On generalized autocommutativity degree of finite groups

Parama Dutta, Rajat Kanti Nath

Department of Mathematical Sciences, Tezpur University, Napaam-784028, Sonitpur, Assam, India.

Abstract

Let $H$ be a subgroup of a finite group $G$ and $\text{Aut}(G)$ be the automorphism group of $G$. In this paper we introduce and study the probability that the autocommutator of a randomly chosen pair of elements, one from $H$ and the other from $\text{Aut}(G)$, is equal to a given element of $G$.

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1. Introduction

Throughout the paper $H$ denotes a subgroup of a finite group $G$ and $\text{Aut}(G)$ denotes automorphism group of $G$. The autocommutativity degree of $G$, denoted by $\text{Pr}(G, \text{Aut}(G))$, is the probability that an automorphism fixes an element of $G$. In other words,

$$\text{Pr}(G, \text{Aut}(G)) = \frac{|\{(x, \alpha) \in G \times \text{Aut}(G) : [x, \alpha] = 1\}|}{|G| |\text{Aut}(G)|}$$

where $[x, \alpha] = x^{-1} \alpha(x)$ is the autocommutator of $x$ and $\alpha$. The study of autocommutativity degree of finite groups was initiated by Sherman [10] in 1975. Many results on $\text{Pr}(G, \text{Aut}(G))$, including some characterizations of $G$ in terms of $\text{Pr}(G, \text{Aut}(G))$, can be found in [1, 3]. In the year 2015, Rismanchian and Sepehrizadeh [9] generalized the concept of autocommutativity degree and studied relative autocommutativity degree of $H$, that is the probability that an automorphism of $G$ fixes an element of $H$. However in the year 2011, Moghaddam et al. [8] also studied this notion. We write $\text{Pr}(H, \text{Aut}(G))$ to denote the relative autocommutativity degree of $H$. Recently, we have obtained several new results on $\text{Pr}(H, \text{Aut}(G))$ in [2]. In this paper, we introduce a new probability concept called the generalized relative autocommutativity degree of $H$ given by the following ratio

$$\text{Pr}_g(H, \text{Aut}(G)) = \frac{|\{(x, \alpha) \in H \times \text{Aut}(G) : [x, \alpha] = g\}|}{|H| |\text{Aut}(G)|}$$

(1.1)

where $g$ is an element of $G$. In other words $\text{Pr}_g(H, \text{Aut}(G))$ is the probability that the autocommutator of a randomly chosen pair of elements, one from $H$ and the other from $\text{Aut}(G)$, is equal to a given element $g \in G$. Clearly, if $g = 1$ (the identity element of $G$) then $\text{Pr}_g(H, \text{Aut}(G)) = \text{Pr}(H, \text{Aut}(G))$. In the forthcoming sections, we obtain some computing
formulae and bounds for $\Pr_g(H, \text{Aut}(G))$. We also obtain some characterizations of groups through $\Pr_g(H, \text{Aut}(G))$.

Let $S(H, \text{Aut}(G)) = \{[x, \alpha] : x \in H \text{ and } \alpha \in \text{Aut}(G)\}$ and $[H, \text{Aut}(G)]$ be the subgroup generated by $S(H, \text{Aut}(G))$. Let $L(H, \text{Aut}(G)) = \{x \in H : [x, \alpha] = 1 \text{ for all } \alpha \in \text{Aut}(G)\}$ and $L(G) = L(G, \text{Aut}(G))$, the absolute center of $G$ defined in [5]. Clearly, $L(H, \text{Aut}(G))$ is a normal subgroup of $H$ contained in $H \cap \mathcal{Z}(G)$. Let $C_{\text{Aut}(G)}(x) = \{\alpha \in \text{Aut}(G) : \alpha(x) = x\}$ for $x \in G$ and $C_{\text{Aut}(G)}(H) = \{\alpha \in \text{Aut}(G) : \alpha(x) = x \text{ for all } x \in H\}$. Then $C_{\text{Aut}(G)}(x)$ is a subgroup of $\text{Aut}(G)$ and $C_{\text{Aut}(G)}(H) = \cap_{x \in H} C_{\text{Aut}(G)}(x)$. Note that if $g \notin S(H, \text{Aut}(G))$ then $\Pr_g(H, \text{Aut}(G)) = 0$, therefore throughout the paper we consider $g \in S(H, \text{Aut}(G))$.

