Common Fixed Point Theorems Satisfying Implicit Relations on 2-cone Banach Space with an Application

D. Dhamodharan*, Nihal Taş and R. Krishnakumar

Abstract
In this paper, we discuss the existence and uniqueness of common fixed-point theorems satisfying implicit relations on 2-cone Banach spaces. Modifying obtained new contractive conditions, we also give an application to the fixed-circle problem.

Keywords: common fixed point; 2-cone Banach space; 2-cone normed space; fixed circle.

AMS Subject Classification (2010): Primary: 47H10; Secondary: 54H25.

1. Introduction and preliminaries

In 2007, Huang and Zhang [3] introduced the concept of a cone metric space and proved fixed point theorems for contraction mappings such as:

Any mapping $T$ of a complete cone metric space $X$ into itself that satisfies, for some $0 \leq k < 1$, the inequality

$$d(Tx, Ty) \leq kd(x, y)$$

for all $x, y \in X$ has a unique fixed point.

In [4], Karapınar established some fixed-point theorems in cone Banach space in 2009. Ahmet Şahiner and Tuba Yiğit initiated the concept of a 2-cone Banach space and proved some fixed-point theorems [16]. Krishnakumar and Dhamodharan proved some common fixed-point theorems on contractive modulus in 2-cone Banach space [5].

In this paper, following the idea which was given in [14], we establish some common fixed-point theorems for a self-mapping satisfying implicit relations which are contractive conditions in 2-cone Banach spaces. Now we recall some known definitions and basic facts.

Definition 1.1. [3] Let $E$ be the real Banach space. A subset $P$ of $E$ is called a cone if and only if

1. $P$ is closed, nonempty and $P \neq 0$
2. $ax + by \in P$ for all $x, y \in P$ and nonnegative real numbers $a, b$
3. $P \cap (-P) = \{0\}$

Given a cone $P \subset E$, we define a partial ordering $\leq$ with respect to $P$ by $x \leq y$ if and only if $y - x \in P$. We will write $x < y$ to indicate that $x \leq y$ but $x \neq y$, while $x, y$ will stand for $y - x \in intP$, where $intP$ denotes the interior of $P$. The cone $P$ is called normal if there is a number $K > 0$ such that $0 \leq x \leq y$ implies $\|x\| \leq K\|y\|$ for all $x, y \in E$. The least positive number satisfying the above is called the normal constant.

From now on we suppose that $E$ is a Banach space, $P$ is a cone in $E$ with $intP = \emptyset$ and $\leq$ is partial ordering with respect to $P$.

Received : 24–01–2019, Accepted : 25–03–2019
Example 1.1. Let $K > 1$ be given. Consider the real vector space with

$$E = \left\{ ax + b : a, b \in R; x \in \left[ 1 - \frac{1}{k}, 1 \right] \right\}$$

with supremum norm and the cone

$$P = \{ ax + b : a \geq 0, b \leq 0 \}$$

in $E$. The cone $P$ is regular and so normal.

**Definition 1.2.** [3] Let $X$ be a nonempty set. If the mapping $d : X \times X \to E$ satisfies

1. $d(x, y) > 0$ and $d(x, y) = 0$ if and only if $x = y$ for all $x, y \in X$,
2. $d(x, y) = d(y, x)$ for all $x, y \in X$,
3. $d(x, y) \leq d(x, z) + d(z, y)$ for all $x, y, z \in X$,

then $(X, d)$ is called a cone metric space (CMS).

Example 1.2. [3] Let $E = \mathbb{R}^2$

$$P = \{ (x, y) : x, y \geq 0 \}.$$

$X = \mathbb{R}$ and $d : X \times X \to E$ such that

$$d(x, y) = (|x - y|, \alpha|x - y|),$$

where $\alpha \geq 0$ is a constant. Then $(X, d)$ is a cone metric space.

**Definition 1.3.** [4] Let $X$ be a vector space over $\mathbb{R}$. If the mapping $\| \cdot \|_c : X \to E$ satisfies

1. $\| x \|_c \geq 0$ for all $x \in X$,
2. $\| x \|_c = 0$ if and only if $x = 0$,
3. $\| x + y \|_c \leq \| x \|_c + \| y \|_c$ for all $x, y \in X$,
4. $\| kx \|_c = |k|\| x \|_c$ for all $k \in \mathbb{R}$ and for all $x \in X$,

then $\| \cdot \|_c$ is called a cone norm on $X$, and the pair $(X, \| \cdot \|_c)$ is called a cone normed space (CNS).

