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# Some Results on the Generalized Mellin Transforms and Applications

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**Abstract**

This paper discusses the generalized Mellin transforms and their properties with examples and applications to integral and partial differential equations. Several simple lemmas and theorems dealing with general properties of the generalized Mellin transform are proved. The main focus of this paper is to develop the method of the generalized Mellin transform to solve partial differential equations and integral equations in applied mathematics.

***Keywords:*** *Fourier transforms, generalized Mellin transforms, integral and partial differential equations, Mellin transforms.*
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## 1. Introduction

We derive the generalized Mellin transform and its inverse from the complex Fourier transform and its inverse [1][2][3], which are defined respectively by

$$\mathscr{F}\{g(\xi);k\}=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}e^{-ik\xi}g(\xi)\,d\xi=G(k) \tag{1.1}$$

$$\mathscr{F}^{-1}\{G(k);\xi\}=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}e^{ik\xi}G(k)\,dk=g(\xi). \tag{1.2}$$

Making the changes of variables $x^n=e^{\xi}$ and $ik=c-p$, where $Re(x^n)>0$ and $c$ is a constant, in (1.1) and (1.2), we get

$$G(ip-ic)=\frac{1}{\sqrt{2\pi}}\int_{0}^{\infty}(x^n)^{p-c}g(\log x^n)\frac{n}{x}dx \tag{1.3}$$

$$g(\log x^n)=\frac{1}{\sqrt{2\pi}}\int_{c-i\infty}^{c+i\infty}x^{nc}x^{-np}G(ip-ic)\,dp. \tag{1.4}$$

Writing $f(x)=\frac{n}{\sqrt{2\pi}}x^{-nc}g(\log x^n)$ and $\overline{f_n}(p)=G(ip-ic)$ we define generalized Mellin transform of the function $f(x)$[4] and the inverse generalized Mellin transform as follows:

$$\mathscr{M}_n\{f(x);p\}=\overline{f_n}(p)=\int_{0}^{\infty}x^{np-1}f(x)\,dx \tag{1.5}$$

$$\mathscr{M}_n^{-1}\{\overline{f_n}(p);x\}=f(x)=\frac{n}{2\pi i}\int_{c-i\infty}^{c+i\infty}x^{-np}\overline{f_n}(p)\,dp \tag{1.6}$$

where $f(x)$ is a real valued function defined on $(0,\infty)$ and the generalized Mellin transform variable $p$ is a complex number. Obviously, $\mathscr{M}_n$ and $\mathscr{M}_n^{-1}$ are linear integral operators.

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### 1.1. Basic operational properties of Mellin transforms

**Lemma 1.1.** *If $\mathscr{M}_n\{f(x);p\}=\overline{f_n}(p)$, then the following operational properties hold true;*

*(i) Shifting Property:* $\mathscr{M}_n\{x^{an}f(x);p\}=\overline{f_n}(a+p)$, $a>0$
*(ii) Scaling Property:* $\mathscr{M}_n\{f(ax);p\}=a^{-np}\overline{f_n}(p)$, $a>0$
*(iii)* $\mathscr{M}_n\{f(x^{an});p\}=\frac{1}{an}\overline{f_n}\left(\frac{p}{an}\right)$
*(iv)* $\mathscr{M}_n\{x^{-n}f(x^{-n});p\}=\overline{f_n}\left(\frac{1}{n}-\frac{p}{n}\right)$
*(v)* $\mathscr{M}_n\left\{(n\log x)^k f(x);p\right\}=\frac{d^k}{dp^k}\overline{f_n}(p)$, $k\in\mathbb{N}$

*Proof.* Shifting property given in (i) is seen by directly the definition of the $\mathscr{M}_n$-transform (1.5). The identities given in (ii), (iii) and (iv), respectively, are obtained by the definition of the generalized Mellin transform (1.5) and substituting $ax=t$, $x^{na}=t$ and $x^{-n}=t$, respectively. The relation given in (v) can easily be proved by using the result

$$\frac{d}{dp}\left(x^{np-1}\right)=n(\log x)\left(x^{np-1}\right),\ Re(x)>0\,.$$

