ON A SOLVABLE SYSTEM OF DIFFERENCE EQUATIONS OF HIGHER-ORDER WITH PERIOD TWO COEFFICIENTS

YASIN YAZLIK AND MERVE KARA

Abstract. We show that the next difference equations system

\[\begin{align*}
x_{n+1} &= \frac{a_n x_{n-k+1} y_{n-k}}{y_n - \alpha_n} + \beta_{n+1}, \quad y_{n+1} = \frac{b_n y_{n-k+1} x_{n-k}}{x_n - \beta_n} + \alpha_{n+1}, \quad n \in \mathbb{N}_0,
\end{align*}\]

where \(\mathbb{N}_0 = \mathbb{N} \cup \{0\}\), the sequences \((a_n)_{n \in \mathbb{N}_0}\), \((b_n)_{n \in \mathbb{N}_0}\), \((\alpha_n)_{n \in \mathbb{N}_0}\), \((\beta_n)_{n \in \mathbb{N}_0}\) are two periodic and the initial conditions \(x_{-i}, y_{-i} \in \{0, 1, \ldots, k\}\), are non-zero real numbers, can be solved. Also, we investigate the behavior of solutions of above mentioned system when \(a_0 = b_1\) and \(a_1 = b_0\).

1. Introduction

Theory of difference equations have attracted attention of many authors in recent years (see, e.g., [1]-[40]). Most of the recent applications of this theory have appeared in many scientific areas such as biology, physics, engineering, economics. Particularly, rational difference equations and their systems of higher order have great importance in applications. It is very worthy to find systems which belong to solvable nonlinear difference equations systems and to solve nonlinear difference equations or systems in closed-form or explicit-form. The found formulas for the solutions of these types of equations or systems can be used easily for description of many features of the solutions of these equations or systems. For this reason, finding of a formula for solution of a nonlinear difference equation is worthy as well as interesting.

In an earlier paper, Elabbasy et al., in [3], considered, among other things, the next difference equation

\[x_{n+1} = \frac{x_n x_{n-1}}{x_n - 1} + 1, \quad n \in \mathbb{N}_0.\] (1)
Quite recently in [14], Haddad et al. considered the following system of difference equations

\[ x_{n+1} = \frac{ax_ny_{n-1}}{y_n - \alpha} + \beta, \quad y_{n+1} = \frac{by_nx_{n-1}}{x_n - \beta} + \alpha, \quad n \in \mathbb{N}_0, \tag{2} \]

where the parameters \( a, b, \alpha, \beta \) and the initial conditions \( x_i, y_i, i = 0, 1 \), are non-zero real numbers, which is an extension of the equation in (1). By using appropriate substitutions on variables, authors reduced system (2) to the first order linear difference equations and investigated the existence and behavior of the solutions of system (2).

Our aim in this study to show that the next difference equations system

\[ x_{n+1} = \frac{a_n x_{n-k+1} y_{n-k}}{y_n - \alpha_n} + \beta_{n+1}, \quad y_{n+1} = \frac{b_n y_{n-k+1} x_{n-k}}{x_n - \beta_n} + \alpha_{n+1}, \quad n \in \mathbb{N}_0, \tag{3} \]

where the sequences \( a_n, b_n, \alpha_n, \beta_n \) are two periodic and the initial conditions \( x_i, y_i, i \in \{0, 1, \ldots, k\}, \) are non-zero real numbers, is solvable in closed form. Also, by using obtained formulas we give the behavior and periodicity of well-defined solutions of system (3) when \( a_0 = b_1 = 0 \) and \( a_1 = b_0 \). Note that system (3) is a natural extension of both Eq. (1) and system (2).

**Lemma 1.** [27] Let \((a_n)_{n \in \mathbb{N}_0}\) and \((b_n)_{n \in \mathbb{N}_0}\) be two sequences of real numbers and the sequences \(y_{km+i}, i = 0, k - 1,\) be solutions of the equations

\[ y_{km+i} = a_{km+i} y_{k(m-1)+i} + b_{km+i}, \quad m \in \mathbb{N}_0. \tag{4} \]

Then, for each fixed \( i \in \{0, 1, \ldots, k - 1\} \) and \( m \geq -1 \), equation (4) has the general solution

\[ y_{km+i} = y_{-k+i} \prod_{j=0}^{m} a_{kj+i} + \sum_{s=0}^{m} b_{ks+i} \prod_{j=s+1}^{m} a_{kj+i}. \]

Further, if \((a_n)_{n \in \mathbb{N}_0}\) and \((b_n)_{n \in \mathbb{N}_0}\) are constant and \( i = 0, k - 1 \), then

\[ y_{km+i} = \begin{cases} a^{m+1} y_{-k+i} + b \frac{1-a^{m+1}}{1-a}, & \text{if } a \neq 1, \\ y_{-k+i} + b (m + 1), & \text{if } a = 1. \end{cases} \]

**Definition 2.** [12] (Periodicity) A solution \( \{x_n\}_{n=-k}^\infty \) of equation \( x_{n+1} = f(x_n, x_{n-1}, \ldots, x_{n-k}), \quad n \in \mathbb{N}_0, \) is called periodic with period \( p \) if there exists an integer \( p \geq 1 \) such that \( x_{n+p} = x_n \) for all \( n \geq -k. \)

In the sequel, as usual, we suppose that \( \prod_{j=0}^{m} A_j = 1 \) and \( \sum_{j=0}^{m} A_j = 0, \) for all \( m < i. \)

2. Solutions of System (3)

Let \((x_n, y_n)_{n \geq -k}\) be a solution of system (3). If one of the initial conditions \( x_{-j}, y_{-j}, j = 0, k \) is equal to zero, then the system (3) is not defined. For example, if \( x_{-k} = 0 \), then \( y_1 = \alpha_1, \) and so \( x_2 \) can not be calculated. For the other \( j = 0, k - 1, \)
the case is same. Now assume that \( x_{-j} \neq 0 \neq y_{-j}, \ j = 0, k \). If one of the terms \( x_{n_0} \) and \( y_{n_0} \), for \( n_0 \geq 1 \), is equal to zero, then from system (3) either \( x_{n_0+k} = \frac{a_{n_0+k-1}x_{n_0} + \beta_{n_0+k}}{y_{n_0+k-1} - \alpha_{n_0+k-1}} + \beta_{n_0+k} = \beta_{n_0+k} \) and so, it follows that \( y_{n_0+k+1} \) is not defined or \( y_{n_0+k+1} = \frac{b_{n_0+k-1}y_{n_0}x_{n_0} - \alpha_{n_0+k-1}y_{n_0}x_{n_0} - 1 + \alpha_{n_0+k}}{x_{n_0+k-1} - \beta_{n_0+k-1}} + \alpha_{n_0+k} = \alpha_{n_0+k} \) and so, it follows that \( x_{n_0+k+1} \) is not defined. Thus, for well-defined solution of system (3), we may assume that none of the terms of \( x_n \) and \( y_n \) is not equal to zero, for every \( n \geq -k \). By means of the change of variables

\[
u_n = \frac{x_n - \beta_n}{x_{n-k}}, \quad v_n = \frac{y_n - \alpha_n}{y_{n-k}}, \quad n \in \mathbb{N}_0,
\]

the system in (3) becomes

\[
u_{n+1} = \frac{\alpha_n}{\nu_n}, \quad v_{n+1} = \frac{b_n}{\nu_n}, \quad n \in \mathbb{N}_0.
\]