**Remark 1.1.** [1] Each cone normed space is cone metric space with metric defined by

$$d(x, y) = \| x - y \|_c.$$
i. the sequence \( \{x_n\} \) converges to \( x \) if and only if \( \|x_n - x\|_c \to 0 \) as \( n \to \infty \),

ii. the sequence \( \{x_n\} \) is Cauchy if and only if \( \|x_n - x_m\|_c \to 0 \) as \( n, m \to \infty \),

iii. the sequence \( \{x_n\} \) converges to \( x \) and the sequence \( \{y_n\} \) converges to \( y \), then \( \|x_n - y_n\|_c \to \|x - y\|_c \).

**Definition 1.7.** [16] Let \( X \) be a linear space over \( \mathbb{R} \) with dimension greater then or equal to 2, \( E \) be Banach space with the norm \( \|\cdot\| \) and \( P \subset E \) be a cone. If the function

\[
\|.,.\| : X \times X \to (E, P, \|\|)
\]

satisfies the following axioms then \( (X, \|.,.\|_c) \) is called a 2-cone normed space:

1. \( \|x, y\|_c \geq 0 \) for all \( x, y \in X \), \( \|x, y\|_c = 0 \) if and only if \( x \) and \( y \) are linearly dependent,
2. \( \|x, y\|_c = \|y, x\|_c \) for all \( x, y \in X \),
3. \( |\alpha x, y\|_c = |\alpha| \|x, y\|_c \) for all \( x, y \in X \) and \( \alpha \in \mathbb{R} \),
4. \( \|x, y + z\|_c \leq \|x, y\|_c + \|y, z\|_c \) for all \( x, y, z \in X \).

If we fix \( \{u_1, u_2, ..., u_d\} \) to be a basis for \( X \), we can give the following lemma.

**Lemma 1.2.** [16] Let \( (X, \|.,.\|_c) \) be a 2-cone normed space. Then a sequence \( \{x_n\} \) converges to \( x \in X \) if and only if for each \( c \in E \) with \( c \gg 0 \) (0 is zero element of \( E \)) there exists an \( N = N(c) \in \mathbb{N} \) such that \( n > N \) implies \( \|x_n - x, u_i\|_c \ll c \) for every \( i = 1, 2, ..., d \).

**Lemma 1.3.** [16] Let \( (X, \|.,.\|_c) \) be a 2-cone normed space. Then a sequence \( \{x_n\} \) converges to \( x \in X \) if and only if

\[
\lim_{n \to \infty} \max_n \|x_n - x, u_i\|_c = 0.
\]

**Definition 1.8.** [16] A 2-cone normed space \( (X, \|.,.\|_c) \) is a 2-cone Banach space if any Cauchy sequence in \( X \) is convergent to an \( x \in X \).

**Theorem 1.1.** [17] Any 2-cone normed space \( X \) is a cone normed spaces and its topology agrees with the norm generated by \( \|.,\|_\infty \), where the function \( \|.,\|_\infty : X \to (E, P, \|\|) \) is defined by

\[
\|.,\|_\infty := \max \{\|x, u_i\|_c : i = 1, 2, ..., d\}.
\]

## 2. Main results

In this section, we prove some common fixed-point theorems on 2-cone Banach spaces. To do this, we define some notions and give some necessary examples.

**Definition 2.1.** Let \( X \) be a 2-cone Banach space (with \( \dim X \geq 2 \)) and \( T \) be a self-mapping of \( X \). If \( T \) satisfies the condition

\[
\|Tx - Ty, u\|_c \leq h_1 \|x - y, u\|_c
\]

for all \( x, y, u \in X \) and some \( 0 < h_1 < 1 \) then it is called 2-Banach contraction.