□

**Lemma 1.2.** *If $\mathscr{M}_n\{f(x);p\}=\overline{f_n}(p)$, then the following operational properties hold true;*
***(i)****Generalized Mellin transforms of derivatives :*

$$\mathscr{M}_n\{f'(x);p\}=-(np-1)\overline{f_n}\left(p-\frac{1}{n}\right) \tag{1.7}$$

*where $\left[x^{np-1}f(x)\right]$ vanishes at $x\to 0$ and $x\to\infty$.*
*More generally, for $r=0,1,2,...m-1$, $Re(p)>\frac{m}{n}$,*

$$\mathscr{M}_n\left\{f^{(m)}(x);p\right\}=(-1)^m\frac{\Gamma(np)}{\Gamma(np-m)}\overline{f_n}\left(p-\frac{m}{n}\right) \tag{1.8}$$

*where $\left[x^{np-r-1}f^{(r)}(x)\right]$ vanishes as $x\to 0$ and $x\to\infty$.*
***(ii)*** *We have*

$$\mathscr{M}_n\{xf'(x);p\}=-np\overline{f_n}(p) \tag{1.9}$$

*where $[x^{np}f(x)]$ vanishes at $x=0$ and as $x\to\infty$.*
*More generally, we have*

$$\mathscr{M}_n\left\{x^m f^{(m)}(x);p\right\}=(-1)^m\frac{\Gamma(np+m)}{\Gamma(np)}\overline{f_n}(p) \tag{1.10}$$

*where $\left[x^{np+1}f^{(r)}(x)\right]$ vanishes at $x=0$ and as $x\to\infty$, for $r=0,1,2,...,m-1$.*
***(iii)*** *Generalized Mellin transforms of differential operators*
*If $\mathscr{M}_n\{f(x);p\}=\overline{f_n}(p)$, then*

$$\mathscr{M}_n\left\{\left(x\frac{d}{dx}\right)^2 f(x);p\right\}=\mathscr{M}_n\left\{x^2f''(x)+xf'(x);p\right\}=(np)^2\overline{f_n}(p)\,. \tag{1.11}$$

***(iv)*** *Generalized Mellin transforms of integrals*

$$\mathscr{M}_n\left\{\int_0^x f(t)\,dt;p\right\}=-\frac{1}{np}\overline{f_n}\left(p+\frac{1}{n}\right) \tag{1.12}$$

*and more generally,*

$$\mathscr{M}_n\{I_m f(x);p\}=(-1)^m\frac{\Gamma(np)}{\Gamma(np+m)}\overline{f_n}\left(p+\frac{m}{n}\right), \tag{1.13}$$

*where $I_m[f(x)]=\int_0^x\int_0^x...\int_0^x f(t)\,dt$. Setting $F(x)=I_m[f(x)]$, we get $F^m(x)=f(x)$.*

*Proof.* i)The relation (1.7) can be proved by using the definition of the $\mathscr{M}_n$-transform and integration by parts and substituting $x^{np-1}=u$. (1.8) can be proven by using the mathematical induction principle.
ii) Using the definition (1.5) and changing the variable of the integration from $x$ to $u$ where $x^{np}=u$, we get

$$\mathscr{M}_n\{xf'(x);p\}\ =\ -np\int_0^\infty x^{np-1}f(x)\,dx=-np\overline{f_n}(p)\,.$$

The relation (1.10) can be proved by using mathematical induction principle. We assume that the following holds true

$$\mathscr{M}_n\left\{x^{m-1}f^{(m-1)}(x);p\right\}=(-1)^{m-1}\frac{\Gamma(np+m-1)}{\Gamma(np)}\overline{f_n}(p)\,. \tag{1.14}$$

Using the definition of the $\mathscr{M}_n$-transform, the relation (1.14) and substituting $x^{np+m-1}=u$, we obtain the relation (1.10).

