Basis properties of root functions of a regular fourth order boundary value problem

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Abstract

In this paper, we consider the following boundary value problem

\[ y^{(4)} + q(x) y = \lambda y, \quad 0 < x < 1, \]
\[ y'''(1) - (-1)^{\sigma} y''(0) + \alpha y(0) = 0, \]
\[ y^{(s)}(1) - (-1)^{\sigma} y^{(s)}(0) = 0, \quad s = 0, 2, \]

where \( \lambda \) is a spectral parameter, \( q(x) \in L^1(0, 1) \) is complex-valued function and \( \sigma = 0, 1 \). The boundary conditions of this problem are regular but not strongly regular. Asymptotic formulae for eigenvalues and eigenfunctions of the considered boundary value problem are established. When \( \alpha \neq 0 \), we proved that all the eigenvalues, except for finite number, are simple and the system of root functions of this spectral problem forms a Riesz basis in the space \( L^2(0, 1) \). Furthermore, we show that the system of root functions forms a basis in the space \( L^p(0, 1), 1 < p < \infty (p \neq 2) \), under the conditions \( \alpha \neq 0 \) and \( q(x) \in W^1_1(0, 1) \).

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1. Introduction

Henceforth, \( L \) denotes the differential operator generated by the differential expression

\[ l(y) = y^{(4)} + q(x) y, \quad x \in (0, 1), \] (1.1)

and boundary conditions

\[ U_3(y) \equiv y'''(1) - (-1)^{\sigma} y''(0) + \alpha y(0) = 0, \]
\[ U_s(y) \equiv y^{(s)}(1) - (-1)^{\sigma} y^{(s)}(0) = 0, \] (1.2)

where \( q(x) \in L^1(0, 1) \) is complex-valued function, \( s = 0, 2 \) and \( \sigma = 0, 1 \). It is easy to verify that boundary conditions (1.2) are regular, but not strongly regular.

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In [11, 14, 15, 16], Kerimov, Kaya and Gunes investigated the following problem
\[ \begin{align*}
y^{(4)} + p_2(x) y'' + p_1(x) y' + p_0(x) y &= \lambda y, \quad 0 < x < 1, \\
y''(1) - (-1)^\sigma y''(0) + \alpha_{2,1} y'(0) + \alpha_{2,0} y(0) &= 0, \\
y'(1) - (-1)^\sigma y'(0) + \alpha_{1,0} y(0) &= 0, \\
y(1) - (-1)^\sigma y(0) &= 0
\end{align*} \]
in various cases. However, the problems in [11, 14, 15, 16] cannot be reduced to eigenvalue problem for the operator (1.1)-(1.2).

In [8, 19, 27], it was proven that the system of root functions of a differential operator with strongly regular boundary conditions forms a basis. Besides, the basicity of root functions of a differential operator with non-strongly regular boundary conditions was investigated in [3, 4, 5, 6, 7, 9, 12, 17, 20, 21, 22, 23, 24, 25, 26, 29, 30, 31, 32, 33]. For more information about these papers, see [11, 14, 15, 16].

We define \( c_0 \) and \( \varepsilon_n \) as follows:
\[ \begin{align*}
c_0 &= \int_0^1 q(\xi) \, d\xi, \\
\varepsilon_n &= \int_0^1 q(\xi) \cdot e^{2(n-\sigma)\pi i\xi} \, d\xi + \int_0^1 q(\xi) \cdot e^{-2(n-\sigma)\pi i\xi} \, d\xi + n^{-1}.
\end{align*} \]

Now, we give two theorems and their corollary and we will prove them.

**Theorem 1.1.** If \( q(x) \in L_1(0, 1) \) is a complex-valued function and \( \alpha \neq 0 \), all eigenvalues of differential operator (1.1)-(1.2), excluding a finite number, are simple and form two sequences \( \{\lambda_{n,1}\} \) and \( \{\lambda_{n,2}\} \) and these eigenvalues have the following asymptotic formulae for sufficiently large numbers \( n \):
\[ \lambda_{n+n_1,1} = ((2n-\sigma)\pi)^4 \left(1 + \frac{c_0}{((2n-\sigma)\pi)^4} + O\left(n^{-4}\varepsilon_n\right)\right), \]
\[ \lambda_{n+n_2,2} = ((2n-\sigma)\pi)^4 \left(1 + \frac{c_0 - 2(-1)^\sigma \alpha}{((2n-\sigma)\pi)^4} + O\left(n^{-4}\varepsilon_n\right)\right), \]
where \( n_1, n_2 \) are certain integers. Moreover, for sufficiently large numbers \( n \), the corresponding eigenfunctions \( u_{n,1}(x) \) and \( u_{n,2}(x) \) have the asymptotic formulae:
\[ \begin{align*}
u_{n+n_1,1}(x) &= \sqrt{2} \sin(2n-\sigma)\pi x + O(\varepsilon_n), \\
u_{n+n_2,2}(x) &= \sqrt{2} \cos(2n-\sigma)\pi x + O(\varepsilon_n).
\end{align*} \]

**Theorem 1.2.** If \( q(x) \in L_1(0, 1) \) is a complex-valued function and \( \alpha \neq 0 \), the root functions of differential operator (1.1)-(1.2) form a Riesz basis in the space \( L_2(0, 1) \). In addition, if \( q(x) \in W_1^1(0, 1) \), then the root functions form a basis in \( L_p(0, 1) \), \( 1 < p < \infty \), where
\[ L_p(0, 1) = \left\{ f : (0, 1) \to \mathbb{C}, \int_0^1 |f(\xi)|^p \, d\xi < \infty \right\}, \]
\[ W_p^n(0, 1) = \left\{ f : (0, 1) \to \mathbb{C}, f^{(n)} \in L_p(0, 1) \right\}. \]

**Corollary 1.3.** If \( q(x) \in L_2(0, 1) \) is a complex-valued function and \( \alpha \neq 0 \), then \( n_1 + n_2 = 1 - \sigma \). Hence, we can choose \( n_1 = 0, n_2 = 1 - \sigma \).
2. Some auxiliary formulae