**Definition 2.2.** Let \( X \) be a 2-cone Banach space (with \( \dim X \geq 2 \)) and \( T \) be a self mapping of \( X \). A mapping \( T \) is said to be 2-Zamfirescu type contraction if it satisfies at least one of the conditions for all \( x, y, u \in X \) and some \( h_1 \in (0, 1), h_2, h_3 \in (0, \frac{1}{2}) : \)

1. \( \|Tx - Ty, u\|_c \leq h_1 \|x - y, u\|_c \)
2. \( \|Tx - Ty, u\|_c \leq h_2 (\|x - Ty, u\|_c + \|y - Tx, u\|_c) \)
3. \( \|Tx - Ty, u\|_c \leq h_3 (\|x - Tx, u\|_c + \|y - Ty, u\|_c) \)

**Definition 2.3.** Let \( X \) be a 2-cone Banach space and \( T \) be a self mapping of \( X \). \( T \) is said to be continuous at \( x \) if for all sequence \( \{x_n\} \) in \( X \) with \( \|x_n, u\|_c \to \|x, u\|_c \) implies that \( \|Tx_n, u\|_c \to \|Tx, u\|_c \).
Lemma 2.1. Let $X$ and $Y$ be two 2-cone Banach spaces and $T$ be a linear map from $X$ into $Y$. The following properties are equivalent:

i. (Continuity at a point) Given $0 \ll c$ there is a $0 \ll s$ such that $\|Tx - Tx_0, u\|_c \ll c$ whenever $\|x - x_0, u\|_c \ll s$ for some $x_0 \in X$.

ii. (Continuity at zero) For $0 \ll c$ there is a $0 \ll s$ such that $\|Tx, u\|_c \ll c$ whenever $\|x, u\|_c \ll s$.

iii. (Continuity at every point of $x$) Given $0 \ll c$ there is a $0 \ll s$ such that $\|Tx - Ty, u\|_c \ll c$ whenever $\|x - y, u\|_c \ll s$ for some $x \in X$.

Proof. Assume that (i) is true. For some $x_0 \in X$ and for every $0 \ll c$ there is a $0 \ll s$ such that $\|Tx - Tx_0, u\|_c \ll c$ whenever $\|x - x_0, u\|_c \ll s$. Then for every $z \in X$ with $\|z, u\|_c \ll s$ we have $\|T(z + x_0) - Tx_0, u\|_c \ll c$ because $\|(z + x_0) - x_0, u\|_c \ll t$, where $T$ is linear map then $\|Tz, u\|_c \ll c$ whenever $\|z, u\|_c \ll s$ and we have shown that (i) implies (ii).

Assume that (ii) is true. For every $x \in X$ and $0 \ll c$, there exists a $0 \ll s$ such that $\|Tz, u\|_c \ll c$ whenever $\|z, u\|_c \ll s$ then we have $\|Ty - x, u\|_c \ll s$. If we take $y - x$ in place of $z$ then we have (ii) implies (iii) since $T$ is linear map. Clearly (iii) implies (i). Thus (i), (ii), and (iii) are equivalent.

Definition 2.4. Let $\Phi$ be the class of continuous functions $\varphi : P^4 \to P$ non-decreasing in the first argument and if $\varphi$ satisfies one of the following conditions for $x, y \in P$:

a. $(a_1) x \leq \varphi (y, x, y, \frac{x + y}{2})$ or $(a_2) x \leq \varphi (x, y, y, x)$.

b. $(b_1) x \leq \varphi (y, \frac{x + y}{2}, 0, x + y)$ or $(b_2) x \leq \varphi (y, y, x, x)$.

then there exists a real number $0 < h < 1$ such that $x \leq hy$.

Now we define the following conditions:

**Condition (I):** Let $X$ be a 2-cone Banach space (with $\dim X \geq 2$) and $S, T$ be two self-mappings of $X$ such that for all $x, y, u \in X$ satisfying the condition:

$$\|Sx - Ty, u\|_c \leq \varphi \left( \|x - y, u\|_c, \|x - Sx, u\|_c, \|y - Ty, u\|_c, \frac{\|x - Ty, u\|_c + \|y - Sx, u\|_c}{2} \right).$$

**Condition (II):** Let $X$ be a 2-cone Banach space (with $\dim X \geq 2$) and $S, T$ be two self-mappings of $X$ such that for all $x, y, u \in X$ satisfying the condition:

$$\|Sx - Ty, u\|_c \leq \varphi \left( \|x - y, u\|_c, \frac{\|x - Sx, u\|_c + \|y - Ty, u\|_c}{2}, 0, \|x - Ty, u\|_c + \|y - Sx, u\|_c \right).$$

Theorem 2.1. Let $X$ be a 2-cone Banach space (with $\dim X \geq 2$) and $S, T$ be two continuous self-mappings of $X$ satisfying the condition (I). Then $S$ and $T$ have a unique common fixed point in $X$.