From (6), we have two independent equations

\[
u_{n+2} = \frac{\alpha_{n+1}}{\nu_n} u_n, \quad v_{n+2} = \frac{b_{n+1}}{\nu_n} v_n, \quad n \in \mathbb{N}_0,
\]

and so

\[
u_{2m+i} = \left( \frac{a_{1-i}}{b_i} \right)^m u_i, \quad \nu_{2m+i} = \left( \frac{b_{1-i}}{a_i} \right)^m v_i, \quad i \in \{0, 1\}, \quad m \in \mathbb{N}_0,
\]

where the sequences \((a_n)_{n \in \mathbb{N}_0}\) and \((b_n)_{n \in \mathbb{N}_0}\) are two periodic. From (5) we have that

\[
x_n = u_n x_{n-k} + \beta_n = u_n u_{n-k} x_{n-2k} + u_n \beta_{n-k} + \beta_n, \quad y_n = v_n y_{n-k} + \alpha_n = v_n v_{n-k} y_{n-2k} + v_n \alpha_{n-k} + \alpha_n, \quad n \in \mathbb{N}_0.
\]

We consider three cases: (a) \( k = 1 \); (b) \( k = 2t + 1 \) (\( t = 1, 2, \ldots \)); and (c) \( k = 2t \) (\( t = 1, 2, \ldots \)).

a) If \( k = 1 \), then, from equations in (9), we can write

\[
x_n = u_n x_{n-1} + \beta_n, \quad y_n = v_n y_{n-1} + \alpha_n, \quad n \in \mathbb{N}_0.
\]

from which it follows that

\[
x_{2n} = u_{2n} x_{2n-1} + \beta_{2n} = \left( \frac{a_1}{b_0} \right)^n u_0 \ x_{2n-1} + \beta_0, \quad n \in \mathbb{N}_0,
\]

\[
x_{2n+1} = u_{2n+1} x_{2n} + \beta_{2n+1} = \left( \frac{a_0}{b_1} \right)^n u_1 \ x_{2n} + \beta_1, \quad n \in \mathbb{N}_0,
\]

\[
y_{2n} = v_{2n} y_{2n-1} + \alpha_{2n} = \left( \frac{b_1}{a_0} \right)^n v_0 \ y_{2n-1} + \alpha_0, \quad n \in \mathbb{N}_0,
\]

\[
y_{2n+1} = v_{2n+1} y_{2n} + \alpha_{2n+1} = \left( \frac{b_0}{a_1} \right)^n v_1 \ y_{2n} + \alpha_1, \quad n \in \mathbb{N}_0.
\]
which implies that

\[
x_{2n+1} = \left( \frac{a_0}{b_1} \right)^n \left( \frac{a_1}{b_0} \right)^n u_0 u_1 x_{2n-1} + \left( \frac{a_0}{b_1} \right)^n u_1 \beta_0 + \beta_1, \quad n \in \mathbb{N}_0, \quad (11)
\]

\[
x_{2n+2} = \left( \frac{a_0}{b_1} \right)^n \left( \frac{a_1}{b_0} \right)^{n+1} u_0 u_1 x_{2n} + \left( \frac{a_1}{b_0} \right)^{n+1} u_0 \beta_1 + \beta_0, \quad n \in \mathbb{N}_0, \quad (12)
\]

\[
y_{2n+1} = \left( \frac{b_0}{a_1} \right)^n \left( \frac{b_1}{a_0} \right)^n v_0 v_1 y_{2n-1} + \left( \frac{b_0}{a_1} \right)^n v_1 \alpha_0 + \alpha_1, \quad n \in \mathbb{N}_0, \quad (13)
\]

\[
y_{2n+2} = \left( \frac{b_0}{a_1} \right)^n \left( \frac{b_1}{a_0} \right)^{n+1} v_0 v_1 y_{2n} + \left( \frac{b_1}{a_0} \right)^{n+1} v_0 \alpha_1 + \alpha_0, \quad n \in \mathbb{N}_0. \quad (14)
\]

Hence, from Lemma 1 for \( k = 1 \) and Eqs. (11)-(14), we can write the solution of system (3) as follows:

\[
x_{2n-1} = x_{-1} \prod_{t=0}^{n-1} d \left( \frac{a_0}{b_1} \right)^t \left( \frac{a_1}{b_0} \right)^t + \sum_{r=0}^{n-1} \beta_0 u_1 \left( \frac{a_0}{b_1} \right)^r \prod_{t=r+1}^{n-1} d \left( \frac{a_0}{b_1} \right)^t \left( \frac{a_1}{b_0} \right)^t,
\]

\[
x_{2n} = x_0 \prod_{t=0}^{n-1} d \left( \frac{a_0}{b_1} \right)^t \left( \frac{a_1}{b_0} \right)^{t+1} + \sum_{r=0}^{n-1} \beta_0 \prod_{t=r+1}^{n-1} d \left( \frac{a_0}{b_1} \right)^t \left( \frac{a_1}{b_0} \right)^{t+1},
\]

\[
y_{2n-1} = y_{-1} \prod_{t=0}^{n-1} \frac{a_0 b_0}{d} \left( \frac{b_0}{a_1} \right)^t \left( \frac{b_1}{a_0} \right)^t + \sum_{r=0}^{n-1} \alpha_0 v_1 \left( \frac{b_0}{a_1} \right)^r \prod_{t=r+1}^{n-1} \frac{a_0 b_0}{d} \left( \frac{b_0}{a_1} \right)^t \left( \frac{b_1}{a_0} \right)^t,
\]

\[
y_{2n} = y_0 \prod_{t=0}^{n-1} \frac{a_0 b_0}{d} \left( \frac{b_0}{a_1} \right)^t \left( \frac{b_1}{a_0} \right)^{t+1} + \sum_{r=0}^{n-1} \alpha_0 \prod_{t=r+1}^{n-1} \frac{a_0 b_0}{d} \left( \frac{b_0}{a_1} \right)^t \left( \frac{b_1}{a_0} \right)^{t+1},
\]

where \( u_0 = \frac{x_0 - \beta_0}{x_{-k}} \), \( u_1 = \frac{x_1 - \beta_1}{x_{1-k}} \), \( v_0 = \frac{y_0 - \alpha_0}{y_{-k}} \), \( v_1 = \frac{y_1 - \alpha_1}{y_{1-k}} \), \( d = u_0 u_1 \) and \( v_0 v_1 = \frac{a_0 b_0}{d} \).

