On Intuitionistic Zero Gradations

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Abstract. In this paper, we introduce the concept of an intuitionistic zero gradation in Lowen fuzzy topological spaces. We study some relations between the notion of zero-dimensionality of fuzzy bitopological spaces and intuitionistic zero gradations and then prove that the property of being intuitionistic zero gradation is invariant under a strongly gradation preserving map.

Keywords and phrases: Fuzzy topology, Gradation of openness, Intuitionistic gradation of openness, Intuitionistic zero gradation, Zero-dimension.

1. INTRODUCTION

Fuzzy topological spaces is defined in somewhat different ways by Chang [4], Lowen [12] and by Hutton [10] over the system of fuzzy sets proposed by Zadeh in 1965 [16]. There was no fuzziness involved in the openness or closedness of fuzzy sets in all of them. Since a fuzzy set is a set without distinct boundaries, it is not always suitable to ask whether it is completely open or not. Regarding to this point, Chattopadhyay et al. [5] introduced the concept of fuzzy topology by the notion of gradation of openness as a function $\tau: I^X \rightarrow I$ (satisfying some axioms) such that for each $r \in [0,1]$, $\tau_r = \{\mu \in I^X: \tau(\mu) \geq r\}$ is a Chang fuzzy topology on $X$. In parallel with these studies, Atanassov [3] introduced the concept of intuitionistic fuzzy set that is a generalization of fuzzy set in Zadeh’s sense. Applications of intuitionistic fuzzy concepts have already been done by Atanassov and others in knowledge engineering, natural language, neural network, medical diagnosis etc [8]. Coker [6] introduced the concept of intuitionistic fuzzy topological spaces by the notion of intuitionistic fuzzy sets that is a generalization of fuzzy topological spaces in Chang’s sense. In [13] Samanta et al. introduced the concept of intuitionistic gradations of openness that is a generalization of the concept of gradation of openness defined by Chattopadhyay.

In this paper, we introduce a concept of intuitionistic zero gradation that is the first basic step to develop the theory of fuzzy inductive dimension on the intuitionistic fuzzy topological spaces. We also prove that the concept of intuitionistic zero gradation is invariant under a strongly gradation preserving map. Some examples of zero- and nonzero gradations are also presented. It is worth pointing out that when we refer to topological dimensions, we use the notation and definitions in [9]. In particular, ind represents the small inductive dimension.

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2. PRELIMINARIES

A fuzzy set in a nonempty set $X$ is a function (membership function) from $X$ into the closed unit interval $I = [0, 1]$. A fuzzy set $\mu$ in $X$ is called crisp if $\mu(x) \in \{0, 1\}$. The family of all fuzzy sets on $X$ is denoted by $\mathcal{F}X$. For every fuzzy subset $\mu \in \mathcal{F}X$, 2010 Mathematics Subject Classification. 54A40, 54F45.

the support of $\mu$ is defined by $\text{supp}(\mu) = \{x \in X; \mu(x) > 0\} = \mu^{-1}(0, 1)$. A fuzzy set $\mu$ is said to be contained in a fuzzy set $\eta$ if $\mu(x) \leq \eta(x)$ for each $x$ in $X$, denoted by $\mu \leq \eta$. The union and intersection of a family of fuzzy sets is defined by $\bigvee \mu = \sup(\mu)$ and $\bigwedge \mu = \inf(\mu)$, respectively.

Definition 2.1. For every $x \in X$ and every $\alpha \in (0, 1)$, the fuzzy set $x_\alpha$ with membership function $x_\alpha(y) = \begin{cases} \alpha & y = x \\ 0 & y \neq x \end{cases}$ is called a fuzzy point. $x_\alpha$ is said to be contained in a fuzzy set $\mu$, denoted by $x_\alpha \in \mu$, if $\alpha < \mu(x)$ [15].

Any point $x_\alpha$ is called a crisp point. We denote a constant fuzzy set whose unique value is $c \in [0, 1]$ by $c_x$. Note that any fuzzy set is the union of all points which are contained in it.

Definition 2.2. (See [12] ) A fuzzy topology is a family $\mathcal{E}$ of fuzzy sets in $X$ which satisfies the following conditions

(i) $\forall c \in I, c_X \in \mathcal{E}$,

(ii) $\forall \mu, \nu \in \mathcal{E} \Rightarrow \mu \wedge \nu \in \mathcal{E}$,

(iii) $\forall \left(\mu_j\right)_{j \in J} \subseteq \mathcal{E} \Rightarrow \bigvee_{j \in J} \mu_j \in \mathcal{E}$

$\mathcal{E}$ is called a Lowen fuzzy topology for $X$, and the pair $(X, \mathcal{E})$ is called a Lowen fuzzy topological space. Open sets, closed sets and clopens are defined as usual.