We denote the set
\[ \left\{ \rho \in \mathbb{C} : 0 \leq \arg \rho \leq \frac{\pi}{4} \right\} \]  
(2.1)
by \( S_0 \) and the different four roots of the algebraic equation \( \omega^4 + 1 = 0 \) by \( \omega_k, k = 1, 4 \). The numbers \( \omega_k, k = 1, 4 \), can be ordered so that the inequalities
\[ \Re(\rho \omega_1) \leq \Re(\rho \omega_2) \leq \Re(\rho \omega_3) \leq \Re(\rho \omega_4) \]  
(2.2)
hold for all \( \rho \in S_0 \), where \( \Re(z) \) denotes the real parts of a complex number \( z \) (see [28, Chapter II, §4.2]). From now on, the numbers \( \omega_k, k = 1, 4 \), will be chosen by satisfying the inequalities (2.2) for all \( \rho \in S_0 \). Then, we get by [28, Chapter II, §4.8] that the numbers \( \omega_1, \omega_2, \omega_3, \omega_4 \) are determined as
\[ \omega_1 = e^{\frac{3\pi i}{4}}, \quad \omega_2 = e^{-\frac{3\pi i}{4}}, \quad \omega_3 = e^{\frac{\pi i}{4}}, \quad \omega_4 = e^{-\frac{\pi i}{4}}. \]  
(2.3)
One can easily see that
\[ \omega_1 = -\omega_4, \quad \omega_2 = -\omega_3. \]  
(2.4)

Lemma 2.1 ([16]). For all \( \rho \in S_0 \), the inequalities
\[ \Re(\rho \omega_1) \leq -\frac{\sqrt{2}}{2} |\rho|, \quad \Re(\rho \omega_4) \geq \frac{\sqrt{2}}{2} |\rho|. \]  
(2.5)
are valid.

Let
\[ T_0 = \{ \rho - c : \rho \in S_0 \}, \]
where \( c \) is a complex number. The inequalities (2.2) and (2.5) will be rewritten in the forms
\[ \Re((\rho + c) \omega_1) \leq \Re((\rho + c) \omega_2) \leq \Re((\rho + c) \omega_3) \leq \Re((\rho + c) \omega_4), \]  
(2.6)
\[ \Re((\rho + c) \omega_1) \leq \frac{\sqrt{2}}{2} |\rho + c|, \quad \Re((\rho + c) \omega_4) \geq \frac{\sqrt{2}}{2} |\rho + c| \]  
(2.7)
for all \( \rho \in T_0 \).

For each \( \rho \in T_0 \), the equation
\[ l(y) + \rho^{3} y = 0 \]  
(2.8)
has four solutions \( y_1(x, \rho), y_2(x, \rho), y_3(x, \rho), y_4(x, \rho) \). These solutions are linearly independent and analytic when \( |\rho| \geq M_0 \), where \( M_0 \) is a positive constant [28, Chapter II, §4.5-4.6]. Besides, the derivatives of these functions satisfy the following integro-differential equations
\[ \frac{d^s y_k(x, \rho)}{dx^s} = \rho^s e^{\rho \omega_k x} + \frac{1}{4 \rho^3} \int_0^x \frac{\partial^s K_1(x, \xi, \rho)}{\partial x^s} q(\xi) y_k(\xi, \rho) d\xi - \frac{1}{4 \rho^3} \int_0^x \frac{\partial^s K_2(x, \xi, \rho)}{\partial x^s} q(\xi) y_k(\xi, \rho) d\xi, \quad s = 0, 3, \]  
(2.9)
where
\[ K_1(x, \xi, \rho) = \sum_{\alpha=1}^{k} \omega_{\alpha} e^{\rho \omega_{\alpha}(x-\xi)}, \quad K_2(x, \xi, \rho) = \sum_{\alpha=k+1}^{4} \omega_{\alpha} e^{\rho \omega_{\alpha}(x-\xi)}. \]  
(2.10)
Let \( z_{k, s}(x, \rho), k = 1, 4, s = 0, 3 \), be functions that satisfy the equations
\[ \frac{d^s z_{k, s}(x, \rho)}{dx^s} = \rho^s e^{\rho \omega_k x} z_{k, s}(x, \rho). \]  
(2.11)
By [28, Chapter II, §4.5], the functions $z_{k,s}(x,\rho)$ are analytic with respect to $\rho$ and satisfy

$$z_{k,s}(x,\rho) = \omega^s_k + O\left(\rho^{-1}\right), \quad s = 0, 3, \quad k = 1, 4.$$  \hspace{1cm} (2.12)

By (2.9)-(2.11), we have

$$z_{k,s}(x,\rho) = \omega^s_k + \frac{\omega^{s+1}_k}{4\rho^3} \int_0^x q(\xi) z_{k,0}(\xi,\rho) \, d\xi + \frac{1}{4\rho^3} \sum_{\alpha=1}^{k-1} \omega^{s+1}_\alpha \int_0^x e^{\rho(\omega\alpha - \omega_k)(x-\xi)} q(\xi) z_{k,0}(\xi,\rho) \, d\xi - \frac{1}{4\rho^3} \sum_{\alpha=k+1}^4 \omega^{s+1}_\alpha \int_x^1 e^{\rho(\omega\alpha - \omega_k)(x-\xi)} q(\xi) z_{k,0}(\xi,\rho) \, d\xi.$$  \hspace{1cm} (2.13)

Note that, by (2.6), we get

$$\Re(\rho (\omega_\alpha - \omega_\beta)) = \Re((\rho + c) (\omega_\alpha - \omega_\beta)) - \Re(c (\omega_\alpha - \omega_\beta)) \leq 2|c|,$$

where $1 \leq \alpha \leq \beta \leq 4$. By using the above inequality and (2.12), we obtain for $k = 1, 4$

$$\int_0^x q(\xi) z_{k,0}(\xi,\rho) e^{\rho(\omega\alpha - \omega_k)(x-\xi)} \, d\xi = O(1), \quad \alpha \leq k,$$

$$\int_x^1 q(\xi) z_{k,0}(\xi,\rho) e^{\rho(\omega\alpha - \omega_k)(x-\xi)} \, d\xi = O(1), \quad \alpha > k.$$