2. Some computing formulae

We begin with the following results.

**Proposition 2.1.** Let $H$ be a subgroup of $G$. If $g \in G$ then

$$\Pr_{g^{-1}}(H, \text{Aut}(G)) = \Pr_g(H, \text{Aut}(G)).$$

**Proof.** Let $A = \{(x, \alpha) \in H \times \text{Aut}(G) : [x, \alpha] = g\}$ and $B = \{(y, \beta) \in H \times \text{Aut}(G) : [y, \beta] = g^{-1}\}$. Then $(x, \alpha) \mapsto (\alpha(x), \alpha^{-1})$ gives a bijection between $A$ and $B$. Therefore, $|A| = |B|$ and hence the result follows from (1.1). \qed

**Proposition 2.2.** Let $G_1$ and $G_2$ be two finite groups such that $\gcd(|G_1|, |G_2|) = 1$. Let $H_1$ and $H_2$ be subgroups of $G_1$ and $G_2$ respectively. If $(g_1, g_2) \in G_1 \times G_2$ then

$$\Pr_{(g_1, g_2)}(H_1 \times H_2, \text{Aut}(G_1 \times G_2)) = \Pr_{g_1}(H_1, \text{Aut}(G_1))\Pr_{g_2}(H_2, \text{Aut}(G_2)).$$

**Proof.** Let $X = \{(x, \alpha_{G_1 \times G_2}) \in (H_1 \times H_2) \times \text{Aut}(G_1 \times G_2) : [x, \alpha_{G_1 \times G_2}] = (g_1, g_2)\}$, $Y = \{(x, \alpha_{G_1}) \in H_1 \times \text{Aut}(G_1) : [x, \alpha_{G_1}] = g_1\}$, and $Z = \{(y, \alpha_{G_2}) \in H_2 \times \text{Aut}(G_2) : [y, \alpha_{G_2}] = g_2\}$. Since $\gcd(|G_1|, |G_2|) = 1$, by [6, Lemma 2.1], we have $\text{Aut}(G_1 \times G_2) = \text{Aut}(G_1) \times \text{Aut}(G_2)$. Therefore, for every $\alpha_{G_1 \times G_2} \in \text{Aut}(G_1 \times G_2)$ there exist unique $\alpha_{G_1} \in \text{Aut}(G_1)$ and $\alpha_{G_2} \in \text{Aut}(G_2)$ such that $\alpha_{G_1 \times G_2} = \alpha_{G_1} \times \alpha_{G_2}$, where $\alpha_{G_1 \times G_2}((x, y)) = (\alpha_{G_1}(x), \alpha_{G_2}(y))$ for all $(x, y) \in H_1 \times H_2$. Also, for all $(x, y) \in H_1 \times H_2$, we have $[(x, y), \alpha_{G_1 \times G_2}] = (g_1, g_2)$ if and only if $[x, \alpha_{G_1}] = g_1$ and $[y, \alpha_{G_2}] = g_2$. These lead to show that $X = Y \times Z$. Therefore

$$\frac{|X|}{|H_1 \times H_2| |\text{Aut}(G_1 \times G_2)|} = \frac{|Y|}{|H_1| |\text{Aut}(G_1)|} \cdot \frac{|Z|}{|H_2| |\text{Aut}(G_2)|}.$$  