b) Suppose \( k = 2t + 1 \) \((t = 1, 2, \ldots)\). Iterating the right-hand side of equations in
we can write

\[ \begin{align*}
X_{4tm+2m+2j+1} &= X_{2j-4t-1} \prod_{s=0}^{m} \frac{u_{2(2t+s+j)+1}}{u_{2(2t+s+j-t)}} \\
&\quad + \sum_{l=0}^{2m+1} \beta_{2(t+j-t)+l} \prod_{s=l+1}^{2m+1} \frac{u_{2(t+j-t)-1}}{u_{2(t+s+j-t)+s}} \\
X_{4tm+2m+2j+2} &= X_{2j-4t-4} \prod_{s=0}^{m} \frac{v_{2(2t+s+j)+1}}{v_{2(2t+s+j-t)}} \\
&\quad + \sum_{l=0}^{2m+1} \alpha_{2(t+j-t)+l} \prod_{s=l+1}^{2m+1} \frac{v_{2(t+j-t)-1}}{v_{2(t+s+j-t)+s+1}} \\
y_{4tm+2m+2j+1} &= y_{2j-4t-1} \prod_{s=0}^{m} \frac{v_{2(2t+s+j)+1}}{v_{2(2t+s+j-t)}} \\
&\quad + \sum_{l=0}^{2m+1} \alpha_{2(t+j-t)+l} \prod_{s=l+1}^{2m+1} \frac{v_{2(t+j-t)-1}}{v_{2(t+s+j-t)+s+1}} \\
y_{4tm+2m+2j+2} &= y_{2j-4t-4} \prod_{s=0}^{m} \frac{v_{2(2t+s+j)+1}}{v_{2(2t+s+j-t)}} \\
&\quad + \sum_{l=0}^{2m+1} \alpha_{2(t+j-t)+l} \prod_{s=l+1}^{2m+1} \frac{v_{2(t+j-t)-1}}{v_{2(t+s+j-t)+s+1}},
\end{align*} \]

where \( j \in \{ t, t+1, \ldots, 3t \} \). Using (8) in (15)-(18), we get

\[ \begin{align*}
X_{4tm+2m+2j+1} &= X_{2j-4t-1} \prod_{s=0}^{m} \frac{a_0}{b_1^{(2t+1)s+j}} \frac{a_1}{b_0^{(2t+1)s+j-t}} \\
&\quad + \sum_{l=0}^{m} \beta_{2(t+j-t)+l} \frac{a_0}{b_1^{m-l+1}} \frac{a_1}{b_0^{m-l+1}} \times \frac{(t+j+1)(m-l)+(2t+1)}{b_1^{(2t+1)(m-l)^{(2t+1)}}} \frac{a_0}{b_1^{(2t+1)(m-l)^{(2t+1)}}} \\
&\quad + \sum_{l=0}^{m} \beta_{2(t+j+1)+l} \frac{a_1}{b_1^{m-l}} \frac{a_0}{b_0^{m-l}} \times \frac{(t+j+1)(m-l)+(2t+1)}{b_1^{(2t+1)(m-l)^{(2t+1)}}} \frac{a_0}{b_1^{(2t+1)(m-l)^{(2t+1)}}} \\
&\quad \times \frac{u_1^{(t+j+1)(m-l)+(2t+1)}}{u_1^{(t+j+1)(m-l)^{(2t+1)}}} \\
x_{4tm+2m+2j+2} &= x_{2j-4t} \prod_{s=0}^{m} \frac{a_0}{b_1^{(2t+1)s+j+1}} \frac{a_1}{b_0^{(2t+1)s+j-t}}
\end{align*} \]
$y_{4t+2m+2j+1} = y_{2j-4t} \prod_{s=0}^{m} \frac{a_0 b_0}{d} \left( \frac{b_0}{a_1} \right)^{(2t+1)s+j} \left( \frac{b_1}{a_0} \right)^{(2t+1)s-j-t} + \sum_{l=0}^{m} \alpha_2(2t+1+l-1) \beta_0^{m-l} \alpha_1^{m-l+1} \\
\times \left( \frac{b_1}{a_0} \right)^{(j+1)(m-l)+(2t+1) \frac{m(m-1)-(l-1)}{2}} \left( \frac{b_0}{a_1} \right)^{j(m-l+1)+(2t+1) \frac{m(m+1)-(l-1)}{2}} + \sum_{l=0}^{m} \alpha_2(2t+1+l+1) \eta_0^{m-l} \eta_1^{m-l} \\
\times \left( \frac{b_0}{a_1} \right)^{(j+1)(m-l)+(2t+1) \frac{m(m+1)-(l+1)}{2}} \\
\times \left( \frac{b_1}{a_0} \right)^{(j+1)(m-l)+(2t+1) \frac{m(m+1)-(l+1)}{2}}$

c) Suppose $k = (2t) \ (t = 1, 2, \ldots)$. Similarly, iterating the right-hand side of equations in (9), we have

$x_{4t+2j} = x_{2j-4t} \prod_{s=0}^{m} \psi(2t+s+j) \psi(2t+s-j-t)$
SOLVABLE SYSTEM OF DIFFERENCE EQUATIONS

\[ x_{4tm+2j+1} = x_{2j-4t+1} \prod_{s=0}^{m} u_s(2ts+j+1)u_s(2ts+j-t) + \sum_{l=0}^{2m+1} \beta_{2(t+l-j)} \prod_{s=l+1}^{2m+1} u_s(2ts+j-t) \]  
\[ y_{4tm+2j+1} = y_{2j-4t+1} \prod_{s=0}^{m} v_s(2ts+j+1)v_s(2ts+j-t) + \sum_{l=0}^{2m+1} \beta_{2(t+l-j)} \prod_{s=l+1}^{2m+1} v_s(2ts+j-t) \]  
\[ x_{4tm+2j+1} = x_{2j-4t+1} \prod_{s=0}^{m} u_s(2ts+j+1)u_s(2ts+j-t) + \sum_{l=0}^{2m+1} \alpha_{2(t+l-j)} \prod_{s=l+1}^{2m+1} u_s(2ts+j-t) \]  
\[ y_{4tm+2j+1} = y_{2j-4t+1} \prod_{s=0}^{m} v_s(2ts+j+1)v_s(2ts+j-t) + \sum_{l=0}^{2m+1} \alpha_{2(t+l-j)} \prod_{s=l+1}^{2m+1} v_s(2ts+j-t) \]

