CERTAIN SUBCLASSES OF BI-UNIVALENT FUNCTIONS RELATED TO k-FIBONACCI NUMBERS

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Abstract. In this paper, we introduce and investigate new subclasses of bi-univalent functions related to k-Fibonacci numbers. Furthermore, we find estimates of first two coefficients of functions in these classes. Also, we obtain the Fekete-Szegő inequalities for these function classes.

1. Introduction

Let \( D = \{ z : |z| < 1 \} \) be the unit disc in the complex plane. The class of all analytic functions

\[
f(z) = z + \sum_{n=2}^{\infty} a_n z^n
\]

in the open unit disc \( D \) with normalization \( f(0) = 0, f'(0) = 1 \) is denoted by \( A \) and the class \( S \subset A \) is the class which consists of univalent functions in \( D \). We say that \( f \) is subordinate to \( F \) in \( D \), written as \( f \prec F \), if and only if \( f(z) = F(\omega(z)) \) for some analytic function \( \omega, |\omega(z)| \leq |z|, z \in D \).

The Koebe one quarter theorem [5] ensures that the image of \( D \) under every univalent function \( f \in A \) contains a disk of radius 1/4. Thus every univalent function \( f \) has an inverse \( f^{-1} \) satisfying

\[
f^{-1}(f(z)) = z, \ (z \in D) \quad \text{and} \quad f(f^{-1}(w)) = w, \ (|w| < r_0(f), \ r_0(f) \geq \frac{1}{4}).
\]

A function \( f \in A \) is said to be bi-univalent in \( D \) if \( f \) is univalent in \( D \) and \( f^{-1} \) has an univalent extension to \( D \). Let \( \Sigma \) denote the class of bi-univalent functions defined in the unit disk \( D \). Someone can see a short history and examples of functions in the class \( \Sigma \) in [14]. Since \( f \in \Sigma \) has the Maclaurin series given by (1), a computation shows that its inverse \( g = f^{-1} \) has the expansion

\[
g(w) = f^{-1}(w) = w - a_2 w^2 + (2a_2^2 - a_3)w^3 + \cdots.
\]
The work of Srivastava et al. [14] essentially revived the investigation of various subclasses of the bi-univalent function class in recent years. In a considerably large number of sequels to the aforementioned work of Srivastava et al. [14], several different subclasses of the bi-univalent function class were introduced and studied analogously by many authors (see, for example, [1, 2, 4, 8, 3, 15, 9]), but only non-sharp estimates on the initial coefficients $|a_2|$ and $|a_3|$ in the Taylor-Maclaurin expansion [1] were obtained in these recent papers.

The object of the present work is to introduce a new subclass of the function class and find estimates on the coefficients $|a_2|$ and $|a_3|$ for functions in this new subclass of the function class using the technique of Srivastava et al. [14].

Recently, Yılmaz Özgür and Sokól [10] introduced the class $\mathcal{SL}_k$ of starlike functions connected with $k-$ Fibonacci numbers as the set of functions $f \in \mathcal{A}$ which is described in the following definition.

**Definition 1.** Let $k$ be any positive real number. The function $f \in \mathcal{A}$ belongs to the class $\mathcal{SL}_k$ if it satisfies the condition that

$$zf'(z) \prec \tilde{p}_k(z), \quad z \in \mathbb{D},$$

where

$$\tilde{p}_k(z) = \frac{1 + \tau_k z^2}{1 - k\tau_k z - \tau_k z^2}, \quad \tau_k = \frac{k - \sqrt{k^2 + 4}}{2}, \quad z \in \mathbb{D}. \quad (3)$$

Later in [7], Güney et al. defined the class $\mathcal{KSL}_k$ as follows:

**Definition 2.** Let $k$ be any positive real number. The function $f \in \mathcal{A}$ belongs to the class $\mathcal{KSL}_k$ if it satisfies the condition that

$$1 + \frac{zf''(z)}{f'(z)} \prec \tilde{p}_k(z), \quad z \in \mathbb{D},$$

where the function $\tilde{p}_k$ is defined in (3).

For $k = 1$, the classes $\mathcal{SL}$ and $\mathcal{KSL}$ of shell-like functions were defined in [12] (see also [13]).

It was proved in [10] that functions in the class $\mathcal{SL}_k$ are univalent in $\mathbb{D}$. Moreover, the class $\mathcal{SL}_k$ is a subclass of the class of starlike functions $\mathcal{S}^*$, even more, starlike of order $k(k^2 + 4)^{-1/2}/2$. The name attributed to the class $\mathcal{SL}_k$ is motivated by the shape of the curve

$$\mathcal{C} = \{\tilde{p}_k(e^{it}) : t \in [0, 2\pi) \setminus \{\pi\}\}.$$

Now we define the classes $\mathcal{SLM}_k^\alpha$ and $\mathcal{SLG}_k^\gamma$, as follows:

**Definition 3.** Let $k$ be any positive real number. The function $f \in \mathcal{A}$ belongs to the class $\mathcal{SLM}_k^\alpha$, $(0 \leq \alpha \leq 1)$ if it satisfies the condition that

$$\alpha \left(1 + \frac{zf''(z)}{f'(z)}\right) + (1 - \alpha) \frac{zf'(z)}{f(z)} \prec \tilde{p}_k(z), \quad z \in \mathbb{D},$$

$$1 + \frac{zf''(z)}{f'(z)} \prec \tilde{p}_k(z), \quad z \in \mathbb{D}.$$
where the function $\tilde{p}_k$ is defined in (3).