Proof. For a given $x_0 \in X$ and $n \geq 1$, take $x_1, x_2 \in X$ such that $x_1 = Sx_0$ and $x_2 = Tx_1$. In general we define a sequence of elements of $X$ such that $x_{2n+1} = Sx_{2n}$ and $x_{2n} = Tx_{2n-1}$ for $n = 0, 1, 2, 3, \ldots$. Now for all $u \in X$, by condition (I), we obtain

$$\|x_{2n+1} - x_{2n}, u\|_c = \|Sx_{2n} - Tx_{2n-1}, u\|_c \leq \varphi \left( \|x_{2n} - x_{2n-1}, u\|_c, \|x_{2n} - Sx_{2n}, u\|_c, \|x_{2n-1} - Tx_{2n-1}, u\|_c, \frac{\|x_{2n} - Ty, u\|_c + \|y - Sx, u\|_c}{2} \right).$$

This completes the proof.
Hence from Definition 2.4 \((a_1)\), we have
\[
\|x_{2n+1} - x_{2n}, u\|_c \leq h \|x_{2n} - x_{2n-1}, u\|_c \text{ where } 0 < h < 1.
\] (2.1)

Similarly, we have
\[
\|x_{2n} - x_{2n-1}, u\|_c \leq h \|x_{2n-1} - x_{2n-2}, u\|_c.
\] (2.2)

Hence, by (2.1) and (2.2), we have
\[
\|x_{2n+1} - x_{2n}, u\|_c \leq h^2 \|x_{2n-1} - x_{2n-2}, u\|_c.
\] By continuing this process, we get
\[
\|x_{2n+1} - x_{2n}, u\|_c \leq h^{2n} \|x_1 - x_0, u\|_c.
\]

For every \(n > m\), we have
\[
\|x_n - x_m, u\|_c \leq \|x_n - x_{n-1}, u\|_c + \|x_{n-1} - x_{n-2}, u\|_c + \cdots + \|x_m + 1 - x_m, u\|_c
\leq (h^{n-1} + h^{n-2} + \cdots + h^m) \|x_1 - x_0, u\|_c
\leq \left( \frac{h^n}{1-h} \right) \|x_1 - x_0, u\|_c.
\]

Since \(0 < h < 1\), by Definition 2.4, \(\left( \frac{h^n}{1-h} \right) < 0\) as \(m \to \infty\). Hence \(\|x_n - x_m, u\|_c < 0\) as \(n, m \to \infty\). This shows that \(\{x_n\}\) is a Cauchy sequence in \(X\). Hence there exists a point \(z\) in \(X\) such that \(x_n \to z\) as \(n \to \infty\). It follows from the continuity of \(S \) and \(T\) that \(Sz = Tz = z\). Thus \(z\) is a common fixed point of \(S \) and \(T\).

**Uniqueness** Let \(w\) be another common fixed point of \(S \) and \(T\), that is \(Sw = Tw = w\). Then, we have
\[
\|z - w, u\|_c = \|Sz - Tw, u\|_c
\leq \varphi \left( \|z - w, u\|_c, \|z - Sz, u\|_c, \|w - Tw, u\|_c \right)
\leq \varphi(\|z - w, u\|_c, 0, 0, \|z - w, u\|_c).
\] (2.3)

By Definition 2.4 \((a_2)\) and the inequality (2.3), we get
\[
\|z - w, u\|_c \leq 0.
\]

Hence \(z = w\) and for all \(u \in X\). Thus \(z\) is a unique common fixed point of \(S \) and \(T\).

\[\square\]

**Corollary 2.1.** Let \(X\) be a 2-cone Banach space \((\text{with } \dim X \geq 2)\) and \(T\) be a self-mapping of \(X\) satisfying the condition
\[
\|Tx - Ty, u\|_c \leq \varphi \left( \|x - y, u\|_c, \|x - Tx, u\|_c, \|y - Ty, u\|_c, \frac{\|x - Ty, u\|_c + \|y - Tx, u\|_c}{2} \right),
\]
for all \(x, y, u \in X\). Then \(T\) has a unique fixed point in \(X\).

