the space corresponding to curvature restriction</th>
</tr>
</thead>
<tbody>
<tr>
<td>Ricci symmetric space</td>
<td>Einstein space</td>
</tr>
<tr>
<td>Ricci recurrent space</td>
<td>Einstein space</td>
</tr>
<tr>
<td>generalized Ricci-recurrent space</td>
<td>Einstein space</td>
</tr>
<tr>
<td>pseudo Ricci-symmetric space</td>
<td>(\eta)-Einstein space</td>
</tr>
<tr>
<td>generalized pseudo</td>
<td>(\eta)-Einstein space</td>
</tr>
<tr>
<td>semi-pseudo Ricci-symmetric space</td>
<td>(\eta)-Einstein space</td>
</tr>
<tr>
<td>generalized semi-pseudo</td>
<td>(\eta)-Einstein space</td>
</tr>
<tr>
<td>almost pseudo</td>
<td>(\eta)-Einstein space</td>
</tr>
<tr>
<td>almost generalized pseudo</td>
<td>(\eta)-Einstein space</td>
</tr>
<tr>
<td>weakly Ricci-symmetric space</td>
<td>(\eta)-Einstein space</td>
</tr>
</tbody>
</table>
Note that if a manifold is Ricci recurrent, i.e. $\nabla_k R_{jl} = \beta_k R_{jl}$, for a non-null one form $\beta_k$ which leaves after transvection $\nabla_k R = \beta_k R$. Consequently, the manifold is Ricci flat as it is known that the scalar curvature of an Einstein manifold is constant. Thus we can state the following corollary.

**Corollary 4.4.** Every locally Ricci recurrent $(LCS)_n$ manifold is Ricci flat.

### 5. Existence of generalized weakly symmetric $(LCS)_3$-manifold

**Example 5.1.** Let $M^3(\phi, \xi, \eta, g)$ be an $(LCS)_n$-manifold $(M^3, g)$ with a $\phi$-basis

$$e_1 = e^2 \left( x \frac{\partial}{\partial x} + y \frac{\partial}{\partial y} \right), e_2 = \phi e_1 = e^3 \frac{\partial}{\partial y}, e_3 = \xi = e^2 \frac{\partial}{\partial z}.$$

Then from Koszul’s formula for Lorentzian metric $g$, we can obtain the Levi-Civita connection as follows

$$\nabla_{e_1} e_3 = -e^2 e_1, \quad \nabla_{e_1} e_2 = 0, \quad \nabla_{e_1} e_1 = -e^2 e_3,$$

$$\nabla_{e_2} e_3 = -e^2 e_2, \quad \nabla_{e_2} e_2 = -e^2 e_3 - e^3 e_1, \quad \nabla_{e_2} e_1 = -e^2 e_2,$$

$$\nabla_{e_3} e_3 = 0, \quad \nabla_{e_3} e_2 = 0, \quad \nabla_{e_3} e_1 = 0.$$

From the above it can be easily seen that $(\phi, \xi, \eta, g)$ is an $(LCS)_3$ structure on $M$. Consequently $M^3(\phi, \xi, \eta, g)$ is an $(LCS)_3$-manifold with $\alpha = -e^2 \neq 0$ and $\rho = 2e^4$. Using the above relations, one can easily calculate the non-vanishing components of the curvature tensor $\bar{R}$ (up to symmetry and skew-symmetry)

$$\bar{R}(e_1, e_2, e_1, e_2) = (1 - e^2) e^2,$$

$$\bar{R}(e_1, e_3, e_1, e_3) = -e^4 = \bar{R}(e_2, e_3, e_2, e_3).$$

Since $\{e_1, e_2, e_3\}$ forms a basis, any vector field $X, Y, U, V \in \chi(M)$ can be written as

$$X = \sum_i a_i e_i, \quad Y = \sum_i b_i e_i, \quad U = \sum_i c_i e_i, \quad V = \sum_i d_i e_i,$$

Then

$$\bar{R}(X, Y, U, V) = \left( [(a_1 b_2 - a_2 b_1)(c_1 d_2 - c_2 d_1)](1 - e^2) e^2 + \right.$$

$$\left. -[(a_1 b_3 - a_3 b_1)(c_1 d_3 - c_3 d_1)]e^4 + \right.$$

$$\left. -[(a_2 b_3 - a_3 b_2)(c_2 d_3 - c_3 d_2)]e^4 \right) = T_1 \text{ (say),}$$

$$\bar{R}(e_1, Y, U, V) = -b_3(c_1 d_3 - c_3 d_1) e^4 + b_2(c_1 d_2 - c_2 d_1)(1 - e^2) e^2 = \lambda_1 \text{ (say),}$$

$$\bar{R}(e_2, Y, U, V) = -b_3(c_2 d_3 - c_3 d_2) e^4 - b_1(c_1 d_2 - c_2 d_1)(1 - e^2) e^2 = \lambda_2 \text{ (say),}$$