iii) Using the relation (1.10), we have

$$\mathscr{M}_n\left\{\left(x\frac{d}{dx}\right)f(x);p\right\}=-np\overline{f_n}(p), \tag{1.15}$$

$$\mathscr{M}_n\left\{\left(x^2\frac{d^2}{dx^2}\right)f(x);p\right\}=np(np+1)\overline{f_n}(p). \tag{1.16}$$

Using the definition (1.5) and its linearity, we obtain the relation (1.11).
iv) We take $F(x)=\int_0^x f(t)\,dt$ so that $F'(x)=f(x)$ with $F(0)=0$. Application of (1.7) with $F(x)$ as defined

$$\mathscr{M}_n\left\{f(x)=F'(x);p\right\}=-(np-1)\mathscr{M}_n\left\{F(x);p-\frac{1}{n}\right\} \tag{1.17}$$

which is replacing $p$ by $p+\frac{1}{n}$, we arrive at the relation (1.12).
Now we assume that $F(x)=I_m[f(x)]$ and $F^m(x)=f(x)$. Using the relation (1.8) and replacing $p$ by $p+\frac{m}{n}$, we obtain the relation (1.13) given in (iv). □

## 1.2. Convolution Type Theorems

**Theorem 1.3.** *If $\mathscr{M}_n\{f(x);p\}=\overline{f_n}(p)$ and $\mathscr{M}_n\{g(x);p\}=\overline{g_n}(p)$, then the following relations hold:*

$$\mathscr{M}_n\{f(x)*g(x);p\}=\mathscr{M}_n\left\{\int_0^\infty f(\xi)g\left(\frac{x}{\xi}\right)\frac{d\xi}{\xi};p\right\}=\overline{f_n}(p)\overline{g_n}(p), \tag{1.18}$$

$$\mathscr{M}_n\{f(x)\circ g(x);p\}=\mathscr{M}_n\left\{\int_0^\infty f(x\xi)g(\xi)\,d\xi;p\right\}=\overline{f_n}(p)\overline{g_n}\left(\frac{1}{n}-p\right). \tag{1.19}$$

*Proof.* Using the definition of the $\mathscr{M}_n$-transform, we have

$$\begin{aligned}\mathscr{M}_n\{f(x)*g(x);p\} &= \mathscr{M}_n\left\{\int_0^\infty f(\xi)g\left(\frac{x}{\xi}\right)\frac{d\xi}{\xi};p\right\} \\ &= \int_0^\infty f(\xi)\frac{d\xi}{\xi}\left[\int_0^\infty x^{np-1}g\left(\frac{x}{\xi}\right)\right]dx,\end{aligned} \tag{1.20}$$

substituting $\frac{x}{\xi}=\eta$, we obtain

$$\begin{aligned}\mathscr{M}_n\{f(x)*g(x);p\} &= \int_0^\infty f(\xi)\,d\xi\left[\int_0^\infty \xi^{np-1}\eta^{np-1}g(\eta)\right]d\eta \\ &= \overline{f_n}(p)\overline{g_n}(p).\end{aligned} \tag{1.21}$$

Similarly, we have

$$\begin{aligned}\mathscr{M}_n\{f(x)\circ g(x);p\} &= \mathscr{M}_n\left\{\int_0^\infty f(x\xi)g(\xi)\,d\xi;p\right\} \\ &= \int_0^\infty g(\xi)\,d\xi\int_0^\infty x^{np-1}f(x\xi)\,dx.\end{aligned} \tag{1.22}$$

Making the change of variable $x\xi=\eta$, we get

$$\begin{aligned}\mathscr{M}_n\{f(x)\circ g(x);p\} &= \int_0^\infty g(\xi)\,d\xi\int_0^\infty \eta^{np-1}\xi^{(1-np)-1}f(\eta)\,d\eta \\ &= \overline{g_n}\left(\frac{1}{n}-p\right)\overline{f_n}(p). \qquad \square\end{aligned} \tag{1.23}$$