**Proof.** The proof of corollary has immediately follows from above Theorem 2.1 by taking \(S = T\). This completes the proof.

\[\square\]

From the above theorem, we obtain the following results as special cases.

**Theorem 2.2.** Let \(X\) be a 2-cone Banach space \((\text{with } \dim X \geq 2)\) and \(T\), \(S\) be two self-mappings of \(X\) satisfying the condition
\[
\|Sx - Ty, u\|_c \leq h_1 \|x - y, u\|_c,
\]
for all \(x, y, u \in X\), \(0 < h_1 < 1\). Then \(T \) and \(S\) have a unique common fixed point in \(X\).

**Theorem 2.3.** Let \(X\) be a 2-cone Banach space \((\text{with } \dim X \geq 2)\) and \(T\), \(S\) be two self-mappings of \(X\) satisfying the condition
\[
\|Sx - Ty, u\|_c \leq h_2(\|x - Ty, u\|_c + \|y - Sx, u\|_c),
\]
for all \(x, y, u \in X\), \(0 < h_2 < \frac{1}{2}\). Then \(T \) and \(S\) have a unique common fixed point in \(X\).
We prove the following theorem using the condition (II).

**Theorem 2.4.** Let \( X \) be a 2-cone Banach space (with \( \dim X \geq 2 \)) and \( S, T \) be two continuous self-mappings of \( X \) satisfying the condition (II). Then \( S \) and \( T \) have a unique common fixed point in \( X \).

**Proof.** For a given \( x_0 \in X \) and \( n \geq 1 \), take \( x_1, x_2 \in X \) such that \( x_1 = Sx_0 \) and \( x_2 = Tx_1 \). In general we define a sequence of elements of \( X \) such that \( x_{2n+1} = Sx_{2n} \) and \( x_{2n} = Tx_{2n-1} \) for \( n = 0, 1, 2, 3, \ldots \). Now for all \( u \in X \), by condition (II), we obtain

\[
\|x_{2n+1} - x_{2n}, u\|_c = \|Sx_{2n} - Tx_{2n-1}, u\|_c \\
\leq \varphi \left( \frac{\|x_{2n+1} - x_{2n}, u\|_c + \|x_{2n} - x_{2n-1}, u\|_c}{2} \right) \\
= \varphi \left( \frac{\|x_{2n+1} - x_{2n}, u\|_c + \|x_{2n} - x_{2n-1}, u\|_c}{2} \right) \\
= \varphi \left( \frac{\|x_{2n+1} - x_{2n}, u\|_c + \|x_{2n} - x_{2n-1}, u\|_c}{2} \right) \\
\leq \varphi \left( \frac{\|x_{2n+1} - x_{2n}, u\|_c + \|x_{2n} - x_{2n-1}, u\|_c}{2} \right)
\]

Hence from Definition 2.4 \((b_1)\), we have

\[
\|x_{2n+1} - x_{2n}, u\|_c \leq h\|x_{2n+1} - x_{2n}, u\|_c \text{ where } 0 < h < 1.
\]

Similarly, we have

\[
\|x_{2n} - x_{2n-1}, u\|_c \leq h\|x_{2n} - x_{2n-1}, u\|_c.
\]

Hence from (2.4) and (2.5), we have

\[
\|x_{2n+1} - x_{2n}, u\|_c \leq h\|x_{2n} - x_{2n-1}, u\|_c
\]

on continuing this process, we get

\[
\|x_{2n+1} - x_{2n}, u\|_c \leq h^{2n}\|x_1 - x_0, u\|_c.
\]

For every \( n > m \), we have

\[
\|x_{n} - x_{m}, u\|_c \leq \|x_{n} - x_{n-1}, u\|_c + \|x_{n-1} - x_{n-2}, u\|_c + \cdots + \|x_{m+1} - x_{m}, u\|_c \\
\leq (h^{n-1} + h^{n-2} + \cdots + h^m)\|x_1 - x_0, u\|_c \\
\leq \left( \frac{h^m}{1 - h} \right)\|x_1 - x_0, u\|_c.
\]

