Generalized Simpson Type Integral Inequalities

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Abstract

In this paper, we have established some generalized Simpson type inequalities for convex functions. Furthermore, inequalities obtained in special case present a refinement and improvement of previously known results.

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1. Introduction

The following inequality is well known in the literature as Simpson’s inequality.

\textbf{Theorem 1.1.} Let \( f : [a, b] \to \mathbb{R} \) be a four times continuously differentiable mapping on \((a, b)\) and \( \| f^{(4)} \|_\infty = \sup |f^{(4)}(x)| < \infty. \) Then, the following inequality holds:

\[
\left| \frac{1}{3} \left[ \frac{f(a) + f(b)}{2} + 2f\left(\frac{a+b}{2}\right) \right] - \frac{1}{b-a} \int_a^b f(x) \, dx \right| \leq \frac{1}{2880} \| f^{(4)} \|_\infty (b-a)^4.
\]

For recent refinements, counterparts, generalizations and new Simpson’s type inequalities, see ([1]-[21]).

In [2], Dragomir et. al. proved the following some recent developments on Simpson’s inequality for which the remainder is expressed in terms of lower derivatives than the fourth.

\textbf{Theorem 1.2.} Suppose \( f : [a, b] \to \mathbb{R} \) is a differentiable mapping whose derivative is continuous on \((a, b)\) and \( f' \in L[a, b]. \) Then, the following inequality holds,

\[
\left| \frac{1}{3} \left[ \frac{f(a) + f(b)}{2} + 2f\left(\frac{a+b}{2}\right) \right] - \frac{1}{b-a} \int_a^b f(x) \, dx \right| \leq \frac{b-a}{3} \| f' \|_1 \tag{1.1}
\]

holds, where \( \| f' \|_1 = \int_a^b |f'(x)| \, dx. \)

The bound of (1.1) for \( L\)-Lipschitzian mapping was given in [2] by \( \frac{5}{36} L(b-a). \) Also, the following inequality was obtained in [2].

\textbf{Theorem 1.3.} Suppose \( f : [a, b] \to \mathbb{R} \) is an absolutely continuous mapping on \([a, b]\) whose derivative belongs to \( L_p[a, b]. \) Then the following inequality holds,

\[
\left| \frac{1}{3} \left[ \frac{f(a) + f(b)}{2} + 2f\left(\frac{a+b}{2}\right) \right] - \frac{1}{b-a} \int_a^b f(x) \, dx \right| \leq \frac{1}{6} \left[ \frac{2^{p+1} + 1}{3(q+1)} \right] \frac{1}{b-a} \| f' \|_p
\]

where \( \frac{1}{p} + \frac{1}{q} = 1. \)

The aim of this paper is to establish new generalization of Simpson’s type inequalities for the class of functions whose derivatives in absolute value are convex. Furthermore, inequalities obtained in special case present a refinement and improvement of previously known results.

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2. Main Results

Let’s begin start the following Lemma which helps us to obtain the main results:

**Lemma 2.1.** Let $f : I \subset \mathbb{R} \to \mathbb{R}$ be an absolutely continuous on $I$ such that $f' \in L_{1}[a, b]$ with $a, b \in I, a < b$. Then the following equality holds:

\[
\frac{(b-w)^{2}}{2(b-a)} \int_{0}^{1} \left( \frac{1}{3} - t \right) f' \left( \frac{1+t}{2}w + \frac{1-t}{2}b \right) dt + \frac{(w-a)^{2}}{2(b-a)} \int_{0}^{1} \left( \frac{t}{3} \right) f' \left( \frac{1+t}{2}w + \frac{1-t}{2}a \right) dt = \frac{1}{6} f(w) + \frac{1}{3(b-a)} \left[ (b-w) f \left( \frac{w+b}{2} \right) + (w-a) f \left( \frac{a+w}{2} \right) \right]
\]

where $w = ha + (1-h)b, h \in [0, 1]$.