**Definition 4.** Let $0 \leq \gamma \leq 1$, and $k$ be any positive real number. The function $f \in A$ belongs to the class $SLG^k_\gamma$ if the following conditions are satisfied:

$$
\left( \frac{zf'(z)}{f(z)} \right)^\gamma \left( 1 + \frac{zf''(z)}{f'(z)} \right)^{1-\gamma} < \tilde{p}_k(z), \quad z \in \mathbb{D},
$$

where the function $\tilde{p}_k$ is defined in (3).

For $k \leq 2$, note that we have

$$
\tilde{p}_k \left( e^{\pm i \arccos(k^2/4)} \right) = k(k^2 + 4)^{-1/2},
$$

and so the curve $C$ intersects itself on the real axis at the point $w_1 = k(k^2 + 4)^{-1/2}$. Thus $C$ has a loop intersecting the real axis also at the point $w_2 = (k^2 + 4)/(2k)$. For $k > 2$, the curve $C$ has no loops and it is like a conchoid, see for details [10]. Moreover, the coefficients of $\tilde{p}_k$ are connected with $k$-Fibonacci numbers.

For any positive real number $k$, the $k$-Fibonacci number sequence $\{F_k,n\}_{n=0}^\infty$ is defined recursively by

$$
F_{k,0} = 0, \quad F_{k,1} = 1 \quad \text{and} \quad F_{k,n+1} = kF_{k,n} + F_{k,n-1} \quad \text{for} \quad n \geq 1.
$$

When $k = 1$, we obtain the well-known Fibonacci numbers $F_n$. It is known that the $n^{th}$ $k$-Fibonacci number is given by

$$
F_{k,n} = \frac{(k - \tau_k)^n - \tau_k^n}{\sqrt{k^2 + 4}},
$$

where $\tau_k = (k - \sqrt{k^2 + 4})/2$. If $\tilde{p}_k(z) = 1 + \sum_{n=1}^\infty \tilde{p}_{k,n}z^n$, then we have

$$
\tilde{p}_{k,n} = (F_{k,n-1} + F_{k,n+1})\tau_k^n, \quad n = 1, 2, 3, \ldots.
$$

Also, Özgür and Sokol showed in [10] that

$$
\tilde{p}_k(z) = \frac{1 + \tau_kz^2}{1 - k\tau_kz - \tau_k^2z^2} = 1 + \sum_{n=1}^\infty \tilde{p}_{k,n}z^n
$$

$$
= 1 + (F_{k,0} + F_{k,2})\tau_kz + (F_{k,1} + F_{k,3})\tau_k^2z^2 + \cdots
$$

$$
= 1 + k\tau_kz + (k^2 + 2)\tau_k^2z^2 + (k^3 + 3k)\tau_k^3z^3 + \cdots,
$$

where $\tau_k = \frac{k - \sqrt{k^2 + 4}}{2}$, $z \in \mathbb{D}$, (see [10]).

Let $\mathcal{P}(\beta)$, $0 \leq \beta < 1$, denote the class of analytic functions $p$ in $\mathbb{D}$ with $p(0) = 1$ and $\Re\{p(z)\} > \beta$. Especially, we use $\mathcal{P}(0) = \mathcal{P}$ as $\beta = 0$.

Now we give the following lemma which will use in proving.

**Lemma 5.** ([11]) Let $p \in \mathcal{P}$ with $p(z) = 1 + c_1z + c_2z^2 + \cdots$, then

$$
|c_n| \leq 2 \quad \text{for} \quad n \geq 1.
$$

(4)
2. Bi-Univalent function class $\mathcal{S}_L M^b_{a,\Sigma}(\tilde{p}_k(z))$

In this section, we introduce three new subclasses of $\Sigma$ associated with shell-like functions connected with Fibonacci numbers and obtain the initial Taylor coefficients $|a_2|$ and $|a_3|$ for the function classes by subordination.