Note that, in this case, the operation $\circ$ is not commutative. Clearly, putting $x=s$, we find

$$\mathscr{M}_n^{-1}\left\{\overline{f_n}\left(\frac{1}{n}-p\right)\overline{g_n}(p);t\right\}=\int_0^\infty g(st)f(t)\,dt. \tag{1.24}$$

Substituting $g(t)=e^{-t}$ and $\overline{g}(p)=\Gamma(p)$, we get the Laplace transform of $f(t)$ as

$$\mathscr{M}_n^{-1}\left\{\overline{f_n}\left(\frac{1}{n}-p\right)\Gamma(p);t\right\}=\int_0^\infty e^{-t}f(t)\,dt=\mathfrak{L}\{f(t);s\}. \tag{1.25}$$

### 1.3. Parseval's Type Property

**Theorem 1.4.** *If $\mathscr{M}_n\{f(x);p\} = \overline{f_n}(p)$ and $\mathscr{M}_n\{g(x);p\} = \overline{g_n}(p)$, then the following relation holds,*

$$\mathscr{M}_n\{f(x)g(x);p\} = \int_0^\infty x^{np-1} f(x)g(x)\,dx \frac{n}{2\pi i}\int_{c-i\infty}^{c+i\infty} \overline{f_n}(s)\,\overline{g_n}(p-s)\,ds.$$

*In particular, when $p=1$, we obtain the Parseval formula for the generalized Mellin transform,*

$$\int_0^\infty x^{n-1} f(x)g(x)\,dx = \frac{n}{2\pi i}\int_{c-i\infty}^{c+i\infty} \overline{f_n}(s)\,\overline{g_n}(1-s)\,ds. \tag{1.26}$$

*Proof.* Using the definition of the $\mathscr{M}_n$-transform and $\mathscr{M}_n^{-1}$-transform, changing the order of integration, we have

$$\begin{aligned}\mathscr{M}_n\{f(x)g(x);p\} &= \frac{n}{2\pi i}\int_0^\infty x^{np-1} g(x)\,dx \int_{c-i\infty}^{c+i\infty} x^{-ns}\overline{f_n}(s)\,ds \\ &= \frac{n}{2\pi i}\int_{c-i\infty}^{c+i\infty} x^{-ns}\overline{f_n}(s)\,ds \int_0^\infty x^{n(p-s)-1} g(x)\,dx \\ &= \frac{n}{2\pi i}\int_{c-i\infty}^{c+i\infty} \overline{f_n}(s)\,\overline{g_n}(p-s)\,ds.\end{aligned}$$

In particular, when $p=1$, the above relation becomes (1.26) as follows,

$$\mathscr{M}_n\{f(x)g(x);1\} = \int_0^\infty x^{n-1} f(x)g(x)\,dx = \frac{n}{2\pi i}\int_{c-i\infty}^{c+i\infty} \overline{f_n}(s)\,\overline{g_n}(1-s)\,ds. \quad \square$$

## 2. Examples

We shall illustrate the above results by several examples like in [5].

**Example 2.1.** *We show the followings*

*(i)* $\mathscr{M}_n\{e^{-mx};p\} = \frac{1}{m^{np}}\Gamma(np)$,

*(ii)* $\mathscr{M}_n\left\{e^{-m|x|};p\right\} = \frac{1}{m^{np}}\Gamma(np)$, *where $Re(np) > 0$.*

**Demonstration:** Since the relation given in (ii) is seen similarly, we only give the proof of the relation given in (i). Using the definition of the $\mathscr{M}_n$-transform and making the change of variable $mx = t$, we get

$$\mathscr{M}_n\{e^{-mx};p\} = \overline{f_n}(p) = \int_0^\infty x^{np-1}e^{-mx}dx = \frac{1}{m^{np}}\int_0^\infty t^{np-1}e^{-t}dt = \frac{1}{m^{np}}\Gamma(np).$$