In Chang’s definition of fuzzy topology the condition (i) should be replaced by (i)’ $0_X, 1 \in \mathcal{E}$. A base or subbase for a fuzzy space have the same meaning in the classic sense.

Definition 2.3. (See [11] ) The triplet $(X, T, T^*)$ is called a fuzzy bitopological space where $T$ and $T^*$ are Lowen fuzzy topologies on $X$. The ordered pair $(T, T^*)$ is called a fuzzy bitopology on $X$. A fuzzy bitopological space $(X, T, T^*)$ is said to be inclusive if $T \subseteq T^*$.

Let $f: (X, T, T^*) \rightarrow (Y, S, S^*)$ be a mapping, where $(X, T, T^*)$ and $(Y, S, S^*)$ are two bitopological spaces of fuzzy subsets. Then $f$ is said to be continuous if $f: (X, T) \rightarrow (Y, S)$ and $f: (X, T^*) \rightarrow (Y, S^*)$ are continuous [1].
Definition 2.4. (See [5] ) A fuzzy topological space is a pair \((X, \tau)\), where \(\tau : \mathbb{I}^X \rightarrow \mathbb{I}\) is a mapping satisfying the following properties

(i) \(\tau(0_X) = \tau(1_X) = 1\),

(ii) \(\tau(\mu_1 \cap \mu_2) \geq \tau(\mu_1) \wedge \tau(\mu_2)\),

(iii) \(\tau(\bigcup_{\mu_i \in \mathbf{F}}) \geq \bigwedge_{\mu_i \in \mathbf{F}} \tau(\mu_i)\).

The map \(\tau\) is called a gradation of openness or a fuzzy topology on \(X\). The real number \(\tau(\mu)\) is the degree of openness of the fuzzy subset \(\mu \in \mathbb{I}^X\) that this degree may range from 0 "completely nonopen set" to 1 "completely open set". For each \(r \in [0, 1]\), \(\tau_r = \{\mu \in \mathbb{I}^X : \tau(\mu) \geq r\}\) is a Chang fuzzy topology on \(X\) that is called the \(r\)-level Chang fuzzy topology on \(X\) with respect to the gradation of openness \(\tau\).

Definition 2.5. (See [3] ) An intuitionistic fuzzy set \(A\) in a set \(X\) is an object having the form

\[ A = \{(x, \mu_A(x), \gamma_A(x)) : x \in X\}, \]

where the function \(\mu_A : X \rightarrow \mathbb{I}\) and \(\gamma_A : X \rightarrow \mathbb{I}\) denote the degree of membership and the degree of nonmembership of each element \(x \in X\) to the set \(A\), respectively, and \(0 \leq \mu_A(x) + \gamma_A(x) \leq 1\) for each \(x \in X\).

It should be noted that the concept of the boundary of a fuzzy set is essential in the definition of inductive dimension. Tarres et al. [7] proposed a definition of fuzzy boundary of a fuzzy set. Throughout this paper we use their definition.

Definition 2.6. (See [7] ) Let \(\mu\) be a fuzzy set in a Lowen fuzzy topological space \(X\). The fuzzy boundary of \(\mu\), denoted by \(\text{Fr}(\mu)\), is defined as the infimum of all closed fuzzy sets \(\sigma\) in \(X\) with the property \(\sigma(x) \geq \mu(x)\) for all \(x \in X\) for which \(\mu(x) - \sigma(x) > 0\).

It is ready to see that a fuzzy set \(\mu\) is clopen if and only if \(\text{Fr}(\mu) = \emptyset_X\). If the definition of the Adnadjiev’s dimension function [2] is particularized, in the case of zero dimensionality, the following definition is obtained.

Definition 2.7. (See [7] ) A Lowen fuzzy topological space \((X, \delta)\) is called zero-dimensional and it is denoted by \(\text{ind}(X) = 0\) if for each fuzzy point \(x_\alpha\) in \(X\) and every fuzzy open set \(\mu\) containing \(x_\alpha\), there exists an open fuzzy set \(\sigma\) in \(X\) with \(\text{Fr}(\sigma) = 0_X\) such that \(x_\alpha \in \sigma \leq \mu\).