Firstly, let $p(z) = 1 + p_1 z + p_2 z^2 + \cdots$, and $p \prec \tilde{p}_k$. Then there exists an analytic function $u$ such that $|u(z)| < 1$ in $U$ and $p(z) = \tilde{p}_k(u(z))$. Therefore, the function

$$h(z) = \frac{1 + u(z)}{1 - u(z)} = 1 + c_1 z + c_2 z^2 + \cdots$$

is in the class $\mathcal{P}(0)$. It follows that

$$u(z) = \frac{c_1 z}{2} + \left( c_2 - \frac{c_1^2}{2} \right) \frac{z^2}{2} + \left( c_3 - c_1 c_2 + \frac{c_1^3}{4} \right) \frac{z^3}{2} + \cdots$$

and

$$\tilde{p}_k(u(z)) = 1 + \tilde{p}_{k,1} \left\{ \frac{c_1 z}{2} + \left( c_2 - \frac{c_1^2}{2} \right) \frac{z^2}{2} + \left( c_3 - c_1 c_2 + \frac{c_1^3}{4} \right) \frac{z^3}{2} + \cdots \right\}$$

$$+ \tilde{p}_{k,2} \left\{ \frac{c_1 z}{2} + \left( c_2 - \frac{c_1^2}{2} \right) \frac{z^2}{2} + \left( c_3 - c_1 c_2 + \frac{c_1^3}{4} \right) \frac{z^3}{2} + \cdots \right\}^2$$

$$+ \tilde{p}_{k,3} \left\{ \frac{c_1 z}{2} + \left( c_2 - \frac{c_1^2}{2} \right) \frac{z^2}{2} + \left( c_3 - c_1 c_2 + \frac{c_1^3}{4} \right) \frac{z^3}{2} + \cdots \right\}^3 + \cdots$$

$$= 1 + \frac{\tilde{p}_{k,1} c_1 z}{2} + \left\{ \frac{1}{2} \left( c_2 - \frac{c_1^2}{2} \right) \tilde{p}_{k,1} + \frac{c_1^3}{4} \tilde{p}_{k,2} \right\} \frac{z^2}{2}$$

$$+ \left\{ \frac{1}{2} \left( c_3 - c_1 c_2 + \frac{c_1^3}{4} \right) \tilde{p}_{k,1} + \frac{1}{2} c_1 \left( c_2 - \frac{c_1^2}{2} \right) \tilde{p}_{k,2} + \frac{c_1^3}{8} \tilde{p}_{k,3} \right\} \frac{z^3}{2} + \cdots$$

And similarly, there exists an analytic function $v$ such that $|v(w)| < 1$ in $\mathbb{D}$ and $p(w) = \tilde{p}_k(v(w))$. Therefore, the function

$$k(w) = \frac{1 + v(w)}{1 - v(w)} = 1 + d_1 w + d_2 w^2 + \cdots$$

is in the class $\mathcal{P}(0)$. It follows that

$$v(w) = \frac{d_1 w}{2} + \left( d_2 - \frac{d_1^2}{2} \right) \frac{w^2}{2} + \left( d_3 - d_1 d_2 + \frac{d_1^3}{4} \right) \frac{w^3}{2} + \cdots$$

and

$$\tilde{p}_k(v(w)) = 1 + \frac{\tilde{p}_{k,1} d_1 w}{2} + \left\{ \frac{1}{2} \left( d_2 - \frac{d_1^2}{2} \right) \tilde{p}_{k,1} + \frac{d_1^2}{4} \tilde{p}_{k,2} \right\} \frac{w^2}{2}$$

$$+ \left\{ \frac{1}{2} \left( d_3 - d_1 d_2 + \frac{d_1^3}{4} \right) \tilde{p}_{k,1} + \frac{1}{2} d_1 \left( d_2 - \frac{d_1^2}{2} \right) \tilde{p}_{k,2} + \frac{d_1^3}{8} \tilde{p}_{k,3} \right\} \frac{w^3}{2} + \cdots$$
Definition 6. For $0 \leq \alpha \leq 1$, a function $f \in \Sigma$ of the form \ref{eq:1} is said to be in the class $\mathcal{SLM}_{\alpha, k}(\bar{p}_k(z))$ if the following subordination hold:

$$\alpha \left(1 + \frac{zf''(z)}{f'(z)}\right) + (1 - \alpha) \left(\frac{zf'(z)}{f(z)}\right) \prec \bar{p}_k(z) = \frac{1 + \tau_k^2 z^2}{1 - k\tau_k z - \tau_k^2 z^2},$$

and

$$\alpha \left(1 + \frac{wg''(w)}{g'(w)}\right) + (1 - \alpha) \left(\frac{wg'(w)}{g(w)}\right) \prec \bar{p}_k(w) = \frac{1 + \tau_k^2 w^2}{1 - k\tau_k w - \tau_k^2 w^2},$$

where $\tau_k = \frac{k - \sqrt{k^2 + 1}}{2}$ where $z, w \in D$ and $g$ is given by \ref{eq:2}.