**Proof.** If we apply integration by parts, we have

\[
I_{1} = \int_{0}^{1} \left( \frac{1}{3} - t \right) f' \left( \frac{1+t}{2}w + \frac{1-t}{2}b \right) dt - \frac{2}{b-w} \int_{0}^{1} \left( \frac{1}{3} - t \right) f' \left( \frac{1+t}{2}w + \frac{1-t}{2}b \right) \left|_{0}^{1} \right. - \frac{1}{b-w} \int_{0}^{1} f' \left( \frac{1+t}{2}w + \frac{1-t}{2}b \right) dt
\]

and similar methods we can show that

\[
I_{2} = \int_{0}^{1} \left( \frac{t}{3} \right) f' \left( \frac{1+t}{2}w + \frac{1-t}{2}a \right) dt - \frac{2}{6(w-a)} f(w) + \frac{2}{3(w-a)} f \left( \frac{a+w}{2} \right) - \frac{2}{(w-a)} \int_{w}^{w} f(u) du.
\]

Therefore, we obtain the following result

\[
\frac{(b-w)^{2}}{2(b-a)} I_{1} + \frac{(w-a)^{2}}{2(b-a)} I_{2} = \frac{b-w}{6(b-a)} f(w) + \frac{b-w}{3(b-a)} f \left( \frac{b+w}{2} \right) - \frac{1}{b-a} \int_{w}^{w} f(u) du
\]

\[
+ \frac{w-a}{6(b-a)} f(w) + \frac{w-a}{3(b-a)} f \left( \frac{a+w}{2} \right) - \frac{1}{b-a} \int_{w}^{w} f(u) du
\]

\[
= \frac{1}{6} f(w) + \frac{1}{3(b-a)} \left[ (b-w) f \left( \frac{w+b}{2} \right) + (w-a) f \left( \frac{a+w}{2} \right) \right]
\]

\[
- \frac{1}{b-a} \int_{w}^{w} f(u) du
\]

which completes the proof. □

**Theorem 2.2.** Let $f : I \subset \mathbb{R} \to \mathbb{R}$ be a differentiable mapping on $I$ such that $f' \in L_{1}[a, b]$ with $a, b \in I, a < b$. If $|f'|$ is a convex function on $[a, b]$ then the following inequality holds:

\[
\left| \frac{1}{6} f(w) + \frac{1}{3(b-a)} \left[ (b-w) f \left( \frac{w+b}{2} \right) + (w-a) f \left( \frac{a+w}{2} \right) \right] - \frac{1}{b-a} \int_{w}^{w} f(u) du \right| \leq \frac{61 \left[ (b-w)^{2} + (w-a)^{2} \right] |f''(w)| + 29 \left[ (b-w)^{2} |f'(b)| + (w-a)^{2} |f'(a)| \right]}{6^{4}(b-a)}
\]

where $w = ha + (1-h)b, h \in [0, 1]$. 


Proof. From Lemma 2.1 and by using the convexity of $|f'|$, we get
\[
\left| \frac{1}{6} f(w) + \frac{1}{3(b-a)} \left[ (b-w) f \left( \frac{w+b}{2} \right) + (w-a) f \left( \frac{a+w}{2} \right) \right] \right| - \frac{1}{b-a} \int_a^b f(x) \, dx
\leq \frac{(b-w)^2}{2(b-a)} \int_0^1 \left| \frac{1}{2} \int_0^t \left[ \frac{1}{2} |f'(w)| + \frac{1}{2} |f'(b)| \right] dt \right|
\]
\[
+ \frac{(w-a)^2}{2(b-a)} \int_0^1 \left| \frac{1}{3} \int_0^t \left[ \frac{1}{3} |f'(w)| + \frac{1}{3} |f'(a)| \right] dt \right|
\]
\[
= \frac{(b-w)^2}{4(b-a)} |f'(w)| \int_0^1 \left| \frac{1}{3} \int_0^t (1+t) dt + \frac{(b-w)^2}{4(b-a)} |f'(b)| \right| \int_0^1 \left| \frac{1}{3} \int_0^t (1-t) dt \right|
\]
\[
+ \frac{(w-a)^2}{4(b-a)} |f'(a)| \int_0^1 \left| \frac{1}{2} \int_0^t (1-t) dt \right|
\]
By simple computation, we have
\[
\int_0^1 \left| \frac{1}{3} \int_0^t (1+t) dt \right| dt = \frac{61}{342}
\] (2.1)
and
\[
\int_0^1 \left| \frac{1}{3} \int_0^t (1-t) dt \right| dt = \frac{29}{342}
\] (2.2)
This completes the proof.