$$\bar{R}(e_3, Y, U, V) = b_1(c_1 d_3 - c_3 d_1) e^4 + b_2(c_2 d_3 - c_3 d_2) e^4 = \lambda_3 \text{ (say),}$$

$$\bar{R}(X, e_1, U, V) = a_3(c_1 d_3 - c_3 d_1) e^4 - a_2(c_1 d_2 - c_2 d_1)(1 - e^2) e^2 = \lambda_4 \text{ (say),}$$

$$\bar{R}(X, e_2, U, V) = a_3(c_2 d_3 - c_3 d_2) e^4 + a_1(c_1 d_2 - c_2 d_1)(1 - e^2) e^2 = \lambda_5 \text{ (say),}$$

$$\bar{R}(X, e_3, U, V) = -a_1(c_1 d_3 - c_3 d_1) e^4 - a_2(c_2 d_3 - c_3 d_2) e^4 = \lambda_6 \text{ (say),}$$

$$\bar{R}(X, Y, e_1, V) = -d_3(a_1 b_3 - a_3 b_1) e^4 + d_2(a_1 b_2 - a_2 b_1)(1 - e^2) e^2 = \lambda_7 \text{ (say),}$$

$$\bar{R}(X, Y, e_2, V) = -d_3(a_2 b_3 - a_3 b_2) e^4 + d_1(a_1 b_2 - a_2 b_1)(1 - e^2) e^2 = \lambda_8 \text{ (say),}$$

$$\bar{R}(X, Y, e_3, V) = -d_3(a_3 b_3 - a_3 b_2) e^4 + d_0(a_1 b_2 - a_2 b_1)(1 - e^2) e^2 = \lambda_9 \text{ (say).}$$
\begin{align*}
\bar{R}(X, Y, e_2, V) &= -d_3(a_2b_3 - a_3b_2)e^{4z} - d_1(a_1b_2 - a_2b_1)(1 - e^{2z})e^{2z} \\
&= \lambda_8 \text{ (say)}, \\
\bar{R}(X, Y, e_3, V) &= d_1(a_1b_3 - a_3b_1)e^{4z} + d_2(a_2b_3 - a_3b_2) = \lambda_9 \text{ (say)}, \\
\bar{R}(X, Y, U, e_1) &= c_3(a_1b_3 - a_3b_1)e^{4z} - c_2(a_1b_2 - a_2b_1)(1 - e^{2z})e^{2z} \\
&= \lambda_{10} \text{ (say)}, \\
\bar{R}(X, Y, U, e_2) &= c_3(a_2b_3 - a_3b_2)e^{4z} + c_1(a_1b_2 - a_2b_1)(1 - e^{2z})e^{2z} \\
&= \lambda_{11} \text{ (say)}, \\
\bar{R}(X, Y, U, e_3) &= -c_1(a_1b_3 - a_3b_1)e^{4z} - c_2(a_2b_3 - a_3b_2)e^{4z} = \lambda_{12} \text{ (say)}, \\
\bar{G}(X, Y, U, V) &= (b_1c_1 + b_2c_2 - b_3c_3)(a_1d_1 + a_2d_2 - a_3d_3) \\
&- (a_1c_1 + a_2c_2 - a_3c_3)(b_1d_1 + b_2d_2 - b_3d_3) = T_2 \text{ (say)}, \\
\bar{G}(e_1, Y, U, V) &= (b_2c_2 - b_3c_3)d_1 - (b_2d_2 - b_3d_3)c_1 = \omega_1 \text{ (say)}, \\
\bar{G}(e_2, Y, U, V) &= (b_1c_1 - b_3c_3)d_2 - (b_1d_1 - b_3d_3)c_2 = \omega_2 \text{ (say)}, \\
\bar{G}(e_3, Y, U, V) &= -(b_1c_1 + b_2c_2)d_3 + (b_1d_1 + b_2d_2)c_3 = \omega_3 \text{ (say)}, \\
\bar{G}(X, e_1, U, V) &= (a_2d_2 - a_3d_3)c_1 - (a_2c_2 - a_3c_3)d_1 = \omega_4 \text{ (say)}, \\
\bar{G}(X, e_2, U, V) &= (a_1d_1 - a_3d_3)c_2 - (a_1c_1 - a_3c_3)d_2 = \omega_5 \text{ (say)}, \\
\bar{G}(X, e_3, U, V) &= -(a_1d_1 + a_2d_2)c_3 + (a_1c_1 + a_2c_2)d_3 = \omega_6 \text{ (say)}, \\
\bar{G}(X, Y, e_1, V) &= (a_2d_2 - a_3d_3)b_1 - (b_2d_2 - b_3d_3)a_1 = \omega_7 \text{ (say)}, \\
\bar{G}(X, Y, e_2, V) &= (a_1d_1 - a_3d_3)b_2 - (b_1d_1 - b_3d_3)a_2 = \omega_8 \text{ (say)}, \\
\bar{G}(X, Y, e_3, V) &= -(a_1d_1 + a_2d_2)b_3 + (b_1d_1 + b_2d_2)a_3 = \omega_9 \text{ (say)}, \\
\bar{G}(X, Y, U, e_1) &= (b_2c_2 - b_3c_3)a_1 - (a_2c_2 - a_3c_3)b_1 = \omega_{10} \text{ (say)}, \\
\bar{G}(X, Y, U, e_2) &= (b_1c_1 - b_3c_3)a_2 - (a_1c_1 - a_3c_3)b_2 = \omega_{11} \text{ (say)}, \\
\bar{G}(X, Y, U, e_3) &= -(b_1c_1 + b_2c_2)a_3 + (a_1c_1 + a_2c_2)b_3 = \omega_{12} \text{ (say)},
\end{align*}