By using the last relations and the formulae (2.12)-(2.13), we get

$$z_{k,s}(x,\rho) = \omega^s_k + O\left(\rho^{-3}\right), \quad s = 0, 3, \quad k = 1, 4.$$  \hspace{1cm} (2.14)

If we now put (2.14) in (2.13), then (2.13) takes the form

$$z_{k,s}(x,\rho) = \omega^s_k + \frac{\omega^{s+1}_k}{4\rho^3} \int_0^x q(\xi) \, d\xi + \frac{1}{4\rho^3} \sum_{\alpha=1}^{k-1} \omega^{s+1}_\alpha \int_0^x e^{\rho(\omega\alpha - \omega_k)(x-\xi)} q(\xi) \, d\xi - \frac{1}{4\rho^3} \sum_{\alpha=k+1}^4 \omega^{s+1}_\alpha \int_x^1 e^{\rho(\omega\alpha - \omega_k)(x-\xi)} q(\xi) \, d\xi + O\left(\rho^{-6}\right).$$
By the last relation, we have

\[ z_{2,s}(0, \rho) = \omega_2 - \frac{\omega_2^{s+1}}{4\rho^3} \int_0^1 q(\xi) e^{2\rho\omega_2} d\xi - \frac{\omega_2^{s+1}}{4\rho^3} \int_0^1 q(\xi) e^{\rho(\omega_2 - \omega_1)} d\xi \]

\[ = \omega_2 - \frac{\omega_2^{s+1}}{4\rho^3} \int_0^1 q(\xi) e^{\rho(\omega_2 - \omega_1)} d\xi + O\left(\rho^{-6}\right), \]

\[ z_{3,s}(0, \rho) = \omega_3 - \frac{\omega_3^{s+1}}{4\rho^3} \int_0^1 q(\xi) e^{\rho(\omega_3 - \omega_1)} d\xi + O\left(\rho^{-6}\right), \]

\[ z_{2,s}(1, \rho) = \omega_2 + \frac{\omega_2^{s+1}}{4\rho^3} \int_0^1 q(\xi) e^\rho(d\xi + O\left(\rho^{-6}\right), \]

\[ z_{3,s}(1, \rho) = \omega_3 + \frac{\omega_3^{s+1}}{4\rho^3} \int_0^1 q(\xi) e^\rho(d\xi + \right), \]

\[ + \frac{\omega_3^{s+1}}{4\rho^3} \int_0^1 q(\xi) e^{2\rho\omega_3(1-\xi)} d\xi + O\left(\rho^{-6}\right), \]

where we assume that \( c_0 = 0 \). The case \( c_0 \neq 0 \) will be investigated later.

3. Proof of Theorem 1.1

Let

\[ \Delta(\rho) = \begin{vmatrix} U_3(y_1) & U_3(y_2) & U_3(y_3) & U_3(y_4) \\ U_2(y_1) & U_2(y_2) & U_2(y_3) & U_2(y_4) \\ U_1(y_1) & U_1(y_2) & U_1(y_3) & U_1(y_4) \\ U_0(y_1) & U_0(y_2) & U_0(y_3) & U_0(y_4) \end{vmatrix}. \] (3.1)

If the vertex \(-c\) in the domain \( T_0 \) is properly chosen, then eigenvalues \( \lambda \) of the operator (1.1)-(1.2) whose absolute values are sufficiently large have the form \( \lambda = -\rho^4 \), where the numbers \( \rho \) are the zeros of the following equation

\[ \Delta(\rho) = 0 \] (3.2)

and in \( T_0 \). Conversely, the set of such numbers \( \rho \) contains all the zeros of (3.2) in \( T_0 \) excluding a finite number [28, Chapter II. § 4.9]. By (2.11), we have

\[ U_s(y_k) = \rho^4 \{ e^{\rho\omega_k} z_{k,s}(1, \rho) - (-1)^\sigma z_{k,s}(0, \rho) \}, \]

\[ U_3(y_k) = \rho^3 \{ e^{\rho\omega_k} z_{k,3}(1, \rho) - (-1)^\sigma z_{k,3}(0, \rho) \} + \alpha_{k,0}(0, \rho) \] (3.3)

for \( s = 0, 2 \) and \( k = 1, 3 \). By (2.7), \( e^{\rho\omega_1} \) exponentially tends to zero and \( e^{\rho\omega_4} \) exponentially tends to infinity. So, the relations

\[ U_s(y_1) = -(-1)^\sigma \rho^s \left[ z_{1,s}(0, \rho) + O\left(\rho^{-7}\right) \right], \quad s = 0, 2, \]

\[ U_3(y_1) = -(-1)^\sigma \rho^3 \left[ z_{1,3}(0, \rho) - (-1)^\sigma \frac{\alpha}{\rho^3} z_{1,0}(0, \rho) + O\left(\rho^{-7}\right) \right], \]

\[ U_s(y_4) = \rho^4 e^{\rho\omega_4} \left[ z_{4,s}(1, \rho) + O\left(\rho^{-7}\right) \right], \quad s = 0, 3 \] (3.4)

are valid by (2.14) and (3.3).
Let
\[
A_{s,k}(\rho) =
\begin{cases}
  z_{1,s}(0, \rho), & \text{if } k = 1, \\
  e^{i\omega_k} z_{k,s}(1, \rho) - (-1)^s z_{k,s}(0, \rho), & \text{if } k = 2, 3, \\
  z_{4,s}(1, \rho), & \text{if } k = 4,
\end{cases}
\]