Remark 2.3. If we choose $h = 0$ in Theorem 2.2, then we have $w = b$ and
\[
\left| \frac{1}{6} f(b) + \frac{1}{3} f \left( \frac{a+b}{2} \right) \right| - \frac{1}{b-a} \int_a^b f(x) \, dx
\leq (b-a) \frac{61 |f'(b)| + 29 |f'(a)|}{64}
\] (2.3)
Similarly, if we choose $h = 1$ in Theorem 2.2, then we have $w = a$ and
\[
\left| \frac{1}{6} f(a) + \frac{1}{3} f \left( \frac{a+b}{2} \right) \right| - \frac{1}{b-a} \int_a^b f(x) \, dx
\leq (b-a) \frac{61 |f'(a)| + 29 |f'(b)|}{64}
\] (2.4)
Combining the inequalities (2.3) and (2.4), we have
\[
\left| \frac{1}{6} f(a) + \frac{1}{3} f \left( \frac{a+b}{2} \right) + f(b) \right| - \frac{1}{b-a} \int_a^b f(x) \, dx
\leq \frac{5}{72} (b-a) \left[ |f'(a)| + |f'(b)| \right]
\]
which is given by Sarikaya et al. in \cite{13} (for $s = 1$).

Theorem 2.4. Let $f : I \subset \mathbb{R} \to \mathbb{R}$ be a differentiable mapping on $I$ such that $f' \in L_1 [a, b]$ where $a, b \in I, a < b$. If $|f'|^q$ is a convex function on $[a, b]$, $q > 1$, then the following inequality holds:
\[
\left| \frac{f(w)}{6} + \frac{1}{3(b-a)} \left[ (b-w) f \left( \frac{w+b}{2} \right) + (w-a) f \left( \frac{a+w}{2} \right) \right] \right|
\]
\[
- \frac{1}{b-a} \int_a^b f(x) \, dx
\]
\[
\leq \left( \frac{1}{12} (b-a) \left( \frac{1 + 2p+1}{3(p+1)} \right) \right)^{\frac{1}{p}}
\times \left\{ (b-w)^2 \left[ \frac{3 |f'(w)|^q + |f'(b)|^q}{4} \right]^{\frac{1}{q}} + (w-a)^2 \left[ \frac{3 |f'(w)|^q + |f'(a)|^q}{4} \right]^{\frac{1}{q}} \right\}
\]
where $w = ha + (1-h)b, h \in [0, 1]$. 

Proof. From Lemma 2.1 and by Hölder’s inequality we get
\[
\left| f\left( w \right) + \frac{1}{3(b-a)} \left[ (b-w) f\left( \frac{w+b}{2} \right) + (w-a) f\left( \frac{a+w}{2} \right) \right] \right| - \frac{1}{b-a} \int_{a}^{b} f(x) \, dx \leq \left( \frac{b-w}{2(b-a)} \right)^{\frac{1}{2}} \left| \int_{0}^{\frac{t}{2}} \left| f'\left( \frac{1+t+w+\frac{1-t}{2}b}{2} \right) \right|^q \, dt \right|^{\frac{1}{2}}
\]

By simple computation, we have
\[
\int_{0}^{1} \left| t - \frac{p}{2} \right|^{p} \, dt = \frac{2(1+2^{p+1})}{6^{p+1}(p+1)}.
\]