Hence, the result follows from (1.1). \qed

In the year 1940, Hall [4] introduced the concept of isoclinism between two groups. Following Hall, Moghaddam et al. [7] have defined autoisoclinism between two groups, in the year 2013. Recently in [2], we generalize the notion of autoisoclinism between two groups. Let $H_1$ and $H_2$ be subgroups of the groups $G_1$ and $G_2$ respectively. The pairs $(H_1, G_1)$ and $(H_2, G_2)$ are said to be autoisoclinic if there exist isomorphisms $\psi : \frac{H_1}{L(H_1, \text{Aut}(G_1))} \rightarrow \frac{H_2}{L(H_2, \text{Aut}(G_2))}$, $\beta : [H_1, \text{Aut}(G_1)] \rightarrow [H_2, \text{Aut}(G_2)]$ and $\gamma : \text{Aut}(G_1) \rightarrow \text{Aut}(G_2)$ such that the following diagram commutes

\[
\begin{array}{ccc}
\frac{H_1}{L(H_1, \text{Aut}(G_1))} \times \text{Aut}(G_1) & \xrightarrow{\psi \times \gamma} & \frac{H_2}{L(H_2, \text{Aut}(G_2))} \times \text{Aut}(G_2) \\
\downarrow^{\alpha(H_1, \text{Aut}(G_1))} & & \downarrow^{\alpha(H_2, \text{Aut}(G_2))} \\
[H_1, \text{Aut}(G_1)] & \xrightarrow{\beta} & [H_2, \text{Aut}(G_2)]
\end{array}
\]
where the maps $a_{(H_i, \text{Aut}(G_i))}: \frac{H_i}{L(H_i, \text{Aut}(G_i))} \times \text{Aut}(G_i) \to [H_i, \text{Aut}(G_i)]$, for $i = 1, 2$, are given by

$$a_{(H_i, \text{Aut}(G_i))}(x_i, L(H_i, \text{Aut}(G_i))), \alpha_i = [x_i, \alpha_i].$$

Such a pair $(\psi \times \gamma, \beta)$ is said to be an autoisoclinism between the pairs of groups $(H_1, G_1)$ and $(H_2, G_2)$. We have the following generalization of [3, Theorem 5.1] and [9, Lemma 2.5].

**Theorem 2.3.** Let $G_1$ and $G_2$ be two finite groups with subgroups $H_1$ and $H_2$ respectively. If $(\psi \times \gamma, \beta)$ is an autoisoclinism between the pairs $(H_1, G_1)$ and $(H_2, G_2)$ then, for $g \in G_1$,

$$\text{Pr}_{\beta(g)}(H_2, \text{Aut}(G_2)) = \text{Pr}_g(H_1, \text{Aut}(G_1)).$$

**Proof.** Let $S_g = \{(x_1 L(H_1, \text{Aut}(G_1)), \alpha_1) \in \frac{H_1}{L(H_1, \text{Aut}(G_1))} \times \text{Aut}(G_1) : [x_1, \alpha_1] = g\}$ and $T_{\beta(g)} = \{(x_2 L(H_2, \text{Aut}(G_2)), \alpha_2) \in \frac{H_2}{L(H_2, \text{Aut}(G_2))} \times \text{Aut}(G_2) : [x_2, \alpha_2] = \beta(g)\}$. Since $(H_1, G_1)$ is autoisoclinic to $(H_2, G_2)$ we have $|S_g| = |T_{\beta(g)}|$. Again, it is clear that

$$|\{(x_1, \alpha_1) \in H_1 \times \text{Aut}(G_1) : [x_1, \alpha_1] = g\}| = |L(H_1, \text{Aut}(G_1))| |S_g|$$

and

$$|\{(x_2, \alpha_2) \in H_2 \times \text{Aut}(G_2) : [x_2, \alpha_2] = \beta(g)\}| = |L(H_2, \text{Aut}(G_2))| |T_{\beta(g)}|.$$  

Hence, the result follows from (1.1), (2.1) and (2.2).