Since \( 0 < h < 1 \), by Definition 2.4, \( \left( \frac{h^m}{1 - h} \right) < 0 \) as \( m \to \infty \). Hence \( \|x_{n} - x_{m}, u\|_c < 0 \) as \( n, m \to \infty \). This shows that \( \{x_n\} \) is a Cauchy sequence in \( X \). Hence there exists a point \( z \) in \( X \) such that \( x_n \to z \) as \( n \to \infty \). It follows from the continuity of \( S \) and \( T \) that \( Sz = Tz = z \). Thus \( z \) is a common fixed point of \( S \) and \( T \).

**Uniqueness** Let \( w \) be another common fixed point of \( S \) and \( T \), that is \( Sw = Tw = w \). Then, we have

\[
\|z - w, u\|_c = \|Sz - Tw, u\|_c \\
\leq \varphi \left( \frac{\|z - w, u\|_c + \|w - Sw, u\|_c + \|z - Sz, u\|_c}{2} \right) \\
\leq \varphi(\|z - w, u\|_c, 0, \|z - w, u\|_c, \|z - w, u\|_c).
\]

By Definition 2.4 \((b_2)\) and the inequality (2.6), we get

\[
\|z - w, u\|_c \leq 0.
\]

Hence \( z = w \) and for all \( u \in X \). Thus \( z \) is a unique common fixed point of \( S \) and \( T \).  \(\square\)
Corollary 2.2. Let $X$ be a 2-cone Banach space (with $\dim X \geq 2$) and $T$ be a self-mapping of $X$ satisfying the condition

$$\|Tx - Ty, u\|_c \leq \varphi \left( \|x - y, u\|_c, \frac{\|x - Tx, u\|_c + \|y - Ty, u\|_c}{2}, \|x - Ty, u\|_c, \|y - Tx, u\|_c \right),$$

for all $x, y, u \in X$. Then $T$ has a unique fixed point in $X$.

Proof. The proof of corollary has immediately follows from above Theorem 2.4 by taking $S = T$. This completes the proof. 

From the above theorem we obtain the following result as a special case.

Theorem 2.5. Let $X$ be a 2-cone Banach space (with $\dim X \geq 2$) and $T, S$ be two self-mappings of $X$ satisfying the condition

$$\|Sx - Ty, u\|_c \leq h_3(\|x - Sx, u\|_c + \|y - Ty, u\|_c),$$

for all $x, y, u \in X$, $0 < h_3 < \frac{1}{2}$. Then $T$ and $S$ have a unique common fixed point in $X$.

From Theorem 2.1 and Theorem 2.4, we obtain the following results as special cases.

Theorem 2.6. Let $X$ be a 2-cone Banach space (with $\dim X \geq 2$) and $T, S$ be two self-mappings of $X$. A mapping $T$ and $S$ are said to be 2-Zamfirescu type contraction satisfying the at least one of the following conditions is true:

1. $\|Sx - Ty, u\|_c \leq h_1\|x - y, u\|_c$
2. $\|Sx - Ty, u\|_c \leq h_2(\|x - Ty, u\|_c + \|y - Sx, u\|_c)$
3. $\|Sx - Ty, u\|_c \leq h_3(\|x - Ty, u\|_c + \|y - Ty, u\|_c)$

for all $x, y, u \in X$, $h_1 \in (0, 1)$ and $h_2, h_3 \in (0, \frac{1}{2})$. Then $T$ and $S$ have a unique common fixed point in $X$.

3. An application to the fixed-circle problem

In this section, we give an application to the fixed-circle problem which is a new geometric approach to fixed-point theory raised by Özgür and Taş [8]. More recently, some different solutions of the problem have been investigated with various techniques on metric spaces or some generalized metric spaces (see [6], [7], [9], [10], [11], [12], [13], [18], [19], [20] and [21] for more details). In this context, we obtain new fixed-circle theorems on 2-cone normed spaces. At first, we recall the notion of an open ball and define a circle on a 2-cone normed space.

Definition 3.1. [17] Let $\| \|^{\infty}_c : X \to (E, P, \| \|)$ and $r \in E$ with $r \gg \theta$. Then the set

$$B_{\{u_1, u_2, \ldots, u_d\}}(x_0, r) = \{ x : \| x - x_0 \|^\infty_c < r \}$$

is called an open ball centered at $x_0$ with radius $r$.