**Example 2.2.** *For the Beta function $B(p,q)$ (see [7]), we obtain*

$$\mathscr{M}_n\left\{\frac{1}{x+1};p\right\} = B(np, 1-np). \tag{2.1}$$

**Demonstration:** Using the definition (1.5) and making the change of variable from $x$ to $t$, where $x = \frac{t}{1-t}$, we have

$$\begin{aligned}\mathscr{M}_n\left\{\frac{1}{x+1};p\right\} &= \overline{f_n}(p) = \int_0^1 t^{np-1}(1-t)^{(1-np)-1}dt \\ &= B(np, 1-np) = \Gamma(np)\Gamma(1-np).\end{aligned}$$

**Example 2.3.** *We show*

$$\mathscr{M}_n\left\{\frac{1}{e^x-1};p\right\} = \Gamma(np)\,\zeta(np) \tag{2.2}$$

*where $Re(np) > 0$, $\zeta(p)$ is the Riemann-Zeta function. It is defined by $\zeta(p) = \sum_{n=1}^\infty \frac{1}{n^p}$, $Re(p) > 0$.*

**Demonstration:** By the relation (i) of Example 2.1, the definitions of the Gamma and the Riemann-Zeta functions, and the equality $\sum_{m=1}^\infty e^{-mx} = \frac{1}{e^x-1}$, we get

$$\overline{f_n}(p) = \int_0^\infty x^{np-1}\frac{1}{e^x-1}dx = \sum_{m=1}^\infty \int_0^\infty x^{np-1}e^{-mx}dx = \sum_{m=1}^\infty \frac{\Gamma(np)}{m^{np}} = \Gamma(np)\,\zeta(np).$$

**Example 2.4.** *We show for $Re(np) > 1$,*

$$\mathscr{M}_n\left\{\frac{2}{e^{2x}-1};p\right\} = 2^{1-np}\Gamma(np)\,\zeta(np). \tag{2.3}$$

**Demonstration:** Using the definition (1.5), the relation (i) given in Example 2.1 and the identity $\frac{1}{e^{2x}-1}=\sum_{m=1}^{\infty}e^{-2mx}$, we have

$$\begin{aligned}\mathscr{M}_n\left\{\frac{2}{e^{2x}-1};p\right\} &= 2\int_0^{\infty}x^{np-1}\frac{1}{e^{2x}-1}dx=2\sum_{m=1}^{\infty}\int_0^{\infty}x^{np-1}e^{-2mx}dx\\ &= 2\sum_{m=1}^{\infty}\frac{\Gamma(np)}{(2m)^{np}}=2^{1-np}\Gamma(np)\zeta(np).\end{aligned}$$

**Example 2.5.** *We show for $Re(np)>1$,*

$$\mathscr{M}_n\left\{\frac{1}{e^{x}+1};p\right\}=\left(1-2^{1-np}\right)\Gamma(np)\zeta(np). \tag{2.4}$$

**Demostration:** Using the relations (2.2) and (2.3), we have

$$\begin{aligned}\mathscr{M}_n\left\{\frac{1}{e^{x}+1};p\right\} &= -\mathscr{M}_n\left\{\frac{1}{e^{2x}-1};p\right\}+\mathscr{M}_n\left\{\frac{1}{e^{x}-1};p\right\}\\ &= \left(1-2^{1-np}\right)\Gamma(np)\zeta(np).\end{aligned}$$

**Example 2.6.** *For the Beta function $B(p,q)$, we show*

$$\mathscr{M}_n\left\{(1+x)^{-m};p\right\}=B(np,m-np) \tag{2.5}$$

**Demonstration:** By definition (1.5) and making the change of variable
$x=\frac{t}{1-t}$, we find

$$\mathscr{M}_n\left\{(1+x)^{-m};p\right\}=B(np,m-np)=\frac{\Gamma(np)\Gamma(m-np)}{\Gamma(m)}.$$

Hence, we get $\mathscr{M}_n^{-1}\left\{\Gamma(np)\Gamma(m-np);p\right\}=\frac{\Gamma(m)}{(1+x)^m}$.