where \( j \in \{ t, t+1, \ldots, 3t-1 \} \). Using (8) in (19)-(22), we get

\[ x_{4tm+2j} = x_{2j-4t+1} \prod_{s=0}^{m} u_s(2ts+j+1)u_s(2ts+j-t) + \sum_{l=0}^{2m+1} \beta_{2(t+l-j)} u_0(2m-l+1) \]  
\[ x_{4tm+2j+1} = x_{2j-4t+1} \prod_{s=0}^{m} u_s(2ts+j+1)u_s(2ts+j-t) + \sum_{l=0}^{2m+1} \beta_{2(t+l-j)} u_1(2m-l+1) \]  
\[ y_{4tm+2j} = y_{2j-4t+1} \prod_{s=0}^{m} v_s(2ts+j+1)v_s(2ts+j-t) + \sum_{l=0}^{2m+1} \alpha_{2(t+l-j)} v_0(2m-l+1) \]  
\[ y_{4tm+2j+1} = y_{2j-4t+1} \prod_{s=0}^{m} v_s(2ts+j+1)v_s(2ts+j-t) + \sum_{l=0}^{2m+1} \alpha_{2(t+l-j)} v_1(2m-l+1) \]
The previous computations prove the next theorem.

**Theorem 3.** Let \( \{x_n, y_n\}_{n \geq -k} \) be a well-defined solution of system (3). Then, we get the next formulas:

(a) If \( k = 1 \), for all \( n \in N_0 \), then we get

\[
x_{2n-1} = x_0 \prod_{t=0}^{n-1} d \left( \frac{a_0}{b_1} \right)^t \left( \frac{a_1}{b_0} \right)^t + \sum_{r=0}^{n-1} \beta_0 u_1 \left( \frac{a_0}{b_1} \right)^r \prod_{t=r+1}^{n-1} d \left( \frac{a_0}{b_1} \right)^t \left( \frac{a_1}{b_0} \right)^t,
\]

\[
x_{2n} = x_0 \prod_{t=0}^{n-1} d \left( \frac{a_0}{b_1} \right)^t \left( \frac{a_1}{b_0} \right)^{t+1} + \sum_{r=0}^{n-1} \beta_0 \prod_{t=r+1}^{n-1} d \left( \frac{a_0}{b_1} \right)^t \left( \frac{a_1}{b_0} \right)^{t+1} + \sum_{r=0}^{n-1} \beta_1 u_0 \prod_{t=r+1}^{n-1} d \left( \frac{a_0}{b_1} \right)^{r+1} \left( \frac{a_1}{b_0} \right)^{t+1} \left( \frac{a_1}{b_0} \right)^t,
\]

\[
y_{2n-1} = y_0 \prod_{t=0}^{n-1} d \left( \frac{a_0}{b_1} \right)^t \left( \frac{b_0}{b_1} \right)^t + \sum_{r=0}^{n-1} \alpha_0 v_1 \left( \frac{b_0}{a_1} \right)^r \prod_{t=r+1}^{n-1} d \left( \frac{b_0}{a_1} \right)^t \left( \frac{b_1}{a_0} \right)^t,
\]

\[
y_{2n} = y_0 \prod_{t=0}^{n-1} d \left( \frac{a_0}{b_1} \right)^t \left( \frac{b_0}{a_1} \right)^{t+1} + \sum_{r=0}^{n-1} \alpha_0 \prod_{t=r+1}^{n-1} d \left( \frac{b_0}{a_1} \right)^t \left( \frac{b_1}{a_0} \right)^{t+1} + \sum_{r=0}^{n-1} \alpha_1 v_0 \left( \frac{b_1}{a_0} \right)^{r+1} \prod_{t=r+1}^{n-1} d \left( \frac{b_0}{a_1} \right)^{t+1} \left( \frac{b_1}{a_0} \right)^t.
\]

(b) If \( k = 2t+1 \) (\( t = 1, 2, \ldots \)), for all \( m \in N_0 \), we have

\[
x_{4tm+2m+2j+1} = x_{2j-4t-1} \prod_{s=0}^{m} d \left( \frac{a_0}{b_1} \right)^{(2t+1)s+j} \left( \frac{a_1}{b_0} \right)^{(2t+1)s+j-t} + \sum_{l=0}^{m} \beta_{2(2t+j+t-l)} \frac{a_0}{b_1}^{m-l} \frac{a_1}{b_0}^{(t+j+1)(m-l)+(2t+1)2(m-l)-(l-1)}.
\]
\[
\begin{align*}
\times \left( \frac{a_0}{b_1} \right)^j (m-l+1)+(2t+1) \frac{m(m+1)-l(l-1)}{2} \\
+ \sum_{l=0}^{m} \beta_2(2t+l+1) u_0^{m-l} u_1^{m-l} \left( \frac{a_1}{b_0} \right)^{(t+j+1)(m-l)+(2t+1) \frac{m(m+1)-l(l+1)}{2}} \times \left( \frac{a_0}{b_1} \right)^j (m-l)+(2t+1) \frac{m(m+1)-l(l+1)}{2}
\end{align*}
\]

\[
\begin{align*}
x_{4tm+2m+2j+2} &= x_{2j-4t} \prod_{s=0}^{m} d \left( \frac{a_0}{b_1} \right)^{(2t+1)s+1} \left( \frac{a_1}{b_0} \right)^{(2t+1)s+j-t} \\
+ \sum_{l=0}^{m} \beta_2(2t+l+t-l+1) u_0^{m-l+1} u_1^{m-l} \left( \frac{a_1}{b_0} \right)^{(j+1)(m-l)+(2t+1) \frac{m(m+1)-l(l+1)}{2}} \times \left( \frac{a_0}{b_1} \right)^{(t+j+1)(m-l)+(2t+1) \frac{m(m+1)-l(l-1)}{2}}
\end{align*}
\]