Note that $\text{Aut}(G)$ acts on $G$ by the action $(\alpha, x) \mapsto \alpha(x)$ where $\alpha \in \text{Aut}(G)$ and $x \in G$. Let $\text{orb}(x) = \{\alpha(x) : \alpha \in \text{Aut}(G)\}$ be the orbit of $x \in G$. Then by orbit-stabilizer theorem, we have

$$|\text{orb}(x)| = \frac{|\text{Aut}(G)|}{|C_{\text{Aut}(G)}(x)|}.$$ 

Now we obtain the following computing formula for $\text{Pr}_g(H, \text{Aut}(G))$ in terms of the order of $C_{\text{Aut}(G)}(x)$ and $\text{orb}(x)$.

**Theorem 2.4.** Let $H$ be a subgroup of $G$. If $g \in G$ then

$$\text{Pr}_g(H, \text{Aut}(G)) = \frac{1}{|H| |\text{Aut}(G)|} \sum_{x \in H \text{orb}(x)} |C_{\text{Aut}(G)}(x)| = \frac{1}{|H|} \sum_{x \in H \text{orb}(x)} \frac{1}{|\text{orb}(x)|}.$$ 

**Proof.** Let $T_{x,g}(H, G) = \{\alpha \in \text{Aut}(G) : [x, \alpha] = g\}$ for any $x \in H$. Then $T_{x,g}(H, G) \neq \emptyset$ if and only if $x \in \text{orb}(x)$. We also have

$$\{(x, \alpha) \in H \times \text{Aut}(G) : [x, \alpha] = g\} = \biguplus_{x \in H} (\{x\} \times T_{x,g}(H, G)),$$

where $\biguplus$ represents the union of disjoint sets. Therefore, by (1.1), we have

$$|H| |\text{Aut}(G)||\text{Pr}_g(H, \text{Aut}(G))| = |\biguplus_{x \in H} (\{x\} \times T_{x,g}(H, G))| = \sum_{x \in H} |T_{x,g}(H, G)|.$$  

(2.3)

Let $\sigma \in T_{x,g}(H, G)$ and $\beta \in \sigma C_{\text{Aut}(G)}(x)$. Then $\beta = \sigma \alpha$ for some $\alpha \in C_{\text{Aut}(G)}(x)$. We have

$$[x, \beta] = [x, \sigma \alpha] = x^{-1} \sigma(\alpha(x)) = [x, \alpha] = g.$$ 

Therefore, $\beta \in T_{x,g}(H, G)$ and so $\sigma C_{\text{Aut}(G)}(x) \subseteq T_{x,g}(H, G)$. Again, let $\gamma \in T_{x,g}(H, G)$ then $\gamma(x) = xg$. We have $\sigma^{-1} \gamma(x) = \sigma^{-1}(xg) = x$ and so $\sigma^{-1} \gamma \in C_{\text{Aut}(G)}(x)$. Therefore, $\gamma \in \sigma C_{\text{Aut}(G)}(x)$ which gives $T_{x,g}(H, G) \subseteq \sigma C_{\text{Aut}(G)}(x)$. Thus, $\sigma C_{\text{Aut}(G)}(x) = T_{x,g}(H, G)$ and hence

$$|T_{x,g}(H, G)| = |C_{\text{Aut}(G)}(x)| = \frac{|\text{Aut}(G)|}{|\text{orb}(x)|}.$$  

(2.4)

Therefore, the result follows from (2.3) and (2.4).

Putting $g = 1$ in Theorem 2.4 we get the following corollary.
Corollary 2.5. Let $H$ be a subgroup of $G$. Then
\[ \text{Pr}(H, \text{Aut}(G)) = \frac{1}{|H| |\text{Aut}(G)|} \sum_{x \in H} |C_{\text{Aut}(G)}(x)| = \frac{\text{orb}(H)}{|H|} \]
where $\text{orb}(H) = \{\text{orb}(x) : x \in H\}$.