Specializing the parameter $\alpha = 0$ and $\alpha = 1$ we have the following:

Definition 7. A function $f \in \Sigma$ of the form \ref{eq:1} is said to be in the class $\mathcal{SL}_{\alpha, k}(\bar{p}_k(z))$ if the following subordination hold:

$$zf'(z) \prec \bar{p}_k(z) = \frac{1 + \tau_k^2 z^2}{1 - k\tau_k z - \tau_k^2 z^2},$$

and

$$wg'(w) \prec \bar{p}_k(w) = \frac{1 + \tau_k^2 w^2}{1 - k\tau_k w - \tau_k^2 w^2},$$

where $\tau_k = \frac{k - \sqrt{k^2 + 1}}{2}$, $z, w \in D$ and $g$ is given by \ref{eq:2}.

Definition 8. A function $f \in \Sigma$ of the form \ref{eq:1} is said to be in the class $\mathcal{KSL}_{\alpha, k}(\bar{p}_k(z))$ if the following subordination hold:

$$1 + zf''(z) \prec \bar{p}_k(z) = \frac{1 + \tau_k^2 z^2}{1 - k\tau_k z - \tau_k^2 z^2},$$

and

$$1 + wg''(w) \prec \bar{p}_k(w) = \frac{1 + \tau_k^2 w^2}{1 - k\tau_k w - \tau_k^2 w^2},$$

where $\tau_k = \frac{k - \sqrt{k^2 + 1}}{2}$, $z, w \in D$ and $g$ is given by \ref{eq:2}.

In the following theorem we determine the initial Taylor coefficients $|a_2|$ and $|a_3|$ for the function class $\mathcal{SLM}_{\alpha, k}(\bar{p}_k(z))$. Later we state the bounds to other classes as a special cases.

Theorem 9. Let $f$ given by \ref{eq:1} be in the class $\mathcal{SLM}_{\alpha, k}(\bar{p}_k(z))$. Then

$$|a_2| \leq \frac{k\sqrt{k}|\tau_k|}{\sqrt{(1 + \alpha)^2 k - (1 + \alpha)(2(1 + \alpha)k^2 + \alpha)\tau_k}}$$

and

$$|a_3| \leq \frac{k|\tau_k| \left\{(1 + \alpha)^2 k - \left[(k^2 + 2)\alpha^2 + (5k^2 + 4)\alpha + 2(k^2 + 1)\right]\tau_k\right\}}{2(1 + 2\alpha)(1 + \alpha) \left[(1 + \alpha)k - (2(1 + \alpha) + \alpha k^2)\tau_k\right]}.$$
Proof. Let \( f \in \mathcal{SLM}_k^{\alpha}(\tilde{p}_k(z)) \) and \( g = f^{-1} \). Considering (11) and (12), we have

\[
\alpha \left( 1 + \frac{zf''(z)}{f'(z)} \right) + (1 - \alpha) \left( \frac{zf'(z)}{f(z)} \right) = \tilde{p}_k(u(z))
\]

(19)

and

\[
\alpha \left( 1 + \frac{wg''(w)}{g'(w)} \right) + (1 - \alpha) \left( \frac{wg'(w)}{g(w)} \right) = \tilde{p}_k(v(w)),
\]

(20)

where \( \tau_k = \frac{k - \sqrt{k^2 + 1}}{2} \), \( z, w \in \mathbb{D} \) and \( g \) is given by (2). We have also

\[
\alpha \left( 1 + \frac{zf''(z)}{f'(z)} \right) + (1 - \alpha) \left( \frac{zf'(z)}{f(z)} \right) = 1 + (1 + \alpha)a_2z + 2(1 + 2\alpha)a_3 - (1 + 3\alpha)a_2^2z^2 + \ldots
\]

\[
= 1 + \frac{\tilde{p}_{k,1}c_1}{2}z + \left[ \frac{1}{2} \left( c_2 - \frac{c_1^3}{4} \right) \tilde{p}_{k,2} - \frac{c_1^2}{4} \tilde{p}_{k,1} \right] z^2 + \ldots
\]

(21)

and

\[
\alpha \left( 1 + \frac{wg''(w)}{g'(w)} \right) + (1 - \alpha) \left( \frac{wg'(w)}{g(w)} \right) = 1 - (1 + \alpha)a_2w + ((3 + 5\alpha)a_2^2 - 2(1 + 2\alpha)a_3)w^2 + \ldots
\]

\[
= 1 + \frac{\tilde{p}_{k,1}d_1}{2}w + \left[ \frac{1}{2} \left( d_2 - \frac{d_1^2}{4} \right) \tilde{p}_{k,2} + \frac{d_1^2}{4} \tilde{p}_{k,1} \right] w^2 + \ldots
\]

(22)

It follows from (21) and (22) that

\[
(1 + \alpha)a_2 = \frac{c_1k\tau_k}{2},
\]

(23)

\[
2(1 + 2\alpha)a_3 - (1 + 3\alpha)a_2^2 = \frac{1}{2} \left( c_2 - \frac{c_1^3}{2} \right) k\tau_k + \frac{c_1^2}{4} (k^2 + 2)\tau_k^2,
\]