\[
\begin{align*}
y_{4tm+2m+2j+1} &= y_{2j-4t-1} \prod_{s=0}^{m} \frac{a_0 b_0}{d} \left( \frac{b_0}{a_1} \right)^{(2t+1)s+j} \left( \frac{b_1}{a_0} \right)^{(2t+1)s+j-t} \\
+ \sum_{l=0}^{m} \alpha_2(2t+l+t-l+1) v_0^{m-l+1} v_1^{m-l} \left( \frac{b_1}{a_0} \right)^{(t+l+1)(m-l)+(2t+1) \frac{m(m+1)-l(l-1)}{2}} \times \left( \frac{b_0}{a_1} \right)^j (m-l+1)+(2t+1) \frac{m(m+1)-l(l-1)}{2}
\end{align*}
\]

\[
\begin{align*}
y_{4tm+2m+2j+2} &= y_{2j-4t} \prod_{s=0}^{m} \frac{a_0 b_0}{d} \left( \frac{b_0}{a_1} \right)^{(2t+1)s+j+1} \left( \frac{b_1}{a_0} \right)^{(2t+1)s+j-t}
\end{align*}
\]
\( c) \) If \( k = 2t \) \((t = 1, 2, \ldots)\), for all \( m \in \mathbb{N}_0\), we have

\[
\begin{align*}
x_{4tm+2j} &= x_{2j-4t} \prod_{s=0}^{m} v_0^2 \left( \frac{a_1}{a_0} \right)^{4t+2j-t} \left( \frac{a_0}{b_0} \right)^{2m+1} + \sum_{l=0}^{2m+1} \beta_{2(tl+j-t)} u_0^{2m-l+1} \\
&\quad \times \left( \frac{a_0}{b_0} \right) \left( 2m+2(l-t) \right) + \frac{t(2m+2l+1)}{4}
\end{align*}
\]

\[
\begin{align*}
y_{4tm+2j} &= y_{2j-4t} \prod_{s=0}^{m} v_0^2 \left( \frac{b_0}{a_0} \right)^{4t+2j-t} \left( \frac{b_0}{a_0} \right)^{2m+1} + \sum_{l=0}^{2m+1} \alpha_{2(tl+j-t)} v_0^{2m-l+1} \\
&\quad \times \left( \frac{a_0}{b_0} \right) \left( 2m+2(l-t) \right) + \frac{t(2m+2l+1)}{4}
\end{align*}
\]

where \( u_0 = \frac{x_0 - b_0}{x - k} \), \( u_1 = \frac{x_1 - b_1}{x_1 - k} \), \( v_0 = \frac{y_0 - b_0}{y - k} \), \( v_1 = \frac{y_1 - a_1}{y_1 - k} \), \( d = u_0 u_1 \) and \( v_0 v_1 = \frac{a_0 b_0}{d} \).

3. THE CASE \( a_0 = b_1 \) AND \( a_1 = b_0 \)

The aim of in this section is to investigate the asymptotic behavior and periodicity of well-defined solutions of system (3) in the case \( a_0 = b_1 \) and \( a_1 = b_0 \). The following corollary gives us the solutions of system (3) when \( a_0 = b_1 \) and \( a_1 = b_0 \).

**Corollary 4.** Suppose that \( \{(x_n, y_n)\}_{n \geq -k} \) is a well-defined solution of system (3) when \( a_0 = b_1 \) and \( a_1 = b_0 \). Then, the solutions of system (3) are given by:
a) If \( k = 1 \), for all \( m \in N_0 \), then we get

\[
x_{2m-1} = \begin{cases} x_{-1} + (\beta_0 u_1 + \beta_1) m & \text{if } d = 1, \\ x_{-1} d^m + (\beta_0 u_1 + \beta_1) \frac{d^{m-1}}{d-1} & \text{if } d \neq 1, \end{cases}
\]

\[
x_{2m} = \begin{cases} x_0 + (\beta_1 u_0 + \beta_0) m & \text{if } d = 1, \\ x_0 d^m + (\beta_1 u_0 + \beta_0) \frac{d^{m-1}}{d-1} & \text{if } d \neq 1, \end{cases}
\]

\[
y_{2m-1} = \begin{cases} y_{-1} + (\alpha_0 v_1 + \alpha_1) m & \text{if } a_0 b_0 = d, \\ y_{-1} \left(\frac{a_0 b_0}{d}\right)^m + (\alpha_0 v_1 + \alpha_1) \left(\frac{a_0 b_0}{d}\right)^{m-1} & \text{if } a_0 b_0 \neq d, \end{cases}
\]

\[
y_{2m} = \begin{cases} y_0 + (\alpha_1 v_0 + \alpha_0) m & \text{if } a_0 b_0 = d, \\ y_0 \left(\frac{a_0 b_0}{d}\right)^m + (\alpha_1 v_0 + \alpha_0) \left(\frac{a_0 b_0}{d}\right)^{m-1} & \text{if } a_0 b_0 \neq d, \end{cases}
\]

b) If \( k = 2t + 1 (t = 1, 2, \ldots) \), for all \( m \in N_0 \) and \( j \in \{t, t+1, \ldots, 3t\} \), we have

\[
x_{4tm+2m+2j+1} = \begin{cases} x_{2j-4t-1} + (\beta_0 u_1 + \beta_1) (m + 1) & \text{if } d = 1, \\ x_{2j-4t-1} d^m + (\beta_0 u_1 + \beta_1) \left(\frac{d^{m+1}}{d-1}\right) & \text{if } d \neq 1, \end{cases}
\]

\[
x_{4tm+2m+2j+2} = \begin{cases} x_{2j-4t} + (\beta_1 u_0 + \beta_0) (m + 1) & \text{if } d = 1, \\ x_{2j-4t} d^m + (\beta_1 u_0 + \beta_0) \left(\frac{d^{m+1}}{d-1}\right) & \text{if } d \neq 1, \end{cases}
\]

\[
y_{4tm+2m+2j+1} = \begin{cases} y_{2j-4t-1} + (\alpha_0 v_1 + \alpha_1) (m + 1) & \text{if } d = a_0 b_0, \\ y_{2j-4t-1} \left(\frac{a_0 b_0}{d}\right)^{m+1} + (\alpha_0 v_1 + \alpha_1) \left(\frac{a_0 b_0}{d}\right)^{m-1} & \text{if } d \neq a_0 b_0, \end{cases}
\]

\[
y_{4tm+2m+2j+2} = \begin{cases} y_{2j-4t} + (\alpha_1 v_0 + \alpha_0) (m + 1) & \text{if } d = a_0 b_0, \\ y_{2j-4t} \left(\frac{a_0 b_0}{d}\right)^{m+1} + (\alpha_1 v_0 + \alpha_0) \left(\frac{a_0 b_0}{d}\right)^{m-1} & \text{if } d \neq a_0 b_0, \end{cases}
\]