As an application of Theorem 2.4 we have the following result.

Proposition 2.6. Let $H$ be a subgroup of $G$. If $\text{orb}(x) = x[H, \text{Aut}(G)]$ for all $x \in H \setminus L(H, \text{Aut}(G))$ then
\[ \text{Pr}_g(H, \text{Aut}(G)) = \begin{cases} 
\frac{1}{|H, \text{Aut}(G)|} \left( 1 + \frac{|H, \text{Aut}(G)| - 1}{|H : L(H, \text{Aut}(G))|} \right), & \text{if } g = 1 \\
\frac{1}{|H, \text{Aut}(G)|} \left( 1 - \frac{1}{|H : L(H, \text{Aut}(G))|} \right), & \text{if } g \neq 1.
\end{cases} \]

Proof. If $g = 1$ then the result follows from [2, Proposition 3.4]. If $g \neq 1$, we have $xg \notin \text{orb}(x)$ for all $x \in L(H, \text{Aut}(G))$. Again, since $g \in S(H, \text{Aut}(G)) \subseteq [H, \text{Aut}(G)]$ therefore $xg \in x[H, \text{Aut}(G)] = \text{orb}(x)$ for all $x \in H \setminus L(H, \text{Aut}(G))$. Now from Theorem 2.4 we have
\[ \text{Pr}_g(H, \text{Aut}(G)) = \frac{1}{|H|} \sum_{x \in H \setminus L(H, \text{Aut}(G))} \frac{1}{\text{orb}(x)} \]
\[ = \frac{1}{|H|} \sum_{x \in H \setminus L(H, \text{Aut}(G))} \frac{1}{|H, \text{Aut}(G)|} \]
\[ = \frac{1}{|H, \text{Aut}(G)|} \left( 1 - \frac{1}{|H : L(H, \text{Aut}(G))|} \right). \]
\[ \square \]

3. Various bounds

In this section, we obtain various bounds for $\text{Pr}_g(H, \text{Aut}(G))$. We begin with the following lower bounds.

Proposition 3.1. Let $H$ be a subgroup of $G$. Then, for $g \in G$, we have
\[ \text{Pr}_g(H, \text{Aut}(G)) \geq \frac{|L(H, \text{Aut}(G))| + |C_{\text{Aut}(G)}(H)||[H, \text{Aut}(G)]|}{|H| |\text{Aut}(G)|}, \quad \text{if } g = 1 \]
\[ \text{Pr}_g(H, \text{Aut}(G)) \geq \frac{|L(H, \text{Aut}(G))||C_{\text{Aut}(G)}(H)|}{|H| |\text{Aut}(G)|}, \quad \text{if } g \neq 1. \]

Proof. Let $\mathcal{C}$ denotes the set $\{(x, \alpha) \in H \times \text{Aut}(G) : [x, \alpha] = g\}$.

If $g = 1$ then $(L(H, \text{Aut}(G)) \times \text{Aut}(G)) \cup (H \times C_{\text{Aut}(G)}(H))$ is a subset of $\mathcal{C}$ and $|[L(H, \text{Aut}(G)) \times \text{Aut}(G)] \cup (H \times C_{\text{Aut}(G)}(H))| = |L(H, \text{Aut}(G))||\text{Aut}(G)| + |C_{\text{Aut}(G)}(H)||H| - |L(H, \text{Aut}(G))||C_{\text{Aut}(G)}(H)|$. Hence, the result follows from (1.1).