(24)

and

\[
- (1 + \alpha)a_2 = \frac{d_1k\tau_k}{2},
\]

(25)

\[
(3 + 5\alpha)a_2^2 - 2(1 + 2\alpha)a_3 = \frac{1}{2} \left( d_2 - \frac{d_1^2}{2} \right) k\tau_k + \frac{d_1^2}{4} (k^2 + 2)\tau_k^2.
\]

(26)

From (23) and (25), we have

\[
c_1 = -d_1,
\]

(27)

and

\[
2a_2^2 = \frac{(c_1^2 + d_1^2)}{4(1 + \alpha)^2} k^2\tau_k^2.
\]

(28)
Now, by summing (24) and (26), we obtain
\[ 2(1 + \alpha)a_2^2 = \frac{1}{2}(c_2 + d_2)k\tau_k - \frac{1}{4}(c_2^2 + d_2^2)k\tau_k + \frac{1}{4}(c_2^2 + d_2^2)(k^2 + 2)\tau_k^2. \] (29)

By putting (28) in (29), we have
\[ 2(1 + \alpha) \left[ (\frac{1}{2} + \frac{\alpha}{2}) \right] a_2^2 = \frac{1}{2}(c_2 + d_2)k^3\tau_k^2. \] (30)

Therefore, using Lemma 5 we obtain
\[ |a_2| \leq \frac{k\sqrt{\tau_k}|\tau_k|}{\sqrt{(1 + \alpha)^2k - (1 + \alpha)(2(1 + \alpha) + \alpha k^2)\tau_k}}. \] (31)

If we can take the parameter \( \alpha = 0 \) and \( \alpha = 1 \) in the above theorem, we have the following the initial Taylor coefficients \( |a_2| \) and \( |a_3| \) for the function classes \( \mathcal{S}\mathcal{L}_k(z) \) and \( \mathcal{K}\mathcal{S}\mathcal{L}_k(z) \), respectively.

Corollary 10. Let \( f \) given by (1) be in the class \( \mathcal{S}\mathcal{L}_k(z) \). Then
\[ |a_2| \leq \frac{k\sqrt{\tau_k}|\tau_k|}{\sqrt{k - 2\tau_k}} \]
and
\[ |a_3| \leq \frac{k|\tau_k|\{k - (k^2 + 1)\tau_k\}}{2(k - 2\tau_k)}. \]

Corollary 11. Let \( f \) given by (1) be in the class \( \mathcal{K}\mathcal{S}\mathcal{L}_k(z) \). Then
\[ |a_2| \leq \frac{k\sqrt{\tau_k}|\tau_k|}{\sqrt{4k - 2(4 + k^2)}\tau_k} \]
and
\[ |a_3| \leq \frac{k|\tau_k|\{k - (k^2 + 1)\tau_k\}}{3(2k - (4 + k^2)\tau_k)}. \]
If we can take the parameter $k = 1$ in the above corollaries, we have the following
the initial Taylor coefficients $|a_2|$ and $|a_3|$ for the function classes $\mathcal{S}\mathcal{L}_\Sigma(\tilde{p}(z))$ and
$\mathcal{K}\mathcal{S}\mathcal{L}_\Sigma(\tilde{p}(z))$, respectively, which were obtained in [5] by Güney et al.

**Corollary 12.** Let $f$ given by (1) be in the class $\mathcal{S}\mathcal{L}_\Sigma(\tilde{p}(z))$. Then

$$|a_2| \leq \frac{|\tau|}{\sqrt{1 - 2\tau}}$$

and

$$|a_3| \leq \frac{|\tau|(1 - 4\tau)}{2(1 - 2\tau)}.$$

**Corollary 13.** Let $f$ given by (1) be in the class $\mathcal{K}\mathcal{S}\mathcal{L}_\Sigma(\tilde{p}(z))$. Then

$$|a_2| \leq \frac{|\tau|}{\sqrt{4 - 10\tau}}$$

and

$$|a_3| \leq \frac{|\tau|(1 - 4\tau)}{3(2 - 5\tau)}.$$

### 3. Bi-Univalent function class $\mathcal{S}\mathcal{L}\mathcal{G}_\Sigma^k(z)$

In this section, we define a new class $\mathcal{S}\mathcal{L}\mathcal{G}_\Sigma^k(z)$ of $\gamma-$ bi-starlike functions associated with shell-like domain.

**Definition 14.** Let $0 \leq \gamma \leq 1$, and $k$ be any positive real number. A function $f \in \Sigma$ of the form (1) is said to be in the class $\mathcal{S}\mathcal{L}\mathcal{G}_\Sigma^k(z)$ if the following subordination hold:

$$\left(\frac{zf'(z)}{f(z)}\right)^\gamma \left(1 + \frac{zf''(z)}{f'(z)}\right)^{1-\gamma} < \tilde{p}_k(z)$$

(1)

and

$$\left(\frac{wg'(w)}{g(w)}\right)^\gamma \left(1 + \frac{wg''(w)}{g'(w)}\right)^{1-\gamma} < \tilde{p}_k(w),$$

(2)

where the function $\tilde{p}_k$ is defined in (3) and $z, w \in D$.