where \( s = 0, 2 \). By the formulae (3.3)-(3.5), it is obvious that
\[
U_s(y_1) = -(-1)^s \rho^s \{ A_{s,1}(\rho) + O(\rho^{-7}) \},
\]
\[
U_s(y_k) = \rho^s A_{s,k}(\rho),
\]
\[
U_s(y_4) = \rho^s e^{i\omega_k} \{ A_{s,4}(\rho) + O(\rho^{-7}) \},
\]
where \( k = 2, 3 \) and \( s = 0, 3 \). We put these formulae of boundary conditions in the equation (3.2). If we divide out the common multipliers \( \rho^3, \rho^2, \rho \) of the rows and also divide out the common multipliers \( -(-1)^s \) and \( e^{i\omega_k} \) of the columns of the determinant \( \Delta(\rho) \), then we get that the equation (3.2) is equivalent to
\[
\Delta_1(\rho) + O(\rho^{-7}) = 0,
\]
where
\[
\Delta_1(\rho) = \begin{vmatrix}
  A_{3,1}(\rho) & A_{3,2}(\rho) & A_{3,3}(\rho) & A_{3,4}(\rho) \\
  A_{2,1}(\rho) & A_{2,2}(\rho) & A_{2,3}(\rho) & A_{2,4}(\rho) \\
  A_{1,1}(\rho) & A_{1,2}(\rho) & A_{1,3}(\rho) & A_{1,4}(\rho) \\
  A_{0,1}(\rho) & A_{0,2}(\rho) & A_{0,3}(\rho) & A_{0,4}(\rho)
\end{vmatrix}.
\]

We now rewrite the formulae (3.5)-(3.6) in [14]. If \( \rho \) is a root of equation (3.7), we get that the equalities
\[
e^{i\omega_2} - (-1)^s = O(\rho^{-3}), \quad e^{i\omega_3} - (-1)^s = O(\rho^{-3})
\]
are valid.

By using the relations (2.14), (2.15) and (3.9) for \( s = 0, 3 \), we have
\[
A_{s,k}(\rho) = A_{s,k}^{(k)}(\rho) + B_{s,k}^{(k)}(\rho) + O(\rho^{-6}), \quad k = 2, 3,
\]
\[
A_{s,k}(\rho) = \omega_k^s + O(\rho^{-3}), \quad k = 1, 4,
\]
where

\[ A_{s,2}^{(2)}(\rho) = \omega_s^3 (e^{\rho \omega_s} - (-1)^s) + \frac{(-1)^s \omega_s^{s+1}}{4\rho^3} \int_0^1 q(\xi) e^{2\rho \omega_s \xi} d\xi, \quad s = 0, 2, \]

\[ A_{s,3}^{(3)}(\rho) = \omega_s^3 (e^{\rho \omega_s} - (-1)^s) + \frac{(-1)^s \omega_s^{s+1}}{4\rho^3} \int_0^1 q(\xi) e^{2\rho \omega_s (1-\xi)} d\xi, \quad s = 0, 2, \]

\[ A_{3,2}^{(2)}(\rho) = \omega_3^3 (e^{\rho \omega_3} - (-1)^3) + \frac{(-1)^3 \omega_3^{3+1}}{4\rho^3} \int_0^1 q(\xi) e^{2\rho \omega_3 \xi} d\xi + \frac{\alpha}{\rho^3}, \]

\[ A_{3,3}^{(3)}(\rho) = \omega_3^3 (e^{\rho \omega_3} - (-1)^3) + \frac{(-1)^3 \omega_3^{3+1}}{4\rho^3} \int_0^1 q(\xi) e^{2\rho \omega_3 (1-\xi)} d\xi + \frac{\alpha}{\rho^3}, \]

and

\[ B_{s,k}^{(k)}(\rho) = \frac{(-1)^s \omega_s^{s+1}}{4\rho^3} \int_0^1 q(\xi) e^{\rho(\omega_s - \omega_k)(1-\xi)} d\xi + \frac{(-1)^s \omega_s^{s+1}}{4\rho^3} \int_0^1 q(\xi) e^{\rho(\omega_s - \omega_k)\xi} d\xi, \quad s = 0, 3, \quad k = 2, 3. \] (3.12)

By the relations (3.9), (3.11) and (3.12), we have

\[ A_{s,k}(\rho) = O\left(\rho^{-3}\right), \quad k = 2, 3, \quad s = 0, 3. \] (3.13)

If we put the equalities (3.10) in the determinant (3.8), then, by using (3.13), we get that the equation (3.7) is equivalent to

\[ \Delta_2(\rho) + O\left(\rho^{-7}\right) = 0, \] (3.14)

where

\[ \Delta_2(\rho) = \begin{vmatrix} \omega_1^3 & A_{3,2}^{(2)}(\rho) + B_{3,2}^{(2)}(\rho) & A_{3,3}^{(3)}(\rho) + B_{3,3}^{(3)}(\rho) & \omega_3^3 \\ \omega_2^3 & A_{3,2}^{(2)}(\rho) + B_{2,2}^{(2)}(\rho) & A_{3,3}^{(3)}(\rho) + B_{2,3}^{(3)}(\rho) & \omega_4^2 \\ \omega_1 & A_{1,2}^{(2)}(\rho) + B_{1,2}^{(2)}(\rho) & A_{1,3}^{(3)}(\rho) + B_{1,3}^{(3)}(\rho) & \omega_4^2 \\ 1 & A_{0,2}^{(2)}(\rho) + B_{0,2}^{(2)}(\rho) & A_{0,3}^{(3)}(\rho) + B_{0,3}^{(3)}(\rho) & 1 \end{vmatrix}. \]