Example 2.8. Let \(\delta\) be the Lowen fuzzy topology on \(X = [0, 1]\) with subbase

Where

\[
\{c_X : c \in [0, 1]\} \cup \{\mu\},
\]

\[
\mu(x) = \begin{cases} 
0 & 0 \leq x \leq \frac{1}{2} \\
\frac{1}{3} & \frac{1}{2} < x \leq 1 
\end{cases}
\]

Clearly any non-constant open fuzzy sets has the form
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\[ \nu(x) = \begin{cases} 
  a & 0 \leq x \leq \frac{1}{2} \\
  b & \frac{1}{2} < x \leq 1 
\end{cases}, \quad 0 \leq a \leq b \leq \frac{1}{3} \]

There exists no fuzzy clopen set \( \sigma \) such that \( \sigma \leq \mu \) because the constant fuzzy sets are the only clopen fuzzy sets. Thus \( \text{ind}(X) \neq 0 \).

It is ready to see that for every nonempty set \( X \) the fuzzy topological space \( (X, \delta) \) is zero-dimensional, where \( \delta = \{ c_X : c \in [0, 1] \} \). A bitopological space \( (X, T, T^*) \) is called zero-dimensional if \( \text{ind}(X, T) = \text{ind}(X, T^*) = 0 \).

Remark 2.9. Let \( (X, \delta) \) be a Lowen fuzzy topological space and \( Y \subseteq X \), then the family \( \delta_Y = \{ \mu|_Y : \mu \in \delta \} \) is a fuzzy topology for \( Y \) and \( (Y, \delta_1) \) is called a subspace of \( (X, \delta) \). If \( \text{ind}(X) = 0 \), then \( \text{ind}(Y) = 0 \). Note that restriction of every clopen subset of \( X \) on subspace \( Y \) is a clopen set in \( Y \).

3. INTUITIONISTIC ZERO GRADATIONS

In Lowen and Chang’s definition of fuzzy topology, fuzziness in the concept of openness of a fuzzy subset has not been considered. The initial request is that the topology be a fuzzy subset of a power set of fuzzy subsets. For this purpose, Chattopadhyay et al. gave an axiomatic definition in [5] so called a gradation of openness and Samanta et al. [13] extended it to the concept of intuitionistic gradations of openness. Here we use the following modified definition.

Definition 3.1. Let \( X \) be a nonempty set and \( \tau, \tau^* : \mathcal{P}^X \to \mathbb{I} \) be two mappings satisfying

(i) \( \tau(c_X) = 1 \) and \( \tau^*(c_X) = 0 \), for all \( c \in [0, 1] \).

(ii) \( \tau(\mu) + \tau^*(\mu) \leq 1 \), for all \( \mu \in \mathcal{P}^X \).

(iii) \( \tau(\mu_1 \land \mu_2) \geq \tau(\mu_1) \land \tau(\mu_2), \tau^*(\mu_1 \land \mu_2) \leq \tau^*(\mu_1) \lor \tau^*(\mu_2), \mu_1, \mu_2 \in \mathcal{P}^X \)

(iv) \( \tau(\bigcup_{i \in I} \mu_i) \geq \bigvee_{i \in I} \tau(\mu_i), \tau^*(\bigcup_{i \in I} \mu_i) \leq \bigvee_{i \in I} \tau^*(\mu_i), \mu_i \in \mathcal{P}^X \)

The ordered pair \( (\tau, \tau^*) \) is called intuitionistic gradation of openness or intuitionistic fuzzy topology on \( X \) and the triplet \( (X, \tau, \tau^*) \) is called an intuitionistic fuzzy topological space. The mappings \( \tau \) and \( \tau^* \) are interpreted as gradation of openness and gradation of nonopenness, respectively. Suppose that \( (\tau_1, \tau_1^*) \) and \( (\sigma, \sigma^*) \) be two intuitionistic fuzzy topologies on a given nonempty set \( X \). If \( (\tau, \tau^*) \supseteq (\sigma, \sigma^*) \) i.e., \( \tau(\mu) \geq \sigma(\mu) \) and \( \tau^*(\mu) \leq \sigma^*(\mu) \) for every \( \mu \in \mathcal{P}^X \), we say that \( (\tau, \tau^*) \) is finer than \( (\sigma, \sigma^*) \).