If $g \neq 1$ then $\mathcal{C}$ is non-empty since $g \in S(H, \text{Aut}(G))$. Let $(y, \beta) \in \mathcal{C}$ then $(y, \beta) \notin L(H, \text{Aut}(G)) \times \text{Aut}(G)$ otherwise $[y, \beta] = 1$. It is easy to see that the coset $(y, \beta)(L(H, \text{Aut}(G)) \times C_{\text{Aut}(G)}(H))$ having order $|L(H, \text{Aut}(G))||C_{\text{Aut}(G)}(H)|$ is a subset of $\mathcal{C}$. Hence, the result follows from (1.1). \[ \square \]

Proposition 3.2. Let $H$ be a subgroup of $G$. If $g \in G$ then
\[ \text{Pr}_g(H, \text{Aut}(G)) \leq \text{Pr}(H, \text{Aut}(G)). \]
The equality holds if and only if $g = 1$. 

Proof. By Theorem 2.4, we have
\[
\Pr_g(H, \text{Aut}(G)) = \frac{1}{|H||\text{Aut}(G)|} \sum_{x \in H, xg \in \text{orb}(x)} |C_{\text{Aut}(G)}(x)|
\]
\[
\leq \frac{1}{|H||\text{Aut}(G)|} \sum_{x \in H} |C_{\text{Aut}(G)}(x)| = \Pr(H, \text{Aut}(G)).
\]
Clearly the equality holds if and only if \( g = 1 \).

Proposition 3.3. Let \( H \) be a subgroup of \( G \). Let \( g \in G \) and \( p \) the smallest prime dividing \( |\text{Aut}(G)| \). If \( g \neq 1 \) then
\[
\Pr_g(H, \text{Aut}(G)) \leq \frac{|H| - |L(H, \text{Aut}(G))|}{p|H|} < \frac{1}{p}.
\]

Proof. By Theorem 2.4, we have
\[
\Pr_g(H, \text{Aut}(G)) = \frac{1}{|H|} \sum_{x \in H \setminus L(H, \text{Aut}(G))} \frac{1}{|\text{orb}(x)|}
\] (3.1)
noting that for \( x \in L(H, \text{Aut}(G)) \) we have \( xg \notin \text{orb}(x) \). Also, for \( x \in H \setminus L(H, \text{Aut}(G)) \) and \( xg \in \text{orb}(x) \) we have \( |\text{orb}(x)| > 1 \). Since \( |\text{orb}(x)| \) is a divisor of \( |\text{Aut}(G)| \) we have \( |\text{orb}(x)| \geq p \). Hence, the result follows from (3.1).

Proposition 3.4. Let \( H_1 \) and \( H_2 \) be two subgroups of \( G \) such that \( H_1 \subseteq H_2 \). Then
\[
\Pr_g(H_1, \text{Aut}(G)) \leq |H_2 : H_1| \Pr_g(H_2, \text{Aut}(G)).
\]
The equality holds if and only if \( xg \notin \text{orb}(x) \) for all \( x \in H_2 \setminus H_1 \).

Proof. By Theorem 2.4, we have
\[
|H_1||\text{Aut}(G)|\Pr_g(H_1, \text{Aut}(G)) = \sum_{x \in H_1, xg \in \text{orb}(x)} |C_{\text{Aut}(G)}(x)|
\]
\[
\leq \sum_{x \in H_2, xg \notin \text{orb}(x)} |C_{\text{Aut}(G)}(x)|
\]
\[
= |H_2||\text{Aut}(G)|\Pr_g(H_2, \text{Aut}(G)).
\]
Hence, the result follows.

We conclude this section with the following result.

Proposition 3.5. Let \( H \) be a subgroup of \( G \). If \( g \in G \) then
\[
\Pr_g(H, \text{Aut}(G)) \leq |G : H| \Pr(G, \text{Aut}(G))
\]
with equality if and only if \( g = 1 \) and \( H = G \).