**Example 2.7.** *We show*

$$\mathscr{M}_n\left\{\left(1-\frac{|x|}{a}\right)H\left(1-\frac{|x|}{a}\right);p\right\}=a^{np}\frac{1}{np(np+1)} \tag{2.6}$$

*where $Re(np)>0, Re(a)>0$ and $H(x)$ denotes the Heaviside Function (see [2]).*

**Demonstration:** Using the definition (1.5) and the definition of Heaviside function we arrive at the relation (2.6).

**Example 2.8.** *We show*

$$\mathscr{M}_n\{\cos(kx);p\}=\frac{\Gamma(np)}{k^{np}}\cos\left(np\frac{\pi}{2}\right), \tag{2.7}$$

$$\mathscr{M}_n\{\sin(kx);p\}=\frac{\Gamma(np)}{k^{np}}\sin\left(np\frac{\pi}{2}\right). \tag{2.8}$$

**Demonstration:** Using the relation (i) given in Example 2.1 and the linearity of the generalized Mellin transform, we get

$$\begin{aligned}\mathscr{M}_n\left\{e^{-ikx};p\right\} &= \mathscr{M}_n\{\cos(kx);p\}-i\mathscr{M}_n\{\sin(kx);p\}\\ &= \frac{\Gamma(np)}{k^{np}}\cos\left(np\frac{\pi}{2}\right)-i\frac{\Gamma(np)}{k^{np}}\sin\left(np\frac{\pi}{2}\right)\end{aligned}$$

Using this relation, we arrive at the relations (2.7) and (2.8).
These results can be used to calculate the Fourier cosine and Fourier sine transforms of $x^{np-1}$. Result (2.7) can be written as

$$\int_0^{\infty}x^{np-1}\cos(kx)\,dx=\frac{\Gamma(np)}{k^{np}}\cos\left(np\frac{\pi}{2}\right),$$

or, equivalently,

$$\begin{aligned}\mathscr{F}_c\left\{\sqrt{\frac{\pi}{2}}x^{np-1};p\right\} &= \frac{\Gamma(np)}{k^{np}}\cos\left(np\frac{\pi}{2}\right),\\ \mathscr{F}_s\left\{\sqrt{\frac{\pi}{2}}x^{np-1};p\right\} &= \frac{\Gamma(np)}{k^{np}}\sin\left(np\frac{\pi}{2}\right).\end{aligned} \tag{2.9}$$

### 2.1. Applications of the Generalized Mellin Transform

**Example 2.9.** *We solve the following boundary value problem for A=constant*

$$\begin{aligned} x^2u_{xx}+xu_x+u_{yy} &= 0,\ 0<x<\infty,\ 0<y<1, && (2.10)\\ u(x,0) &= 0, && (2.11)\\ u(x,1) &= \begin{cases} A\ ,\ 0\leq x\leq 1\\ 0, \quad x>1\end{cases} \end{aligned}$$

**Demonstration:** Applying the generalized Mellin transform of $u(x,y)$ with respect to $x$ defined by

$$\bar{u}(p,y)=\int_0^{\infty}x^{p-1}u(x,y)dx, \tag{2.12}$$

we reduce the given system into the form

$$\frac{d^2\overline{u_n}(p,y)}{dy^2}+(np)^2\overline{u_n}(p,y)=0,\quad 0<y<1, \tag{2.13}$$

$$\bar{u}(p,0)=0,\quad \overline{u_n}(p,1)=\int_0^{\infty}x^{p-1}Adx=\frac{A}{np}. \tag{2.14}$$

The solution of the transformed problem (2.13)-(2.14) is

$$\overline{u_n}(p,y)=\frac{A}{np\sin(np)}\sin(npy),\quad 0<Re(np)<1. \tag{2.15}$$