c) If \( k = 2t (t = 1, 2, \ldots) \), for all \( m \in N_0 \) and \( j \in \{t, t+1, \ldots, 3t-1\} \), we have

\[
x_{4tm+2j+1} = \begin{cases} x_{2j-4t+1} + \beta_1 (2m + 2) & \text{if } u_1 = 1, \\ x_{2j-4t+1} u_1^{2m+2} + \beta_1 \left(\frac{u_1}{u_1-1}\right)^{2m+2-1} & \text{if } u_1 \neq 1, \end{cases}
\]

\[
x_{4tm+2j} = \begin{cases} x_{2j-4t} + \beta_0 (2m + 2) & \text{if } u_0 = 1, \\ x_{2j-4t} (u_0)^{2m+2} + \beta_0 \left(\frac{u_0}{u_0-1}\right)^{2m+2-1} & \text{if } u_0 \neq 1, \end{cases}
\]

\[
y_{4tm+2j+1} = \begin{cases} y_{2j-4t+1} + \alpha_1 (2m + 2) & \text{if } v_1 = 1, \\ y_{2j-4t+1} (v_1)^{2m+2} + \alpha_1 \left(\frac{v_1}{v_1-1}\right)^{2m+2-1} & \text{if } v_1 \neq 1, \end{cases}
\]
where \( u_0 = \frac{x_0-\beta_0}{x_1-k}, u_1 = \frac{x_1-\beta_1}{x_{1-k}}, v_0 = \frac{y_0-\alpha_0}{y_1-k}, v_1 = \frac{y_1-\alpha_1}{y_{1-k}}, d = u_0u_1 \) and \( v_0v_1 = \frac{a_0b_0}{d} \).

The next theorem gives the limiting properties of solutions of system (3) in the case \( a_0 = b_1, a_1 = b_0 \).

**Theorem 5.** Suppose that \( \{(x_n,y_n)\}_{n\geq-k} \) is a well-defined solution of system (3) with \( a_0 = b_1, a_1 = b_0 \). Then, the next statements are true.

1. Let \( k = 1 \).
   a. If \((d-1)x_{-1} + (\beta_0u_1 + \beta_1) \neq 0\), then we get
   
   \[
   \lim_{n \to \infty} |x_{2n-1}| = \begin{cases} 
   \frac{\beta_0u_1 + \beta_1}{d-1} & \text{if } |d| < 1, \\
   \infty & \text{if } |d| > 1.
   \end{cases}
   \]
   Otherwise, if \((d-1)x_{-1} + (\beta_0u_1 + \beta_1) = 0\) and \(d \neq 1\), then \(x_{2n-1} = x_{-1}\) for all \(n \in N_0\).

   b. If \(\beta_0u_1 + \beta_1 \neq 0\) and \(d = 1\), then \(|x_{2n-1}| \to \infty\), as \(n \to \infty\). Otherwise, if \(\beta_0u_1 + \beta_1 = 0\) and \(d = 1\), then \(x_{2n-1} = x_{-1}\), for all \(n \in N_0\).

   c. If \((d-1)x_{0} + (\beta_1u_0 + \beta_0) \neq 0\), then we have
   
   \[
   \lim_{n \to \infty} |x_{2n}| = \begin{cases} 
   \frac{\beta_1u_0 + \beta_0}{d-1} & \text{if } |d| < 1, \\
   \infty & \text{if } |d| > 1.
   \end{cases}
   \]
   Otherwise, if \((d-1)x_{0} + (\beta_1u_0 + \beta_0) = 0\) and \(d \neq 1\), then \(x_{2n} = x_{0}\) for all \(n \in N_0\).

   d. If \(\beta_1u_0 + \beta_0 \neq 0\) and \(d = 1\), then \(|x_{2n}| \to \infty\), as \(n \to \infty\). Otherwise, if \(\beta_1u_0 + \beta_0 = 0\) and \(d = 1\), then \(x_{2n} = x_{0}\), for all \(n \in N_0\).

   e. If \(\left(\frac{a_0b_0}{d} - 1\right)y_{-1} + (a_0v_1 + \alpha_1) \neq 0\), then we get
   
   \[
   \lim_{n \to \infty} |y_{2n-1}| = \begin{cases} 
   \frac{(a_0v_1 + \alpha_1)d}{d-a_0b_0} & \text{if } |d| > a_0b_0, \\
   \infty & \text{if } |d| < a_0b_0.
   \end{cases}
   \]
   Otherwise, if \(\left(\frac{a_0b_0}{d} - 1\right)y_{-1} + (a_0v_1 + \alpha_1) = 0\) and \(d \neq a_0b_0\), then \(y_{2n-1} = y_{-1}\) for all \(n \in N_0\).

   f. If \(\alpha_0v_1 + \alpha_1 \neq 0\) and \(d = a_0b_0\), then \(|y_{2n-1}| \to \infty\), as \(n \to \infty\). Otherwise, if \(\alpha_0v_1 + \alpha_1 = 0\) and \(d = a_0b_0\), then \(y_{2n-1} = y_{-1}\), for all \(n \in N_0\).

   g. If \(\left(\frac{a_0b_0}{d} - 1\right)y_{0} + (\alpha_1v_0 + \alpha_0) \neq 0\), then we get
   
   \[
   \lim_{n \to \infty} |y_{2n}| = \begin{cases} 
   \frac{(\alpha_1v_0 + \alpha_0)d}{d-a_0b_0} & \text{if } |d| > a_0b_0, \\
   \infty & \text{if } |d| < a_0b_0.
   \end{cases}
   \]
Otherwise, if \( \frac{a_{n}b_{n}}{d} - 1 \) \( y_{0} + (\alpha_{1}v_{0} + \alpha_{0}) = 0 \) and \( d \neq a_{0}b_{0} \), then \( y_{2n} = y_{0} \) for all \( n \in N_{0} \).

(h) If \( \alpha_{1}v_{0} + \alpha_{0} \neq 0 \) and \( d = a_{0}b_{0} \), then \( |y_{2n}| \to \infty \) as \( n \to \infty \). Otherwise, if \( \alpha_{1}v_{0} + \alpha_{0} = 0 \) and \( d = a_{0}b_{0} \), then \( y_{2n} = y_{0} \) for all \( n \in N_{0} \).

(2) Let \( k = 2t + 1 \) \( (t = 1, 2, \ldots) \) and \( j \in \{t, t + 1, \ldots, 3t\} \).