By the definition of \( B_{s,k}^{(k)}(\rho) \) (see: (3.12)), it can be easily proven that the columns

\[ (B_{3,2}^{(2)}(\rho), B_{2,2}^{(2)}(\rho), B_{1,2}^{(2)}(\rho), B_{0,2}^{(2)}(\rho))^T \]

and

\[ (B_{3,3}^{(3)}(\rho), B_{2,3}^{(3)}(\rho), B_{1,3}^{(3)}(\rho), B_{0,3}^{(3)}(\rho))^T \]

are two linear combinations of the first and fourth columns of the determinant \( \Delta_2(\rho) \). Consequently, the determinant \( \Delta_2(\rho) \) can be rewritten as follows:

\[ \Delta_2(\rho) = \begin{vmatrix} \omega_1^3 & A_{3,2}^{(2)}(\rho) & A_{3,3}^{(3)}(\rho) & \omega_3^3 \\ \omega_2^3 & A_{2,2}^{(2)}(\rho) & A_{2,3}^{(3)}(\rho) & \omega_4^2 \\ \omega_1 & A_{1,2}^{(2)}(\rho) & A_{1,3}^{(3)}(\rho) & \omega_4^2 \\ 1 & A_{0,2}^{(2)}(\rho) & A_{0,3}^{(3)}(\rho) & 1 \end{vmatrix}. \] (3.15)
Thus, the equation (3.18) has the unique root $G$ can be easily proved by using the proof of Riemann-Lebesque Lemma. Equation (3.18) in (3.14) is reduced to $c$ can be easily obtained by using (3.20). By putting $\varepsilon$ where

$$
(3.23).
$$

Consider the equation (3.18). By Rouche’s theorem, we can get that the roots of the equation (3.18) in $O(n)$ in [28, Chapter II, § 4.9]. Besides, the equation (3.18) has a unique root in $O(n)$ of $G$. Assume that $\tilde{\rho}$ is the unique root of (3.18) in $G_n$. By the equalities (40) and (41) in [14], we obtain

$$
\tilde{\rho} = -\frac{(2n-\sigma)\pi}{\omega_2} + r, \quad r = O\left(n^{-3}\right).
$$

If we use the formulae (3.20) in (3.17), we obtain

$$
\varepsilon(\rho) = O\left(\varepsilon_n\right),
$$

where $\varepsilon_n$ is the sequence defined in (1.4).

Now, we find more accurate formula for the number $r$. The following formulae

$$
\frac{1}{\rho^3} = \frac{\omega_2}{(2n-\sigma)^3\pi^3} + O\left(n^{-7}\right),
$$

$$
\overline{\rho}^\omega = (-1)^\sigma \left\{1 + r\omega_2 + O\left(n^{-6}\right)\right\}
$$

can be easily obtained by using (3.20). By putting $\rho = \tilde{\rho}$ in (3.18) and using the relations (3.21) and (3.23), we have

$$
r = O\left(n^{-3}\varepsilon_n\right).
$$

Thus, the equation (3.18) has the unique root

$$
\tilde{\rho}_{n,1} = -\frac{(2n-\sigma)\pi}{\omega_2} + O\left(n^{-3}\varepsilon_n\right)
$$

in $O(n^{-1})$-neighbourhood $G_n$ of $z_n = -(2n-\sigma)\pi/\omega_2$, $n = n_0, n_0 + 1, \ldots$ by (3.20) and (3.24).

Similarly, we conclude that the equation (3.19) has the unique root

$$
\tilde{\rho}_{n,2} = -\frac{1}{\omega_2} \left\{(2n-\sigma)\pi i - \frac{(-1)^\sigma i\alpha}{2(2n-\sigma)^3\pi^3}\right\} + O\left(n^{-3}\varepsilon_n\right)
$$

in $O(n^{-1})$-neighbourhood $G_n$ of the point $z_n, n = n_0, n_0 + 1, \ldots$ by the formulae (3.20)-(3.23).
Now, we investigate the eigenfunction \( \tilde{u}_{n,1}(x) \) corresponding to the eigenvalue \( \lambda = -\left( \tilde{\rho}_{n,1} \right)^4 \). We use the following determinant for this eigenfunction

\[
\tilde{u}_{n,1}(x) = \frac{(-1)^{\sigma} e^{-\rho \omega_1 \sqrt{2}}}{4\omega_2 i\alpha \rho^3} \begin{vmatrix}
 y_1(x,\rho) & y_2(x,\rho) & y_3(x,\rho) & y_4(x,\rho) \\
 U_3(y_1) & U_3(y_2) & U_3(y_3) & U_3(y_4) \\
 U_2(y_1) & U_2(y_2) & U_2(y_3) & U_2(y_4) \\
 U_1(y_1) & U_1(y_2) & U_1(y_3) & U_1(y_4) \\
\end{vmatrix},
\]

where \( \rho = \tilde{\rho}_{n,1} \) and \( n \) is sufficiently large positive integer. Easily, we can rewrite

\[
\tilde{u}_{n,1}(x) = -\frac{\rho^3 \sqrt{2}}{4\omega_2 i\alpha} \times \begin{vmatrix}
 (-1)^{\sigma} y_1(x,\rho) & y_2(x,\rho) & y_3(x,\rho) & e^{-\rho \omega_1 y_4(x,\rho)} \\
 A_{3,1}(\rho) & A_{3,2}(\rho) & A_{3,3}(\rho) & A_{3,4}(\rho) \\
 A_{2,1}(\rho) & A_{2,2}(\rho) & A_{2,3}(\rho) & A_{2,4}(\rho) \\
 A_{1,1}(\rho) & A_{1,2}(\rho) & A_{1,3}(\rho) & A_{1,4}(\rho) \\
\end{vmatrix},
\]

where \( \rho = \tilde{\rho}_{n,1} \). If we calculate the determinant in (3.29) by using (3.10), (3.13) and (3.28), then we have

\[
\tilde{u}_{n,1}(x) = -\frac{\rho^3 \sqrt{2}}{4\omega_2 i\alpha} \left\{ y_3(x,\rho) E_2(\rho) - y_2(x,\rho) E_3(\rho) \right\} + O(\rho^{-3}) ,
\]

where \( \rho = \tilde{\rho}_{n,1} \) and

\[
E_k(\rho) = \begin{vmatrix}
 \omega_1^3 & A_{3,k}(\rho) & \omega_4^3 \\
 \omega_1^2 & A_{2,k}(\rho) & \omega_4^2 \\
 \omega_1 & A_{1,k}(\rho) & \omega_4 \\
\end{vmatrix}, \quad k = 2,3.
\]