Since \(|f'|^q\) is a convex on \([a,b]\), we know that for \(t \in [0,1]\)
\[
\int_{0}^{1} \left| f'\left( \frac{1+t+w+\frac{1-t}{2}b}{2} \right) \right|^q \, dt \leq \int_{0}^{1} \left[ \left( 1+\frac{t}{2} \right) |f'(w)|^q + \frac{1-t}{2} |f'(b)|^q \right] \, dt = \frac{3|f'(w)|^q + |f'(b)|^q}{4}
\]
and
\[
\int_{0}^{1} \left| f'\left( \frac{1+t+w+\frac{1-t}{2}a}{2} \right) \right|^q \, dt \leq \int_{0}^{1} \left[ \left( 1+\frac{t}{2} \right) |f'(w)|^q + \frac{1-t}{2} |f'(a)|^q \right] \, dt = \frac{3|f'(w)|^q + |f'(a)|^q}{4}
\]

Using the equations (2.7)-(2.9) in (2.6), then we obtain the (2.5).

\[\text{Remark 2.5.} \quad \text{If we choose} \quad h = 0 \quad \text{in Theorem 2.4, then we have} \quad w = b \quad \text{and}
\]
\[
\left| f\left( b \right) + \frac{1}{3(f\left( a + b \right) - \frac{1}{b-a} \int_{a}^{b} f(x) \, dx \right) \leq \left( \frac{b-a}{12} \right)^{\frac{1}{2}} \left[ |f'(a)|^q + 3|f'(b)|^q \right]^{\frac{1}{4}}
\]

Similarly, if we choose \(h = 1\) in Theorem 2.4, then we have \(w = a\) and
\[
\left| f\left( a \right) + \frac{1}{3(f\left( a + b \right) - \frac{1}{b-a} \int_{a}^{b} f(x) \, dx \right) \leq \left( \frac{b-a}{12} \right)^{\frac{1}{2}} \left[ 3|f'(a)|^q + |f'(b)|^q \right]^{\frac{1}{4}}
\]

Combining the inequalities (2.10) and (2.11), we have
\[
\left| \frac{1}{6} \left( f(a) + 4f\left( \frac{a+b}{2} \right) + f(b) \right) - \frac{1}{b-a} \int_{a}^{b} f(x) \, dx \right| \leq \left( \frac{b-a}{12} \right)^{\frac{1}{2}} \times \left[ \left| f'(a)|^q + 3|f'(b)|^q \right|^{\frac{1}{4}} + \left[ \left| f'(a)|^q + |f'(b)|^q \right|^{\frac{1}{4}} \right] \right]
\]

which is proved by Sarikaya et al. in [13, (for s = 1)].
Theorem 2.6. Let \( f : I \subset \mathbb{R} \to \mathbb{R} \) be a differentiable mapping on \( I \) such that \( f' \in L_1[a, b] \) where \( a, b \in I, a < b \). If \(|f'|^q \) is a convex function on \([a, b], q \geq 1\), then the following inequality holds:

\[
\left| \frac{f(w)}{6} + \frac{1}{3(b-a)} \left[ (b-w) f\left( \frac{w+b}{2} \right) + (w-a) f\left( \frac{a+w}{2} \right) \right] \right| - \frac{1}{b-a} \frac{b-w}{2} f(x) dx \leq \frac{1}{2(b-a)} \left( \frac{5}{36} \right)^{\frac{1}{q}} \left[ (b-w)^2 \left( \frac{61|f'(w)|^q + 29|f'(b)|^q}{3^{3/2}} \right)^{\frac{1}{q}} + (w-a)^2 \left( \frac{61|f'(w)|^q + 29|f'(a)|^q}{3^{3/2}} \right)^{\frac{1}{q}} \right] \]

where \( w = ha + (1-h)b, h \in [0, 1] \).