The inverse generalized Mellin transform gives,

$$u_n(x,y)=\frac{An}{2\pi i}\int_{c-i\infty}^{c+i\infty}\frac{x^{-np}}{np}\frac{\sin(npy)}{\sin(np)}dp \tag{2.16}$$

where $\bar{u}(p,y)$ is analytic in the vertical strip $0<Re(np)=c<\pi$. The integrand of (2.16) has simple poles at $p=\frac{k\pi}{n}, k\in\mathbb{N}$ which as lie inside a semicircular contour in the right half plane. Evaluating of (2.16) by theory of residues gives the solution for $x>1$ as

$$u(x,y)=\frac{A}{\pi}\sum_{k=1}^{\infty}(-1)^k\frac{x^{-kp}}{k}\sin(k\pi y).$$

**Example 2.10.** *We solve the following integral equation*

$$\int_0^{\infty}f(\xi)K(x\xi)d\xi=g(x), x>0 \tag{2.17}$$

**Demonstration:** Application of the generalized Mellin transform with respect to $x$ to given equation combined with the relation (1.19) gives $\overline{f_n}\left(\frac{1}{n}-p\right)\overline{k_n}(p)=\overline{g_n}(p)$. Replacing $p$ by $\frac{1}{n}-p$, we get $\overline{f_n}(p)\overline{k_n}\left(\frac{1}{n}-p\right)=\overline{g_n}\left(\frac{1}{n}-p\right)$. Thus the solution $\overline{f_n}(p)=\overline{g_n}\left(\frac{1}{n}-p\right)\overline{h_n}(p)$ is obtained, where $\overline{h_n}(p)=\dfrac{1}{\overline{k_n}\left(\frac{1}{n}-p\right)}$.
The inverse generalized Mellin transform combined with the relation (1.19) leads

$$f(x)=\mathscr{M}_n^{-1}\left\{\overline{g_n}\left(\frac{1}{n}-p\right)\overline{h_n}(p);x\right\}=\int_0^{\infty}h(x\xi)g(\xi)d\xi$$

provided $h(x)=\mathscr{M}_n^{-1}\left\{\overline{h_n}(p);x\right\}$ exists. Thus, the problem is formally solved. If, in particular, $\overline{h_n}(p)=\overline{k_n}(p)$, then the solution becomes $f(x)=\int_0^{\infty}g(\xi)k(x\xi)d\xi$ where $\overline{k_n}(p)\overline{k_n}\left(\frac{1}{n}-p\right)=1$ and $\overline{h_n}(p)\overline{k_n}\left(\frac{1}{n}-p\right)=1$.

**Example 2.11.** *We solve the following integral equation*

$$\int_0^{\infty}f(\xi)g\left(\frac{x}{\xi}\right)\frac{d\xi}{\xi}=h(x) \tag{2.18}$$

*where $f(x)$ is unknown, $g(x)$ and $h(x)$ are given functions.*

**Demonstration:** Applying the generalized Mellin Transform to given integral equation with respect to $x$, we obtain $\overline{f_n}(p)\overline{g_n}(p)=\overline{h_n}(p)$. Then, applying the inverse generalized Mellin Transform to this equality, we obtain the formal solution as $f(x)=\int_0^{\infty}h(\xi)K\left(\frac{x}{\xi}\right)\frac{d\xi}{\xi}$ by the convolution property (1.18).

**Example 2.12.** *(Potential in an Infinite Wedge) Find the potential $\phi(r,\theta)$ that satisfies the Laplace equation $r^2\phi_{rr}+r\phi_r+\phi_{\theta\theta}=0$ in an infinite wedge $0<r<\infty,\ -\alpha<\theta<\alpha$ with the boundary conditions*

$$\begin{aligned} &\phi(r,\alpha)=\phi(r,-\alpha)=H(a-r) \qquad 0\leq r<\infty\\ &\phi(r,\theta)\to 0 \text{ as } r\to\infty \text{ for all } \theta \text{ in } -\alpha<\theta<\alpha. \end{aligned} \tag{2.19}$$