Definition 3.2. (1) Let $\| \|^{\infty}_c : X \to (E, P, \| \|)$ and $r \in E$ with $r \gg \theta$ or $r = \theta$. Then the set

$$C^2_{x_0, r} = C_{\{u_1, u_2, \ldots, u_d\}}(x_0, r) = \{ x : \| x - x_0 \|^c = r \}$$

is called a circle centered at $x_0$ with radius $r$.

(2) Let $\| \|^{\infty}_c : X \to (E, P, \| \|)$ and $r \in E$ with $r \gg \theta$ or $r = \theta$. Then the set

$$B_{\{u_1, u_2, \ldots, u_d\}}(x_0, r) = B_{\{u_1, u_2, \ldots, u_d\}}(x_0, r) \cup C^2_{x_0, r}$$

is called a closed ball centered at $x_0$ with radius $r$.

(3) The circle $C^2_{x_0, r}$ (or the closed ball $B_{\{u_1, u_2, \ldots, u_d\}}(x_0, r)$) is called the fixed circle (or fixed disc) of a self-mapping $T$ if $Tx = x$ for all $x \in C^2_{x_0, r}$ (or $x \in B_{\{u_1, u_2, \ldots, u_d\}}(x_0, r)$), respectively.

We give the following fixed-circle (or fixed-disc) results:
Theorem 3.1. Let $X$ be a $2$-cone normed space (with $\dim X \geq 2$), $T : X \to X$ be a self-mapping, $x_0 \in X$ and
\[
r = \inf_{x \in X} \{ \|Tx - x, u\|_c : Tx \neq x \}. \tag{3.1}
\]
If $T$ satisfies the following conditions, then $C_{x_0}^2$ is a fixed circle of $T$:
(1) If $Tx \neq x$ then
\[
\|Tx - x, u\|_c \leq \varphi \left( \frac{\|x - x_0, u\|_c}{\|x - x_0, u\|_c + \|x - Tx_0, u\|_c}, \frac{\|Tx - x_0, u\|_c}{\|x - Tx_0, u\|_c + \|Tx - x_0, u\|_c} \right),
\]
where $\varphi \in \Phi$.
(2) $Tx_0 = x_0$.

Proof. Case 1: Let $r = \theta$. Then we have
\[
\|x - x_0\|_c = \theta \implies \max \{ \|x - x_0, u_i\|_c : i = 1, 2, \ldots, d \} = 0
\]
\[
\implies \|x - x_0, u_i\|_c = 0 \text{ for all } i = 1, 2, \ldots, d
\]
\[
\implies x = x_0
\]
\[
\implies C_{x_0}^2 = \{x_0\}.
\]
Using the condition (2), we know $Tx_0 = x_0$ and so $C_{x_0}^2$ is a fixed circle of $T$.

Case 2: Let $r \gg \theta$ and $x \in C_{x_0}^2$ with $Tx \neq x$. By the definition of $r$, we have $r \leq \|Tx - x, u\|_c$. Using the conditions (1), (2) and the property of $\varphi$, we obtain
\[
\|Tx - x, u\|_c \leq \varphi \left( \frac{\|x - x_0, u\|_c, \|Tx - x, u\|_c, \|x - Tx_0, u\|_c, \|x - x_0, u\|_c}{\|x - x_0, u\|_c + \|Tx - x, u\|_c} \right)
\]
\[
\leq \varphi \left( r, \|Tx - x, u\|_c, \frac{r + \|Tx - x, u\|_c}{2} \right).
\]
From Definition 2.4 $(a_1)$, we have
\[
\|Tx - x, u\|_c \leq hr, h \in (0, 1),
\]
which is a contradiction with the definition of $r$. Therefore, it should be $Tx = x$. Consequently, $C_{x_0}^2$ is a fixed circle of $T$. \qed