(a) If \((d - 1) x_{2j - 4t - 1} + (\beta_{0}u_{1} + \beta_{1}) \neq 0 \), then we have

\[
\lim_{m \to \infty} |x_{4tm + 2m + 2j + 1}| = \begin{cases} 
\frac{\beta_{0}u_{0} + \beta_{1}}{d - 1} & \text{if } |d| < 1, \\
\infty & \text{if } |d| > 1.
\end{cases}
\]

Otherwise, if \((d - 1) x_{2j - 4t - 1} + (\beta_{0}u_{1} + \beta_{1}) = 0 \) and \( d \neq 1 \), then \( x_{4tm + 2m + 2j + 1} = x_{2j - 4t - 1} \) for all \( m \in N_{0} \).

(b) If \( \beta_{0}u_{1} + \beta_{1} \neq 0 \) and \( d = 1 \), then \( |x_{4tm + 2m + 2j + 1}| \to \infty \), as \( m \to \infty \). Otherwise, if \( \beta_{0}u_{1} + \beta_{1} = 0 \) and \( d = 1 \), then \( x_{4tm + 2m + 2j + 1} = x_{2j - 4t - 1} \), for all \( m \in N_{0} \).

(c) If \((d - 1) x_{2j - 4t} + (\beta_{1}u_{0} + \beta_{0}) \neq 0 \), then we get

\[
\lim_{m \to \infty} |x_{4tm + 2m + 2j + 2}| = \begin{cases} 
\frac{\beta_{1}u_{0} + \beta_{0}}{d - 1} & \text{if } |d| < 1, \\
\infty & \text{if } |d| > 1.
\end{cases}
\]

Otherwise, if \((d - 1) x_{2j - 4t} + (\beta_{1}u_{0} + \beta_{0}) = 0 \) and \( d \neq 1 \), then \( x_{4tm + 2m + 2j + 2} = x_{2j - 4t} \) for all \( m \in N_{0} \).

(d) If \( \beta_{1}u_{0} + \beta_{0} \neq 0 \) and \( d = 1 \), then \( |x_{4tm + 2m + 2j + 2}| \to \infty \), as \( m \to \infty \). Otherwise, if \( \beta_{1}u_{0} + \beta_{0} = 0 \) and \( d = 1 \), then \( x_{4tm + 2m + 2j + 2} = x_{2j - 4t} \), for all \( m \in N_{0} \).

(e) If \( \frac{a_{n}b_{n}}{d} - 1 \) \( y_{2j - 4t - 1} + (\alpha_{0}v_{1} + \alpha_{1}) \neq 0 \), then we have

\[
\lim_{m \to \infty} |y_{4tm + 2m + 2j + 1}| = \begin{cases} 
\frac{(\alpha_{0}v_{1} + \alpha_{1})d}{d - a_{0}b_{0}} & \text{if } |d| > a_{0}b_{0}, \\
\infty & \text{if } |d| < a_{0}b_{0}.
\end{cases}
\]

Otherwise, if \( \frac{a_{n}b_{n}}{d} - 1 \) \( y_{2j - 4t - 1} + (\alpha_{0}v_{1} + \alpha_{1}) = 0 \) and \( d \neq a_{0}b_{0} \), then \( y_{4tm + 2m + 2j + 1} = y_{2j - 4t - 1} \) for all \( m \in N_{0} \).

(f) If \( \alpha_{0}v_{1} + \alpha_{1} \neq 0 \) and \( d = a_{0}b_{0} \), then \( |y_{4tm + 2m + 2j + 1}| \to \infty \), as \( m \to \infty \). Otherwise, if \( \alpha_{0}v_{1} + \alpha_{1} = 0 \) and \( d = a_{0}b_{0} \), then \( y_{4tm + 2m + 2j + 1} = y_{2j - 4t - 1} \), for all \( m \in N_{0} \).

(g) If \( \frac{a_{n}b_{n}}{d} - 1 \) \( y_{2j - 4t} + (\alpha_{1}v_{0} + \alpha_{0}) \neq 0 \), then we get

\[
\lim_{m \to \infty} |y_{4tm + 2m + 2j + 2}| = \begin{cases} 
\frac{(\alpha_{1}v_{0} + \alpha_{0})d}{d - a_{0}b_{0}} & \text{if } |d| > a_{0}b_{0}, \\
\infty & \text{if } |d| < a_{0}b_{0}.
\end{cases}
\]

Otherwise, if \( \frac{a_{n}b_{n}}{d} - 1 \) \( y_{2j - 4t} + (\alpha_{1}v_{0} + \alpha_{0}) = 0 \) and \( d \neq a_{0}b_{0} \), then \( y_{4tm + 2m + 2j + 2} = y_{2j - 4t} \) for all \( m \in N_{0} \).
We will only prove the items (a) and (b) for Proof.

(a) Assume that \(d - 1 \neq 0\) and \(d = a_0 b_0\), then \(|y_{4tm+2m+2j+2}| \to \infty\), as \(m \to \infty\).

Otherwise, if \(a_1 v_0 + a_0 = 0\) and \(d = a_0 b_0\), then \(y_{4tm+2m+2j+2} = y_{2j-4t}\) for all \(m \in N_0\).

(b) Assume that \(a_1 v_0 + a_0 = 0\) and \(d = a_0 b_0\), then \(|y_{4tm+2m+2j+2}| \to \infty\), as \(m \to \infty\).

Otherwise, if \(a_1 v_0 + a_0 = 0\) and \(d = a_0 b_0\), then \(y_{4tm+2m+2j+2} = y_{2j-4t}\) for all \(m \in N_0\).

Proof. We will only prove the items (a) and (b) for \(k = 2t + 1(t=1,2,\ldots)\) and \(j \in \{ t, t + 1, \ldots, 3t - 1 \}\).

(a) Assume that \((d - 1) x_{2j - 4t + 1} + \beta_1 \neq 0\). It is easy to see from Corollary 4 that \(x_{4tm+2m+2j+1} \neq 0\). Clearly, if \(|d| < 1\), then \(|d|^{u+1} \to 0\).
as \( m \to \infty \). On the other hand, if \( |d| > 1 \), then \( |d|^{m+1} \to \infty \) as \( m \to \infty \). From Eq. (27), we get

\[
\lim_{m \to \infty} |x_{4tm+2m+2j+1}| = \lim_{m \to \infty} \left| \frac{(d-1)x_{2j-4t-1} + (\beta_0 u_1 + \beta_1)}{d-1}d^{m+1} \right|
\]

\[
+ \left( \frac{(\beta_0 u_1 + \beta_1)}{1-d} \right) \lim_{m \to \infty} d^{m+1}
\]

\[
= \left\{ \begin{array}{ll}
\frac{|\beta_0 u_1 + \beta_1|}{d-1} & \text{if } |d| < 1 \\
\infty & \text{if } |d| > 1
\end{array} \right.
\]

Now on the other hand \((d-1)x_{2j-4t-1} + (\beta_0 u_1 + \beta_1) = 0\) and \( d \neq 1 \). Then we have

\[
x_{4tm+2m+2j+1} = x_{2j-4t-1}d^{m+1} + (\beta_0 u_1 + \beta_1) \left( \frac{d^{m+1} - 1}{d-1} \right)
\]

\[
= x_{2j-4t-1}d^{m+1} + \left( \frac{d^{m+1} - 1}{d-1} \right) (- (d-1)x_{2j-4t-1})
\]

\[
= x_{2j-4t-1}d^{m+1} - (d^{m+1} - 1)x_{2j-4t-1}
\]

\[
= x_{2j-4t-1},
\]

which completes the proof of (a).