and the components which can be obtained from these by the symmetry properties. Now, we calculate the covariant derivatives of the non-vanishing components of the curvature tensor as follows

\begin{align*}
(\nabla_{e_1}\bar{R})(X, Y, U, V) &= e^{2z}[a_1\lambda_3 + a_3\lambda_1 + b_1\lambda_6 + b_3\lambda_4] \\
&+ c_1\lambda_9 + c_3\lambda_7 + d_1\lambda_{12} + b_3\lambda_{10}], \\
(\nabla_{e_2}\bar{R})(X, Y, U, V) &= e^{2z}[(a_1 + a_3)\lambda_2 + a_2\lambda_3 + (b_1 + b_3)\lambda_5 + b_2\lambda_6] \\
&+ (c_1 + c_3)\lambda_8 + c_2\lambda_9 + (d_1 + d_3)\lambda_{11} + d_2\lambda_{12}] \\
&+ e^2[a_2\lambda_1 + b_2\lambda_4 + c_2\lambda_7 + d_2\lambda_{10}], \\
(\nabla_{e_3}\bar{R})(X, Y, U, V) &= 2[(a_1b_2 - a_2b_1)(c_1d_2 - c_2d_1)](1 - 2e^{2z})e^{4z} \\
&- 4[(a_1b_3 - a_3b_1)(c_1d_3 - c_3d_1)]e^{6z} \\
&- 4[(a_2b_3 - a_3b_2)(c_2d_3 - c_3d_2)]e^{6z}.
\end{align*}

For the following choice of the 1-forms

\begin{align*}
A_1^*(e_1) &= \frac{e^{2z}[a_1\lambda_3 + a_3\lambda_1 + b_1\lambda_6 + b_3\lambda_4]}{T_1}, \\
A_2^*(e_1) &= \frac{c_1\lambda_9 + c_3\lambda_7 + d_1\lambda_{12} + b_3\lambda_{10}}{T_2}, \\
A_1^*(e_2) &= \frac{- e^{2z}\{(a_1 + a_3)\lambda_2 + a_2\lambda_3 + (b_1 + b_3)\lambda_5 + b_2\lambda_6(c_1 + c_3)\lambda_8 + c_2\lambda_9 + d_1\}}{T_1},
\end{align*}
\[ A_2^i(e_2) = -e^{2z}\left\{d_3\lambda_{11} + d_2\lambda_{12}\right\} + e^z\left\{a_2\lambda_1 + b_2\lambda_4 + c_2\lambda_7 + d_2\lambda_{10}\right\}, \]
\[ A_1^i(e_3) = -4, \]
\[ B_1^i(e_3) = \frac{1}{a_3\lambda_3 + b_3\lambda_6}, \]
\[ B_2^i(e_3) = \frac{1}{a_3\omega_3 + b_3\omega_6}, \]
\[ D_1^i(e_3) = -\frac{1}{c_3\lambda_9 + d_3\lambda_{12}}, \]
\[ D_2^i(e_3) = -\frac{1}{c_3\omega_9 + d_3\omega_{12}}, \]
\[ A_2^i(e_3) = -2(\frac{a_1b_2 - a_2b_1)(c_1d_2 - c_2d_1)e^{2z}}{T_2}, \]

one can easily verify the relations
\[
(\nabla_{e_i}\bar{R})(X, Y, U, V) = A_1^i(e_i)\bar{R}(X, Y, U, V) \\
+ B_1^i(X)\bar{R}(e_i, Y, U, V) + B_1^i(Y)\bar{R}(X, e_i, U, V) \\
+ D_1^i(U)\bar{R}(X, Y, e_i, V) + D_1^i(V)\bar{R}(X, Y, U, e_i) \\
+ A_2^i(e_i)\bar{G}(X, Y, U, V) \\
+ B_2^i(X)\bar{G}(e_i, Y, U, V) + B_2^i(Y)\bar{G}(X, e_i, U, V) \\
+ D_2^i(U)\bar{G}(X, Y, e_i, V) + D_2^i(V)\bar{G}(X, Y, U, e_i)
\]

for \(i = 1, 2, 3\).

From the above, we can state the following theorem.

**Theorem 5.2.** There exists an \((LCS)_3\)-manifold \((M^3, g)\) which is a generalized weakly symmetric.

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**References**


On generalized weakly symmetric \((LCS)_n\)-manifolds