**Remark 15.** Taking $\gamma = 1$, we get $\mathcal{S}\mathcal{L}\mathcal{G}_\Sigma^1(z) \equiv \mathcal{S}\mathcal{L}\mathcal{G}_\Sigma(z)$ the class as given in Definition 7 satisfying the conditions given in (13) and (14).

**Remark 16.** Taking $\gamma = 0$, we get $\mathcal{S}\mathcal{L}\mathcal{G}_\Sigma^0(z) \equiv \mathcal{K}\mathcal{S}\mathcal{L}_\Sigma(z)$ the class as given in Definition 8 satisfying the conditions given in (15) and (16).

**Theorem 17.** Let $f$ given by (1) be in the class $\mathcal{S}\mathcal{L}\mathcal{G}_\Sigma^k(z)$. Then

$$|a_2| \leq \frac{k\sqrt{2k|\tau_k|}}{\sqrt{2(2 - \gamma)^2k - (4(2 - \gamma)^2 + (\gamma^2 - 5\gamma + 4)k^2)|\tau_k|}}$$
and

\[ |a_3| \leq \frac{k|\tau_k|}{2(3-2\gamma)(2k(2-\gamma)^2-(4(2-\gamma)^2+(\gamma^2-5\gamma+4)k^2k^2)\tau_k]}. \]

**Proof.** Let \( f \in \mathcal{S}\mathcal{L}\mathcal{G}_{\gamma,\Sigma}^{k}(\tilde{p}_k(z)) \) and \( g = f^{-1} \) given by (2). Considering (1) and (2), we have

\[ \left( \frac{zf'(z)}{f(z)} \right)^\gamma \left( 1 + \frac{zf''(z)}{f'(z)} \right)^{1-\gamma} = \tilde{p}_k(u(z)) \]

and

\[ \left( \frac{wg'(w)}{g(w)} \right)^\gamma \left( 1 + \frac{wg''(w)}{g'(w)} \right)^{1-\gamma} = \tilde{p}_k(v(w)), \]

where the function \( \tilde{p}_k \) is defined in (3), \( z, w \in \mathbb{D} \) and \( g \) is given by (2). We also have

\[ \left( \frac{zf'(z)}{f(z)} \right)^\gamma \left( 1 + \frac{zf''(z)}{f'(z)} \right)^{1-\gamma} = 1 + (2-\gamma)a_2 z + \left( 2(3-2\gamma)a_3 + \frac{1}{2} [(\gamma-2)^2 - 3(4 - 3\gamma)] a_2^2 \right) z^2 + \ldots \]

and

\[ \left( \frac{wg'(w)}{g(w)} \right)^\gamma \left( 1 + \frac{wg''(w)}{g'(w)} \right)^{1-\gamma} = 1 - (2-\gamma)a_2 w + \left( 8(1-\gamma) + \frac{1}{2} \gamma(\gamma+5)\right) a_2^2 - 2(3-2\gamma)a_3 \right) w^2 + \ldots \]

Equating the coefficients in (5) and (6), with (7)-(10), respectively, we get,

\[ (2-\gamma)a_2 = \frac{c_1 k \tau_k}{2}, \]

\[ 2(3-2\gamma)a_3 + \frac{1}{2} [(\gamma-2)^2 - 3(4 - 3\gamma)] a_2^2 = \frac{1}{2} \left( c_2 - \frac{c_1^2}{4} \right) k \tau_k + \frac{c_1^2}{4} (k^2 + 2) \tau_k^2, \]

and

\[ -(2-\gamma)a_2 = \frac{d_1 k \tau_k}{2}, \]

\[ -2(3-2\gamma)a_3 + [8(1-\gamma) + \frac{1}{2} \gamma(\gamma+5)] a_2^2 = \frac{1}{2} \left( d_2 - \frac{d_1^2}{4} \right) k \tau_k + \frac{d_1^2}{4} (k^2 + 2) \tau_k^2 \]

From (7) and (9), we have

\[ a_2 = \frac{c_1 k \tau_k}{2(2-\gamma)} = -\frac{d_1 k \tau_k}{2(2-\gamma)}, \]

which implies

\[ c_1 = -d_1 \]
and
\[ a_2^2 = \frac{(c_1^2 + d_1^2)k^2\tau_k^2}{8(2 - \gamma)^2}. \]

Now, by summing (8) and (10), we obtain
\[ (\gamma^2 - 3\gamma + 4)a_2^2 = \frac{1}{2}(c_2 + d_2)k\tau_k - \frac{1}{4}(c_1^2 + d_1^2)k\tau_k + \frac{1}{4}(c_1^2 + d_1^2)(k^2 + 2)\tau_k. \]