By the last formula and (3.10), we get that the determinant \( E_k(\rho) \) can be rewritten as follows

\[
E_k(\rho) = \omega_1^3 \begin{vmatrix}
 A_{3,k}(\rho) & \omega_4^3 \\
 A_{2,k}(\rho) & \omega_4^2 \\
 A_{1,k}(\rho) & \omega_4 \\
\end{vmatrix} + O\left( \rho^{-6} \right), \quad k = 2,3,
\]

where \( \rho = \tilde{\rho}_{n,1} \). By (3.12), the second determinant above is zero, i.e.,

\[
E_k(\rho) = \omega_1^3 \begin{vmatrix}
 A_{3,k}(\rho) & \omega_4^3 \\
 A_{2,k}(\rho) & \omega_4^2 \\
 A_{1,k}(\rho) & \omega_4 \\
\end{vmatrix} + O\left( \rho^{-6} \right), \quad k = 2,3,
\]

where \( \rho = \tilde{\rho}_{n,1} \). The following formulae

\[
A_{1,k}(\rho) = A_{2,k}(\rho) = O\left( \rho^{-3} \right), \quad A_{3,k}(\rho) = \frac{\alpha}{\rho^3} + O\left( \rho^{-3} \right)
\]
are directly obtained by using (3.11) and (3.18), where \( k = 2, 3, \rho = \tilde{\rho}_{n,1} \) and \( \varepsilon = \varepsilon_n \). If we calculate the determinant in (3.31) by using the last relations, we get

\[
E_k (\rho) = -\frac{2\omega_0\alpha}{\rho^4} + O \left( \rho^{-2}\varepsilon \right),
\]

where \( k = 2, 3 \) and \( \rho = \tilde{\rho}_{n,1} \) and \( \varepsilon = \varepsilon_n \). Consequently, we have

\[
\tilde{u}_{n,1} (x) = \frac{\sqrt{2}}{2i} \left( y_3 (x, \tilde{\rho}_{n,1}) - y_2 (x, \tilde{\rho}_{n,1}) \right) + O (\varepsilon_n)
\]

by (3.30). On the other hand, we can write

\[
y_2 (x, \tilde{\rho}_{n,1}) = e^{-2(n-\sigma)\pi ix} + O (n^{-1}), \quad y_3 (x, \tilde{\rho}_{n,1}) = e^{2(n-\sigma)\pi ix} + O (n^{-1}),
\]

\[
(\tilde{\rho}_{n,1})^{-1} = O (n^{-1}),
\]

by (2.11), (2.12) and (3.25). Finally, we have the expression

\[
\tilde{u}_{n,1} (x) = \sqrt{2} \sin (2n - \sigma) \pi x + O (\varepsilon_n).
\]

(3.32)

Now, we also investigate the eigenfunction \( \tilde{u}_{n,2} (x) \) corresponding to the eigenvalue \( \lambda = - (\tilde{\rho}_{n,2})^4 \) by using the following determinant

\[
\tilde{u}_{n,2} (x) = \frac{(1) e^{-\rho\alpha\sqrt{2}}}{4i\alpha} \left| \begin{array}{cccc}
y_1 (x, \rho) & y_2 (x, \rho) & y_3 (x, \rho) & y_4 (x, \rho) \\
U_2 (y_1) & U_2 (y_2) & U_2 (y_3) & U_2 (y_4) \\
U_1 (y_1) & U_1 (y_2) & U_1 (y_3) & U_1 (y_4) \\
U_0 (y_1) & U_0 (y_2) & U_0 (y_3) & U_0 (y_4)
\end{array} \right|,
\]

where \( \rho = \tilde{\rho}_{n,2} \). In a similar way, we get

\[
\tilde{u}_{n,2} (x) = \sqrt{2} \cos (2n - \sigma) \pi x + O (\varepsilon_n).
\]

(3.33)

We now prove the formulae (1.5) and (1.6). By the relation \( \lambda = -\rho^4 \), we have

\[
\tilde{\lambda}_{n,1} = - (\tilde{\rho}_{n,1})^4 = ((2n - \sigma) \pi)^4 \left\{ 1 + O \left( n^{-4}\varepsilon_n \right) \right\},
\]

\[
\tilde{\lambda}_{n,2} = - (\tilde{\rho}_{n,2})^4 = ((2n - \sigma) \pi)^4 \left\{ 1 - \frac{2(-1)^\sigma \alpha}{((2n - \sigma) \pi)} + O \left( n^{-4}\varepsilon_n \right) \right\}.
\]

The above formulae are valid in case of \( c_0 = 0 \). Now, assume that \( c_0 \neq 0 \) (see (1.3)). Consider the eigenvalue problem with the differential expression

\[
y^{(4)} + q (x) y = \lambda y
\]

(see (1.1)). We can rewrite this problem as

\[
y^{(4)} + (q (x) - c_0) y = (\lambda - c_0) y.
\]

One can easily see that the integral of \( q (x) - c_0 \) on the line \([0, 1]\) is zero. Then, by the above proof, for the eigenvalues \( \lambda - c_0 \), the formulae

\[
\tilde{\lambda}_{n,1} - c_0 = ((2n - \sigma) \pi)^4 \left\{ 1 + O \left( n^{-4}\varepsilon_n \right) \right\},
\]

\[
\tilde{\lambda}_{n,2} - c_0 = ((2n - \sigma) \pi)^4 \left\{ 1 - \frac{2(-1)^\sigma \alpha}{((2n - \sigma) \pi)} + O \left( n^{-4}\varepsilon_n \right) \right\}.
\]

(3.34)

are valid and the eigenfunctions \( y \) do not change. On the other hand, the construction of the integers \( n_1 \) and \( n_2 \) is similar to the way in [11, 16, 14, 15]. Hence, the formulae (1.5) and (1.6) can be obtained by (3.32), (3.33) and (3.34).
4. Proofs of Theorem 1.2 and Corollary 1.3

First, we prove that the root functions of the operator \( L \) form a Riesz basis in \( L_2 (0, 1) \) provided \( q (x) \in L_1 (0, 1) \).