(b) Let \( d = 1 \). If \( \beta_0 u_1 + \beta_1 \neq 0 \), then from Eq. (27) we get

\[
x_{4tm+2m+2j+1} = x_{2j-4t-1} + (\beta_0 u_1 + \beta_1)(m + 1) \neq 0.
\]

Letting \( m \to \infty \) in above equations implies that \( |x_{4tm+2m+2j+1}| \to \infty \). On the other hand, If \( \beta_0 u_1 + \beta_1 = 0 \), then obviously, for all \( m \in N_0 \),

\[
x_{4tm+2m+2j+1} = x_{2j-4t-1} + (\beta_0 u_1 + \beta_1)(m + 1) = x_{2j-4t-1} + 0 = x_{2j-4t-1},
\]

which completes the proof of (b).

\[\square\]

**Corollary 6.** Suppose that \( \{(x_n, y_n)\}_{n \geq -k} \) is a well-defined solution of system (3) with \( a_0 = b_1, a_1 = b_0, u_0 u_1 = d, v_0 v_1 = \frac{a_0 b_0}{d} \). Then, the next statements are true.

1. Let \( k = 1 \).
(a) If $d = -1$, then for all $m \in N_0$, we get
\[
\begin{align*}
x_{4m-1} &= x_{-1}, \\
x_{4m} &= x_0, \\
x_{4m+1} &= -x_{-1} + (\beta_0 u_1 + \beta_1), \\
x_{4m+2} &= -x_0 + (\beta_1 u_0 + \beta_0).
\end{align*}
\]

(b) If $d = -a_0 b_0$, then for all $m \in N_0$, we have
\[
\begin{align*}
y_{4m-1} &= y_{-1}, \\
y_{4m} &= y_0, \\
y_{4m+1} &= -y_{-1} + (a_0 v_1 + \alpha_1), \\
y_{4m+2} &= -y_0 + (\alpha_1 v_0 + a_0).
\end{align*}
\]

(c) If $d = 1$, $u_1 = u_0 = 1$, $\beta_0 + \beta_1 = 0$, then for all $m \in N_0$, we get
\[
\begin{align*}
x_{4m-1} &= x_{-1}, \\
x_{4m} &= x_0.
\end{align*}
\]

(d) If $d = a_0 b_0$, $v_1 = v_0 = 1$, $\alpha_0 + \alpha_1 = 0$, then for all $m \in N_0$, we have
\[
\begin{align*}
y_{4m-1} &= y_{-1}, \\
y_{4m} &= y_0.
\end{align*}
\]

(2) Let $k = 2t + 1, (t = 1, 2, \ldots)$ and $j \in \{t, t+1, \ldots, 3t\}$.

(a) If $d = -1$, then for all $m \in N_0$, we have
\[
\begin{align*}
x_{8tm+4m+2j+1} &= -x_{2j-4t-1} + (\beta_0 u_1 + \beta_1), \\
x_{8tm+4m+4t+2j+3} &= x_{2j-4t-1}, \\
x_{8tm+4m+2j+2} &= -x_{2j-4t} + (\beta_1 u_0 + \beta_0), \\
x_{8tm+4m+4t+2j+4} &= x_{2j-4t}.
\end{align*}
\]

(b) If $d = -a_0 b_0$, then for all $m \in N_0$, we get
\[
\begin{align*}
y_{8tm+4m+2j+1} &= -y_{2j-4t-1} + (a_0 v_1 + \alpha_1), \\
y_{8tm+4m+4t+2j+3} &= y_{2j-4t-1}, \\
y_{8tm+4m+2j+2} &= -y_{2j-4t} + (\alpha_1 v_0 + a_0), \\
y_{8tm+4m+4t+2j+4} &= y_{2j-4t}.
\end{align*}
\]

(c) If $d = 1$, $u_1 = u_0 = 1$, $\beta_0 + \beta_1 = 0$ then for all $m \in N_0$, we have
\[
\begin{align*}
x_{4tm+2m+2j+1} &= x_{2j-4t-1}, \\
x_{4tm+2m+2j+2} &= x_{2j-4t}.
\end{align*}
\]
(d) If \( d = a_0 b_0 \), \( v_1 = v_0 = 1 \), \( \alpha_0 + \alpha_1 = 0 \) then for all \( m \in N_0 \), we get:

\[
\begin{align*}
\frac{y_{4tm+2m+2j+1}}{y_{4tm+2m+2j+2}} &= y_{2j-4t-1}, \\
\frac{y_{4tm+2m+2j+2}}{y_{4tm+2m+2j+1}} &= y_{2j-4t}.
\end{align*}
\]

(3) Let \( k = 2t \) (\( t = 1, 2, \ldots \)) and \( j \in \{ t, t+1, \ldots, 3t-1 \} \). If \( u_1 = u_0 = v_1 = v_0 = 1 \) or \( u_1 = u_0 = v_1 = v_0 = 1 \) and \( \beta_1 = \beta_0 = \alpha_1 = \alpha_0 = 0 \), then for all \( m \in N_0 \), we have:

\[
\begin{align*}
x_{4tm+2j+1} &= x_{2j-4t+1}, \\
x_{4tm+2j} &= x_{2j-4t}, \\
y_{4tm+2j+1} &= y_{2j-4t+1}, \\
y_{4tm+2j} &= y_{2j-4t}.
\end{align*}
\]

References


Current address: Yasin Yazlık: Nevşehir Hacı Bektaş Veli University, Faculty of Science and Art, Department of Mathematics, Nevşehir, Turkey

E-mail address: yyazlik@nevsehir.edu.tr

ORCID Address: http://orcid.org/0000-0001-6369-540X

Current address: Merve KARA: Aksaray University, Ortaköy Vocational School, Aksaray, Turkey

E-mail address: mervekara@aksaray.edu.tr

ORCID Address: http://orcid.org/0000-0001-8081-0254