Proceeding similarly as in the earlier proof of Theorem 9 and using Lemma 5, we obtain
\[ |a_2| \leq \frac{k\sqrt{2k|\tau_k|}}{\sqrt{2(2 - \gamma)^2k - (4(2 - \gamma)^2 + (\gamma^2 - 5\gamma + 4)k^2)\tau_k}}. \] (11)

Now, so as to find the bound on \( |a_3| \), let's subtract from (8) and (10). So, we find
\[ 4(3 - 2\gamma)a_3 - 4(3 - 2\gamma)a_2^2 = \frac{1}{2}(c_2 - d_2)k\tau_k. \]

Hence, we get
\[ 4(3 - 2\gamma)|a_3| \leq 2k|\tau_k| + 4(3 - 2\gamma)|a_2|^2. \]

Then, in view of (11), we obtain
\[ |a_3| \leq \frac{k|\tau_k|[2(2 - \gamma)^2k - (4(2 - \gamma)^2 + (\gamma^2 - 13\gamma + 16)k^2)\tau_k]}{2(3 - 2\gamma)[2k(2 - \gamma)^2 - (4(2 - \gamma)^2 + (\gamma^2 - 5\gamma + 4)k^2)\tau_k]}. \]

**Remark 18.** By taking \( \gamma = 1 \) and \( \gamma = 0 \) in the above theorem, we have the initial Taylor coefficients \( |a_2| \) and \( |a_3| \) for the function classes \( \mathcal{SL}_k^{\Sigma}(\tilde{p}_k(z)) \) and \( \mathcal{KSL}_k^{\Sigma}(\tilde{p}_k(z)) \), as stated in Corollary 10 and Corollary 11 respectively. Further note that by taking \( k = 1 \) we have the initial Taylor coefficients \( |a_2| \) and \( |a_3| \) for the function classes \( \mathcal{SL}_k^{\Sigma}(\tilde{p}(z)) \) and \( \mathcal{KSL}_k^{\Sigma}(\tilde{p}(z)) \), as stated in Corollary 12 and Corollary 13 respectively.

4. Fekete-Szegő inequalities for the above function classes

Due to Zaprawa [16], we will give Fekete-Szegő inequalities for the above function classes in this section. The first theorem is the solution of the Fekete-Szegő problem in \( \mathcal{SLM}_{\alpha,\Sigma}(\tilde{p}_k(z)) \) and it looks like the following:

**Theorem 19.** Let \( f \) given by (1) be in the class \( \mathcal{SLM}_{\alpha,\Sigma}(\tilde{p}_k(z)) \) and \( \mu \in \mathbb{R} \). Then we have
\[
|a_3 - \mu a_2^2| \leq \begin{cases} 
\frac{k|\tau_k|}{2(1 + 2\alpha)}, & |\mu - 1| \leq \frac{4(1 + \alpha)[(1 + \alpha)k - (2(1 + \alpha) + \alpha k^2)\tau_k]}{8(1 + 2\alpha)k^2|\tau_k|}, \\
\frac{1 - \mu|k^3\tau_k^2}{(1 + \alpha)(1 + \alpha)k - (2(1 + \alpha) + \alpha k^2)\tau_k}, & |\mu - 1| \geq \frac{4(1 + \alpha)[(1 + \alpha)k - (2(1 + \alpha) + \alpha k^2)\tau_k]}{8(1 + 2\alpha)k^2|\tau_k|}.
\end{cases}
\]
Proof. From (30) and (32) we obtain
\begin{equation}
a_3 - \mu a_2^2 = (1 - \mu) \frac{k^3 \tau_k^2(c_2 + d_2)}{4(1 + \alpha) [(1 + \alpha)k - (2(1 + \alpha) + \alpha k^2)\tau_k]} + \frac{k\tau_k(c_2 - d_2)}{8(1 + 2\alpha)} \quad (1)
\end{equation}

\begin{equation}
= \left( \frac{(1 - \mu)k^3 \tau_k^2}{4(1 + \alpha) [(1 + \alpha)k - (2(1 + \alpha) + \alpha k^2)\tau_k]} + \frac{k\tau_k}{8(1 + 2\alpha)} \right) c_2 + \left( h(\mu) + \frac{k|\tau_k|}{8(1 + 2\alpha)} \right) d_2 \quad (2)
\end{equation}

So we have
\begin{equation}
a_3 - \mu a_2^2 = \left( h(\mu) - \frac{k|\tau_k|}{8(1 + 2\alpha)} \right) c_2 + \left( h(\mu) + \frac{k|\tau_k|}{8(1 + 2\alpha)} \right) d_2, \quad (3)
\end{equation}

where
\begin{equation}
h(\mu) = \frac{(1 - \mu)k^3 \tau_k^2}{4(1 + \alpha) [(1 + \alpha)k - (2(1 + \alpha) + \alpha k^2)\tau_k]}. \quad (4)
\end{equation}

Then, by taking modulus of (2), we conclude that
\begin{equation}
|a_3 - \mu a_2^2| \leq \begin{cases} 
\frac{k|\tau_k|}{2(1 + 2\alpha)}, & |\mu| \leq \frac{k|\tau_k|}{8(1 + 2\alpha)}, \\
\frac{4|h(\mu)|}{k|\tau_k|}, & |\mu| \geq \frac{k|\tau_k|}{8(1 + 2\alpha)}. 
\end{cases} \quad (5)
\end{equation}

Taking \( \mu = 1 \), we have the following corollary.