Corollary 3.1. Let $X$ be a $2$-cone normed space (with $\dim X \geq 2$), $T : X \to X$ be a self-mapping, $x_0 \in X$ and $r$ be defined as in (3.1). If $T$ satisfies the following conditions, then $T$ fixes the closed ball $B_{\{u_1, u_2, \ldots, u_d\}}[x_0, \rho]$ with $\rho \leq r$ (or $B_{\{u_1, u_2, \ldots, u_d\}}[x_0, r]$ is the fixed disc of $T$):
(1) If $Tx \neq x$ then
\[
\|Tx - x, u\|_c \leq \varphi \left( \frac{\|x - x_0, u\|_c, \|Tx - x, u\|_c, \|x - Tx_0, u\|_c, \|x - x_0, u\|_c}{\|x - x_0, u\|_c + \|Tx - x, u\|_c} \right),
\]
where $\varphi \in \Phi$.
(2) $Tx_0 = x_0$.

Proof. The proof can be easily seen by the similar arguments used in the proof of Theorem 3.1. \qed

Theorem 3.2. Let $X$ be a $2$-cone normed space (with $\dim X \geq 2$), $T : X \to X$ be a self-mapping, $x_0 \in X$ and $r$ be defined as in (3.1). If $T$ satisfies the following conditions, then $C_{x_0}^2$ is a fixed circle of $T$:
(1) If $Tx \neq x$ then
\[
\|Tx - x, u\|_c \leq \varphi \left( \frac{\|x - x_0, u\|_c, \|Tx - x, u\|_c, \|x - Tx_0, u\|_c, \|x - x_0, u\|_c}{\|Tx - x, u\|_c + \|x - Tx_0, u\|_c} \right),
\]
where $\varphi \in \Phi$.
(2) $Tx_0 = x_0$. 

Proof. **Case 1:** Let \( r = \theta \). Then we have \( C^2_{x_0, r} = \{ x_0 \} \). Using the condition (2), we know \( T x_0 = x_0 \) and so \( C^2_{x_0, r} \) is a fixed circle of \( T \).

**Case 2:** Let \( r \geq \theta \) and \( x \in C^2_{x_0, r} \) with \( T x \neq x \). By the definition of \( r \), we have \( r \leq \| T x - x, u \|_c \). Using the conditions (1), (2) and the property of \( \varphi \), we obtain

\[
\| T x - x, u \|_c \leq \varphi \left( \frac{\| x - x_0, u \|_c + \| T x - x, u \|_c}{2} \right) 
\]

From Definition 2.4 (b), we have

\[
\| T x - x, u \|_c \leq hr, h \in (0, 1),
\]

which is a contradiction with the definition of \( r \). Therefore, it should be \( T x = x \). Consequently, \( C^2_{x_0, r} \) is a fixed circle of \( T \).

**Corollary 3.2.** Let \( X \) be a 2-cone normed space (with \( \dim X \geq 2 \)), \( T : X \to X \) be a self-mapping, \( x_0 \in X \) and \( r \) be defined as in (3.1). If \( T \) satisfies the following conditions, then \( T \) fixes the closed ball \( B_{\{ u_1, u_2, \ldots, u_d \}}[x_0, \rho] \) with \( \rho \leq r \) (or \( B_{\{ u_1, u_2, \ldots, u_d \}}[x_0, r] \) is the fixed disc of \( T \)):

1. If \( T x \neq x \) then
   \[
   \| T x - x, u \|_c \leq \varphi \left( \frac{\| x - x_0, u \|_c + \| T x - x, u \|_c}{2} \right),
   \]
   where \( \varphi \in \Phi \).
2. \( T x_0 = x_0 \).

**Proof.** The proof can be easily seen by the similar arguments used in the proof of Theorem 3.2.

**References**


Affiliations

D. Dhamodharan
Address: Jamal Mohamed College (Autonomous), Dept. of Mathematics, Tiruchirappalli-620020, Tamil Nadu-India.
E-mail: dharan_raj28@yahoo.co.in
ORCID ID: 0000-0003-4859-4816

Nihal Taş
Address: Balıkesir University, Dept. of Mathematics, 10145, Balıkesir-Turkey.
E-mail: nihaltas@balikesir.edu.tr
ORCID ID: 0000-0002-4535-4019

R. Krishnakumar
Address: Urumu Dhanalakshmi College, Dept. of Mathematics, Tiruchirappalli-620019, Tamil Nadu-India.
E-mail: srksacet@yahoo.co.in
ORCID ID: 0000-0001-5927-0150