Let

\[
v_{1,1} (x), v_{1,2} (x), \ldots, v_{n,1} (x), v_{n,2} (x), \ldots
\]

be the biorthogonal system of the following system

\[
u_{1,1} (x), u_{1,2} (x), \ldots, u_{n,1} (x), u_{n,2} (x), \ldots
\]

i.e. \( (u_{n,j}, v_{m,s}) = \delta_{n,m} \delta_{j,s}, n, m = 1, 2, \ldots, j, s = 1, 2 \). By [19, p.84] or [28, p.99], (4.1) is the root functions of the adjoint differential operator \( L^* \). \( L^* \) consists of the differential expression and boundary conditions

\[
l^* (z) = z^{iv} + \bar{q} (x) z, \quad U^0_{g} (z) \equiv z (1) - (-1)^{\sigma} z (0) = 0, \quad U^1_{g} (z) \equiv z' (1) - (-1)^{\sigma} z' (0) = 0, \quad U^2_{g} (z) \equiv z'' (1) - (-1)^{\sigma} z'' (0) = 0, \quad U^3_{g} (z) \equiv z''' (1) - (-1)^{\sigma} z''' (0) + \bar{\alpha} z (0) = 0.
\]

(4.3) shows that the differential operator \( L^* \) provides the conditions of Theorem 1.1. So, the formulae

\[
\begin{align*}
v_{n+n_1,1} (x) &= r_{n+n_1,1} (\sin (2n - \sigma) \pi x + O (\varepsilon_n)), \quad v_{n+n_2,1} (x) = r_{n+n_2,1} (\cos (2n - \sigma) \pi x + O (\varepsilon_n)) \quad (4.4)
\end{align*}
\]

are valid for sufficiently large numbers \( n \), where the numbers \( r_{n+n,j}, j = 1, 2 \) are determined by the inner product \( \langle v_{n+j,n+j}, v_{n+j,n+j} \rangle = 1 \). By these equality and (1.6), (4.4), we have

\[
\begin{align*}
r_{n+n_1,j} &= \sqrt{2} + O (\varepsilon_n), \quad j = 1, 2, \\
\end{align*}
\]

for sufficiently large numbers \( n \). Consequently, if we put the last equality in (4.4), we get

\[
\begin{align*}
v_{n+n_1,1} (x) &= \sqrt{2} \sin (2n - \sigma) \pi x + O (\varepsilon_n), \quad v_{n+n_2,1} (x) = \sqrt{2} \cos (2n - \sigma) \pi x + O (\varepsilon_n) \quad (4.5)
\end{align*}
\]

Each of the systems (4.1) and (4.2) is complete in \( L_2 (0, 1) \) [2]. Furthermore, by (1.6) and (4.5), we get that the sequence of the multiplication of the norms of the elements of the systems (4.1) and (4.2) is bounded i.e. \( \| u_n \| \| v_n \| \leq M \) for all \( n \in \mathbb{N} \), where \( M \) is a constant. On the other hand, since all the eigenvalues, excluding a finite number, are simple, then there are at most finitely many associate functions in the root functions of \( L \). Hence, the system (4.2) is a Riesz basis in \( L_2 (0, 1) \) by the main theorem in [18].

Now, we prove Corollary 1.3 by the assumption \( q (x) \in L_2 (0, 1) \). Let

\[
\begin{align*}
g_0 (x) &= 1, \quad g_{2n-1} (x) = \sqrt{2} \sin 2n \pi x, \quad g_{2n} (x) = \sqrt{2} \cos 2n \pi x, \quad (4.6)
\end{align*}
\]

where \( n = 1, 2, \ldots \). The systems (4.6) and (4.7) are seperately orthonormal bases in \( L_2 (0, 1) \). Since \( q (x) \in L_2 (0, 1) \), then the sum of the squares of the absolute values of Fourier coefficients is convergent. Then, we can easily obtain the following

\[
\sum_{n=1}^{\infty} \varepsilon_n^2 < +\infty. \quad (4.8)
\]

Now, we assume \( \sigma = 0 \). In the case \( \sigma = 1 \), proof can be obtained in a similar method by using (4.7). Let \( n_1 \geq 0 \) and \( n_2 \geq 0 \). By (1.6), (4.6) and (4.8), we obtain

\[
\sum_{n=1}^{\infty} \left( \| u_{n+n_1,1} - g_{2n-1} \|^2 + \| u_{n+n_2,2} - g_{2n} \|^2 \right) \leq \text{const} \sum_{n=1}^{\infty} \varepsilon_n^2 < +\infty. \quad (4.9)
\]
One can easily see that \( n_1 + n_2 \) root functions of \( L \) and one function in the system (4.6) are absent in (4.9). Let \( n_1 + n_2 > 1 \). By (4.9), the system \( S \) generated by all functions excluding \( n_1 + n_2 - 1 \) functions in the system (4.2) is quadratically close to the system (4.6). Since (4.6) is a Riesz basis in \( L_2(0, 1) \), then \( S \) is also a Riesz basis in \( L_2(0, 1) \) \([10]\).

This contradicts the basicity of the system (4.2). Similarly, let \( n_1 = n_2 = 0 \). Since (4.2) forms a Riesz basis in \( L_2(0, 1) \), then again by (4.9), the system \( \{g_k(x)\}_{k=0}^{\infty} \) is a Riesz basis in \( L_2(0, 1) \). Obviously, the latter contradicts the basicity of \( \{g_k(x)\}_{k=0}^{\infty} \) in \( L_2(0, 1) \). All other cases can be checked in a similar method.

Hence, the equality \( n_1 + n_2 = 1 \) is valid. So, we can assume that \( n_1 = 0, n_2 = 1 - \sigma \) without loss of generality. Then, we obtain

\[
\begin{align*}
  u_{n,1}(x) &= \sqrt{2} \sin(2n - \sigma) \pi x + O(\varepsilon_n), \\
  u_{n+1-\sigma,2}(x) &= \sqrt{2} \cos(2n - \sigma) \pi x + O(\varepsilon_n), \\
  v_{n,1}(x) &= \sqrt{2} \sin(2n - \sigma) \pi x + O(\varepsilon_n), \\
  v_{n+1-\sigma,2}(x) &= \sqrt{2} \cos(2n - \sigma) \pi x + O(\varepsilon_n).
\end{align*}
\]

by (1.6) and (4.5).