**Corollary 20.** If \( f \in SL\mathcal{M}^k_\alpha,\Sigma(\tilde{p}_k(z)) \), then
\begin{equation}
|a_3 - a_2^2| \leq \frac{k|\tau_k|}{2(1 + 2\alpha)}. \quad (6)
\end{equation}

The second theorem is the solution of the Fekete-Szegö problem in \( SL\mathcal{G}^k_{\gamma,\Sigma}(\tilde{p}_k(z)) \) and it looks like the following:

**Theorem 21.** Let \( f \) given by \( (1) \) be in the class \( SL\mathcal{G}^k_{\gamma,\Sigma}(\tilde{p}_k(z)) \) and \( \mu \in \mathbb{R} \). Then we have
\begin{equation}
|a_3 - \mu a_2^2| \leq \begin{cases} 
\frac{k|\tau_k|}{2(3 - 2\gamma)}, & |\mu| \leq \frac{2(2 - \gamma)^2k - (4(2 - \gamma)^2 + (\gamma^2 - 5\gamma + 4)k^2)\tau_k}{4(3 - 2\gamma)k^2|\tau_k|}, \\
\frac{2|1 - \mu|k^3 \tau_k^2}{2(2 - \gamma)^2k - (4(2 - \gamma)^2 + (\gamma^2 - 5\gamma + 4)k^2)\tau_k}, & |\mu| \geq \frac{2(2 - \gamma)^2k - (4(2 - \gamma)^2 + (\gamma^2 - 5\gamma + 4)k^2)\tau_k}{4(3 - 2\gamma)k^2|\tau_k|}. 
\end{cases} \quad (7)
\end{equation}

Taking \( \mu = 1 \), we have the following corollary.

**Corollary 22.** If \( f \in SL\mathcal{G}^k_{\gamma,\Sigma}(\tilde{p}_k(z)) \), then
\begin{equation}
|a_3 - a_2^2| \leq \frac{k|\tau_k|}{2(3 - 2\gamma)}. \quad (8)
\end{equation}
If we can take the parameter $\alpha = 0$ and $\alpha = 1$ in the Theorem 19 or $\gamma = 1$ and $\gamma = 0$ in the Theorem 21, we have the following the Fekete-Szegö inequalities for the function classes $\mathcal{SL}_k^\gamma(\tilde{p}_k(z))$ and $\mathcal{KSL}_k^\gamma(\tilde{p}_k(z))$, respectively.

**Corollary 23.** Let $f$ given by (1) be in the class $\mathcal{SL}_k^\gamma(\tilde{p}_k(z))$ and $\mu \in \mathbb{R}$. Then we have

$$|a_3 - \mu a_2^2| \leq \begin{cases} \frac{k|\tau_1|}{2}, & |\mu - 1| \leq \frac{k-2\tau_1}{2k^2|\tau_1|}, \\
\frac{|1-\mu|k^3\tau_1^2}{k-2\tau_1}, & |\mu - 1| \geq \frac{k-2\tau_1}{2k^2|\tau_1|}. \end{cases}$$

**Corollary 24.** Let $f$ given by (1) be in the class $\mathcal{KSL}_k^\gamma(\tilde{p}_k(z))$ and $\mu \in \mathbb{R}$. Then we have

$$|a_3 - \mu a_2^2| \leq \begin{cases} \frac{k|\tau_1|}{6}, & |\mu - 1| \leq \frac{2k-(k^2+4)\tau_1}{3k^2|\tau_1|}, \\
\frac{|1-\mu|k^3\tau_1^2}{2(2k-(k^2+4)\tau_1)}, & |\mu - 1| \geq \frac{2k-(k^2+4)\tau_1}{3k^2|\tau_1|}. \end{cases}$$

5. **Concluding Remarks and Observations**

In our present investigation, we have introduced new classes $\mathcal{SL}_k^{\alpha,\gamma}(\tilde{p}_k(z))$ and $\mathcal{SL}_k^{\beta,\gamma}(\tilde{p}_k(z))$ of bi-univalent functions in the open unit disk $U$. For the initial Taylor-Maclaurin coefficients of functions belonging to these classes, we have studied the problem of finding the upper bound associated with the Fekete-Szegö inequality. We have also considered several results which are closely related to our investigation in this paper.

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