Proof. By Proposition 3.2, we have
\[
\Pr_g(H, \text{Aut}(G)) \leq \Pr(H, \text{Aut}(G))
\]
\[
= \frac{1}{|H||\text{Aut}(G)|} \sum_{x \in H} |C_{\text{Aut}(G)}(x)|
\]
\[
\leq \frac{1}{|H||\text{Aut}(G)|} \sum_{x \in G} |C_{\text{Aut}(G)}(x)|
\]
\[
= |G : H| \Pr(G, \text{Aut}(G)).
\]
Hence, the result follows from Corollary 2.5.
4. Characterizations through \( \Pr_{g}(H, \text{Aut}(G)) \)

In this section, we obtain some characterizations of groups through \( \Pr_{g}(H, \text{Aut}(G)) \). The following lemma is useful in this regard.

**Lemma 4.1.** Let \( H \) be a subgroup of \( G \). If \( p \) is the smallest prime divisor of \( |\text{Aut}(G)| \) and \( |[H, \text{Aut}(G)]| = p \) then \( \text{orb}(x) = x[H, \text{Aut}(G)] \) for all \( x \in H \setminus L(H, \text{Aut}(G)) \).

**Proof.** We have \( \text{orb}(x) \subseteq x[H, \text{Aut}(G)] \) for all \( x \in H \). Also, \( |\text{orb}(x)| \) is a divisor of \( |\text{Aut}(G)| \) for all \( x \in H \). Therefore, \( |\text{orb}(x)| \geq p \) for all \( x \in H \setminus L(H, \text{Aut}(G)) \). Hence, \( |\text{orb}(x)| = |x[H, \text{Aut}(G)]| = p \) for all \( x \in H \setminus L(H, \text{Aut}(G)) \) and the result follows. \( \square \)

Now we derive the following characterizations.

**Theorem 4.2.** Let \( H \) be a subgroup of a finite group \( G \) and \( g \in G \). Let \( p \) be the smallest prime dividing \( |\text{Aut}(G)| \) and \( |[H, \text{Aut}(G)]| = p \). If \( g \neq 1 \) and \( \Pr_{g}(H, \text{Aut}(G)) = \frac{n-1}{np} \) or \( g = 1 \) and \( \Pr_{g}(H, \text{Aut}(G)) = \frac{n+p-1}{np} \) (where \( n \) is a positive integer) then \( \frac{H}{L(H, \text{Aut}(G))} \) is isomorphic to a group of order \( n \). In particular,

1. if \( n = q \) or \( q^{2} \) for some prime \( q \) then \( \frac{H}{L(H, \text{Aut}(G))} \cong \mathbb{Z}_q, \mathbb{Z}_q^2 \) or \( \mathbb{Z}_q \times \mathbb{Z}_q \).
2. if \( H \) is abelian and \( n = q_{1}^{k_{1}} q_{2}^{k_{2}} \cdots q_{m}^{k_{m}} \), where \( q_{i} 's \) are primes not necessarily distinct, then \( \frac{H}{L(H, \text{Aut}(G))} \cong \mathbb{Z}_{q_{1}} \times \mathbb{Z}_{q_{2}} \times \cdots \times \mathbb{Z}_{q_{m}} \).

**Proof.** If \( g \neq 1 \) and \( \Pr_{g}(H, \text{Aut}(G)) = \frac{n-1}{np} \) then, by Lemma 4.1 and Proposition 2.6, we have

\[
\frac{n-1}{np} = \frac{1}{p} \left( 1 - \frac{1}{|H : L(H, \text{Aut}(G))|} \right)
\]

which gives \( |H : L(H, \text{Aut}(G))| = n \).

If \( g = 1 \) and \( \Pr_{g}(H, \text{Aut}(G)) = \frac{n+p-1}{np} \) then, by Lemma 4.1 and Proposition 2.6, we have

\[
\frac{n+p-1}{np} = \frac{1}{p} \left( 1 + \frac{p-1}{|H : L(H, \text{Aut}(G))|} \right)
\]

which also gives \( |H : L(H, \text{Aut}(G))| = n \).

Hence, \( \frac{H}{L(H, \text{Aut}(G))} \) is isomorphic to a group of order \( n \).