Proof. From Lemma 2.1 and by using power main inequality, we get

\[
\left| \frac{f(w)}{6} + \frac{1}{3(b-a)} \left[ (b-w) f\left( \frac{w+b}{2} \right) + (w-a) f\left( \frac{a+w}{2} \right) \right] \right| - \frac{1}{b-a} \frac{b-w}{2} f(x) dx \leq \frac{(b-w)^2}{2(b-a)} \left( \frac{5}{36} \right)^{\frac{1}{q}} \left( \int_0^1 \left| \int_0^{t/2} f'\left( \frac{1+t}{2}w + 1-tb \right) dx \right| dt \right)^{\frac{1}{q}} + \frac{(w-a)^2}{2(b-a)} \left( \int_0^1 \left| \int_0^{t/2} f'\left( \frac{1+t}{2}w + 1-ta \right) dx \right| dt \right)^{\frac{1}{q}}.
\]

Since \(|f'|^q \) is a convex function on \([a, b]\), by using the equalities (2.1) and (2.2), we obtain

\[
\left| \frac{f(w)}{6} + \frac{1}{3(b-a)} \left[ (b-w) f\left( \frac{w+b}{2} \right) + (w-a) f\left( \frac{a+w}{2} \right) \right] \right| - \frac{1}{b-a} \frac{b-w}{2} f(x) dx \leq \frac{(b-w)^2}{2(b-a)} \left( \frac{5}{36} \right)^{\frac{1}{q}} \times \left( \int_0^1 \left| \int_0^{t/2} f'\left( \frac{1+t}{2}w + 1-tb \right) dx \right| dt \right)^{\frac{1}{q}} + \frac{(w-a)^2}{2(b-a)} \left( \frac{5}{36} \right)^{\frac{1}{q}} \times \left( \int_0^1 \left| \int_0^{t/2} f'\left( \frac{1+t}{2}w + 1-ta \right) dx \right| dt \right)^{\frac{1}{q}}
\]

\[
= \frac{1}{2(b-a)} \left( \frac{5}{36} \right)^{\frac{1}{q}} \left[ (b-w)^2 \left( \frac{61}{3^{3/2}} |f'(w)|^q + \frac{29}{3^{3/2}} |f'(b)|^q \right)^{\frac{1}{q}} + (w-a)^2 \left( \frac{61}{3^{3/2}} |f'(w)|^q + \frac{29}{3^{3/2}} |f'(a)|^q \right)^{\frac{1}{q}} \right]
\]

which completes the proof.

Remark 2.7. If we choose \( h = 0 \) in Theorem 2.6, then we have \( w = b \) and

\[
\left| \frac{f(b)}{6} + \frac{1}{3} f\left( \frac{a+b}{2} \right) - \frac{1}{b-a} \frac{b-w}{2} f(x) dx \right| \leq \frac{b-a}{2} \left( \frac{5}{36} \right)^{\frac{1}{q}} \left( \frac{29}{3^{3/2}} |f'(b)|^q + \frac{61}{3^{3/2}} |f'(b)|^q \right)^{\frac{1}{q}}.
\]
If we choose \( h = 1 \) in Theorem 2.6, then we have \( w = a \) and

\[
\left| \frac{f(a)}{6} + \frac{1}{3} f \left( \frac{a+b}{2} \right) - \frac{1}{b-a} \int_a^b f(x) \, dx \right| \leq \frac{b-a}{2} \left( \frac{5}{36} \right)^{1/4} \left( \frac{61|f''(a)|^q + 29|f''(b)|^q}{3^{4/3}} \right)^{1/4}.
\]

Combining the inequalities (2.12) and (2.13), we have

\[
\frac{1}{6} \left[ f(a) + 4f \left( \frac{a+b}{2} \right) + f(b) \right] - \frac{1}{b-a} \int_a^b f(x) \, dx \leq \frac{b-a}{2} \left( \frac{5}{36} \right)^{1/4} \left[ \left( \frac{29|f''(a)|^q + 61|f''(b)|^q}{3^{4/3}} \right)^{1/4} \right. \\
\left. + \left( \frac{61|f''(a)|^q + 29|f''(b)|^q}{3^{4/3}} \right)^{1/4} \right],
\]

which is given by Sarikaya et al. in [13, (for \( s = 1 \)].

References