**Demonstration:** We apply the generalized Mellin transform of the potential $\phi(r,\theta)$ defined by $\mathscr{M}_n\{\phi(r,\theta);p\} = \overline{\phi_n}(p,\theta) = \int_0^\infty r^{np-1}\phi(r,\theta)\,dr$ of the given differential system, then we have

$$(np)(np+1)\overline{\phi_n}(p,\theta) + (-np)\overline{\phi_n}(p,\theta) + \frac{d^2\overline{\phi_n}(p,\theta)}{d\theta^2} = 0 \tag{2.20}$$

$$\begin{array}{l} \frac{d^2\overline{\phi_n}(p,\theta)}{d\theta^2} + (np)^2\overline{\phi_n}(p,\theta) = 0 \\ \overline{\phi_n}(p,\alpha) = \overline{\phi_n}(p,-\alpha) = \frac{a^{np}}{np} \end{array} \tag{2.21}$$

The general solution of the transformed problem (2.21) as follows

$$\overline{\phi_n}(p,\theta) = \frac{a^{np}}{np\cos(np\alpha)}\cos(np\theta). \tag{2.22}$$

Applying the inverse generalized Mellin transform to (2.22), we have

$$\phi(r,\theta) = \frac{n}{2\pi i}\int_{c-i\infty}^{c+i\infty} r^{-np}\frac{a^{np}}{np\cos(np\alpha)}\cos(np\theta)\,dp. \tag{2.23}$$

The integrand of the (2.23) has the simple poles at $np = 0$ and $np = (2k+1)\frac{\pi}{2\alpha}$, $k = 0,1,2,...$ Evaluating (2.23) by theory of residues gives the solution, we get

$$\phi(r,\theta) = n\left\{1 + \frac{1}{\beta}\sum_{k=0}^{\infty}\frac{(-1)^{(k+1)}}{(2k+1)}\left(\frac{a}{r}\right)^{(2k+1)\beta}\cos((2k+1)\beta\theta)\right\} \tag{2.24}$$

where $\beta = \frac{\pi}{2\alpha}$.

**Example 2.13.** *(Potential in an Infinite Wedge) Find the potential $\phi(r,\theta)$ that satisfies the Laplace equation $r^2\phi_{rr} + r\phi_r + \phi_{\theta\theta} = 0$ in an infinite wedge $0 < r < \infty$, $0 < \theta < \alpha$ with the boundary conditions*

$$\begin{array}{c} \phi(r,0) = 0 \quad \phi(r,\alpha) = f(r) \\ \phi(r,\theta) \to 0 \text{ as } r \to \infty \text{ for all } \theta \text{ in } 0 < \theta < \alpha. \end{array} \tag{2.25}$$

**Demonstration:** We apply the generalized Mellin transform of the potential $\phi(r,\theta)$ defined by $\mathscr{M}_n\{\phi(r,\theta);p\} = \overline{\phi_n}(p,\theta) = \int_0^\infty r^{np-1}\phi(r,\theta)\,dr$ of the given diferential system, then we obtain

$$\frac{d^2\overline{\phi_n}(p,\theta)}{d\theta^2} + (np)^2\overline{\phi_n}(p,\theta) = 0, \tag{2.26}$$

$$\overline{\phi_n}(p,0) = 0, \quad \overline{\phi_n}(p,\alpha) = \overline{f_n}(p)$$

The general solution of the transformed problem (2.26) is

$$\overline{\phi_n}(p,\theta) = \frac{\overline{f_n}(p)}{\sin(np\alpha)}\sin(np\theta).$$

The inverse generalized Mellin transform leads to the solution

$$\phi(r,\theta) = \frac{n}{2\pi i}\int_{c-i\infty}^{c+i\infty} r^{-np}\frac{\overline{f_n}(p)}{\sin(np\alpha)}\sin(np\theta)\,dp$$

where $\overline{f_n}(p) = M_n\{f(r);p\} = \int_0^\infty r^{np-1}f(r)\,dr$.

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