1. If \( n = q \) or \( q^{2} \) for some prime \( q \) then \( |H : L(H, \text{Aut}(G))| = q \) or \( q^{2} \). Therefore \( \frac{H}{L(H, \text{Aut}(G))} \) is abelian. Hence, the result follows from fundamental theorem of finite abelian groups.

2. If \( H \) is abelian and \( n = q_{1}^{k_{1}} q_{2}^{k_{2}} \cdots q_{m}^{k_{m}} \), where \( q_{i} 's \) are primes not necessarily distinct then \( \frac{H}{L(H, \text{Aut}(G))} \) is an abelian group of order \( q_{1}^{k_{1}} q_{2}^{k_{2}} \cdots q_{m}^{k_{m}} \). Hence, the result follows from fundamental theorem of finite abelian groups. \( \square \)

Putting \( H = G \), in Theorem 4.2, we have the following corollary.

**Corollary 4.3.** Let \( G \) be a finite group and \( g \in G \). Let \( p \) be the smallest prime dividing \( |\text{Aut}(G)| \) and \( |[G, \text{Aut}(G)]| = p \). If \( g \neq 1 \) and \( \Pr_{g}(G, \text{Aut}(G)) = \frac{n-1}{np} \) or \( g = 1 \) and \( \Pr_{g}(G, \text{Aut}(G)) = \frac{n+p-1}{np} \) (where \( n \) is a positive integer) then \( \frac{G}{L(G)} \) is isomorphic to a group of order \( n \). In particular,

1. if \( n = q \) or \( q^{2} \) for some prime \( q \) then \( \frac{G}{L(G)} \cong \mathbb{Z}_q, \mathbb{Z}_q^2 \) or \( \mathbb{Z}_q \times \mathbb{Z}_q \).
2. if \( G \) is abelian and \( n = q_{1}^{k_{1}} q_{2}^{k_{2}} \cdots q_{m}^{k_{m}} \), where \( q_{i} 's \) are primes not necessarily distinct, then \( \frac{G}{L(G)} \cong \mathbb{Z}_{q_{1}} \times \mathbb{Z}_{q_{2}} \times \cdots \times \mathbb{Z}_{q_{m}} \).

We conclude the paper with the following result which gives converse of Theorem 4.2.
Theorem 4.4. Let $H$ be a subgroup of a finite group $G$ and $g \in G$. Let $p$ be the smallest prime dividing $|\text{Aut}(G)|$ and $|[H, \text{Aut}(G)]| = p$. If $\frac{H}{L(H, \text{Aut}(G))}$ is isomorphic to a group of order $n$ then

$$\text{Pr}_g(H, \text{Aut}(G)) = \begin{cases} \frac{n-1}{n}p, & \text{if } g \neq 1 \\ \frac{np-1}{np}, & \text{if } g = 1. \end{cases}$$

Proof. If $p$ is the smallest prime dividing $|\text{Aut}(G)|$ and $|[H, \text{Aut}(G)]| = p$ then, by Lemma 4.1, we have $\text{orb}(x) = x[H, \text{Aut}(G)]$ for all $x \in H \setminus L(H, \text{Aut}(G))$. Therefore, by Proposition 2.6, we have

$$\text{Pr}_g(H, \text{Aut}(G)) = \begin{cases} \frac{1}{p} \left( 1 + \frac{p-1}{|H : L(H, \text{Aut}(G))|} \right), & \text{if } g = 1 \\ \frac{1}{p} \left( 1 - \frac{1}{|H : L(H, \text{Aut}(G))|} \right), & \text{if } g \neq 1. \end{cases}$$

If $\frac{H}{L(H, \text{Aut}(G))}$ is isomorphic to a group of order $n$ then $|H : L(H, \text{Aut}(G))| = n$ and hence the result follows.

Note that putting $H = G$ in Theorem 4.4, we get the converse of Corollary 4.3.

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