Now, we show that the root functions of \( L \) form a basis in the Lebesgue space \( L_p(0, 1) \) when \( q(x) \in W^1_2(0, 1) \), where \( 1 < p < \infty, p \neq 2 \). We prove the basicity in \( L_p(0, 1) \) in the case \( \sigma = 0 \). In the case \( \sigma = 1 \), the proof is similar. Since the function \( q(x) \) is in the space \( W^1_2(0, 1) \), then it is differentiable and its derivative is integrable. So, we get

\[ \varepsilon_n = O\left(n^{-1}\right) \]

by using (1.3). Thus, the formulae (4.10) turn into

\[
\begin{align*}
  u_{n,1}(x) &= \sqrt{2} \sin(2n - \sigma) \pi x + O(n^{-1}), \\
  u_{n+1-\sigma,2}(x) &= \sqrt{2} \cos(2n - \sigma) \pi x + O(n^{-1}), \\
  v_{n,1}(x) &= \sqrt{2} \sin(2n - \sigma) \pi x + O(n^{-1}), \\
  v_{n+1-\sigma,2}(x) &= \sqrt{2} \cos(2n - \sigma) \pi x + O(n^{-1}).
\end{align*}
\]

For each \( p \in (1, \infty) \), (4.6) is a basis in \( L_p(0, 1) \) \([1, \text{Chapter VIII, §20, Theorem 2}]\). Then, there exists \( M_p > 0 \) such that the inequality

\[
\left\| \sum_{n=0}^{N} (f, g_n) g_n \right\|_p \leq M_p \|f\|_p, \quad N = 1, 2, \ldots,
\]

holds for each function \( f(x) \in L_p(0, 1) \), where \( \|\cdot\|_p \) is the norm of the normed space \( L_p(0, 1) \) \([13, \text{Chapter I, §4, Theorem 6}]\). Let \( p \in (1, 2) \). Since (4.2) is a complete system in \( L_2(0, 1) \), then it is also complete in \( L_p(0, 1) \). Besides, one can easily see that the inequality

\[ \|\langle f, v_{n,j} \rangle u_{n,j}\|_p \leq \text{const} \|f\|_p, \]

where \( j = 1, 2 \) and \( n = 1, 2, \ldots \).

By theorem 6 in \([13, \text{Chapter VIII, §4}]\), for the basicity of this system in \( L_p(0, 1) \), we must prove that there exists a constant \( M > 0 \) such that the inequality

\[
\left\| \sum_{n=1}^{m} (f, v_{n,j}) u_{n,j} \right\|_p \leq M \|f\|_p, \quad m = 1, 2, \ldots,
\]

holds for \( f(x) \in L_p(0, 1) \). Instead of the above inequality, it is enough to prove the following

\[
J_m(f) = \left\| \sum_{n=1}^{m} \left( (f, v_{n,1}) u_{n,1} + (f, v_{n+1,2}) u_{n+1,2} \right) \right\|_p \leq M' \|f\|_p, \quad (4.13)
\]

where \( M' \) is a positive constant and \( m = 1, 2, \ldots \).
By (4.6) and (4.11), we have
\[ J_m(f) \leq J_{m,1}(f) + J_{m,2}(f) + J_{m,3}(f) + J_{m,4}(f), \]
where
\[ J_{m,1}(f) = \left\| \sum_{n=1}^{2m} (f, g_n) g_n \right\|_p, \quad J_{m,2}(f) = \left\| \sum_{n=1}^{2m} (f, O(n^{-1})) g_n \right\|_p, \]
\[ J_{m,3}(f) = \left\| \sum_{n=1}^{2m} (f, O(n^{-1})) g_n \right\|_p, \quad J_{m,4}(f) = \left\| \sum_{n=1}^{2m} (f, O(n^{-1})) O(n^{-1}) g_n \right\|_p. \]

By (4.12),
\[ J_{m,1}(f) \leq \text{const} \|f\|_p. \]

By Theorem 2.8 (Riesz theorem) [34, Chapter XII, §2], the relations
\[ J_{m,2}(f) \leq \text{const} \sum_{n=1}^{2m} |(f, g_n)| n^{-1} \leq \]
\[ \leq \text{const} \left( \sum_{n=1}^{2m} |(f, g_n)|^q \right)^{1/q} \left( \sum_{n=1}^{2m} n^{-p} \right)^{1/p} \leq \text{const} \|f\|_p, \]
holds, where \( 1/p + 1/q = 1 \). Moreover,
\[ J_{m,3}(f) \leq \left\| \sum_{n=1}^{2m} (f, O(n^{-1})) g_n \right\|_2 = \left( \sum_{n=1}^{2m} \left| (f, O(n^{-1})) \right|^2 \right)^{1/2} \leq \]
\[ \text{const} \|f\|_1 \left( \sum_{n=1}^{2m} n^{-2} \right)^{1/2} \leq \text{const} \|f\|_p. \]

Further,
\[ J_{m,4} \leq \text{const} \|f\|_1 \sum_{n=1}^{2m} n^{-2} \leq \text{const} \|f\|_p. \]

The inequalities (4.14)-(4.18) prove the inequality (4.13). The basicity of (4.2) in \( L^p(0,1) \) is obtained when \( 1 < p < 2 \).

Assume that the relations \( 2 < p < \infty \) and \( 1/p + 1/q = 1 \) hold. Then, \( 1 < q < 2 \) and the biorthogonal system (4.1) is the root functions of the adjoint operator \( L^* \). Above, we show that the system of root functions of such operator is a basis of \( L^q(0,1) \). So, the system (4.2) being biorthogonal system of (4.1) is a basis in \( L^p(0,1) \).

**References**