Example 3.2. Let \( X = [0, 1] \), we consider two fuzzy sets \( \eta_1 \) and \( \eta_2 \) in \( X \) as follows

\[ \eta_1(x) = \begin{cases} 
  0 & x \in [0, \frac{1}{2}] \\
  1 & x \in \left( \frac{1}{2}, 1 \right] 
\end{cases} \quad \text{and} \quad \eta_2(x) = \begin{cases} 
  a & x \in [0, \frac{1}{2}] \\
  b & x \in \left( \frac{1}{2}, 1 \right] 
\end{cases} \]
where $0 \leq a \leq b \leq \frac{1}{2}$, define $\tau, \tau^*: \mathcal{P}^X \to \mathbb{I}$ by

$$
\tau(\mu) = \begin{cases} 
1 & \mu = c_X \\
0.5 & \mu = \eta_1, \eta_2 \\
0 & \text{otherwise}
\end{cases}, \quad \tau^*(\mu) = \begin{cases} 
0 & \mu = c_X \\
0.2 & \mu = \eta_1, \eta_2 \\
1 & \text{otherwise}
\end{cases}
$$

Then $(\tau, \tau^*)$ is an intuitionistic gradation of openness on $X$.

**Definition 3.3.** The ordered pair $(\xi, \xi^*)$ of mappings from $\mathcal{P}^X$ to $\mathbb{I}$ is called an intuitionistic gradation of closedness on $X$ if it satisfies:

(i) $\xi(c_X) = 1$ and $\xi^*(c_X) = 0$, for all $c \in \mathbb{I}$.

(ii) $\xi(\mu) + \xi^*(\mu) \leq 1$, for all $\mu \in \mathcal{P}^X$.

(iii) $\xi(\mu_1 \cup \mu_2) \geq \xi(\mu_1) \land \xi^*(\mu_2), \xi^*(\mu_1 \cup \mu_2) \leq \xi^*(\mu_1) \lor \xi^*(\mu_2), \mu_1, \mu_2 \in \mathcal{P}^X$.

(iv) $\xi(\bigcap_{i \in J} \mu_i) \geq \bigwedge_{i \in J} \xi(\mu_i), \xi^*(\bigcap_{i \in J} \mu_i) \leq \bigvee_{i \in J} \xi^*(\mu_i), \mu_i \in \mathcal{P}^X$

**Example 3.4.** Define the mappings $\xi, \xi^*: \mathcal{P}^X \to \mathbb{I}$ as

$$
\xi(\mu) = \begin{cases} 
\alpha & \mu = c_X \\
\beta & \text{otherwise}
\end{cases}, \quad \xi^*(\mu) = \begin{cases} 
0 & \mu = c_X \\
1 & \text{otherwise}
\end{cases}
$$

where $\alpha, \beta \in \mathbb{I}$ and $\alpha + \beta \leq 1$. Then $(\xi, \xi^*)$ is an intuitionistic gradation of closedness on $X$.

**Proposition 3.5.** For each $r \in (0, 1]$, $\tau_r = \{\mu \in \mathcal{P}^X : \tau(\mu) \geq r\}$ and $\tau^*_r = \{\mu \in \mathcal{P}^X : \tau^*(\mu) \leq 1 - r\}$ are Lowen fuzzy topologies on $X$, where $(X, \tau, \tau^*)$ is an intuitionistic fuzzy topological space.

**Proof.** At the first we show that $\tau^*_r$ is a Lowen fuzzy topology on $X$. Since $\tau^*(c_X) = 0 \leq 1 - r$, so $c_X \in \tau^*_r$. Let $\mu_1$ and $\mu_2$ are two fuzzy sets in $\tau^*_r$. Hence $\tau^*(\mu_1) \leq 1 - r$ and $\tau^*(\mu_2) \leq 1 - r$. Since $\tau^*$ is a gradation of nonopenness, so $\tau^*(\mu_1 \cap \mu_2) \leq \tau^*(\mu_1) \lor \tau^*(\mu_2) \leq 1 - r$. Therefore, $\mu_1 \cap \mu_2 \in \tau^*_r$. To check the third condition, let $\mu_i$ be a family of fuzzy sets in $\tau^*_r$ such that $\tau^*\mu_i \leq 1 - r$. By hypothesis $\tau^*(\bigcup_i \mu_i) \leq \bigvee_i \tau^*(\mu_i) \leq 1 - r$. Hence, $\bigcup_i \mu_i \in \tau^*_r$. Similarly, $\tau_r$ is a Lowen fuzzy topology on $X$.

It is ready to see that the families $\{\tau_r\}$ and $\{\tau^*_r\}$ are two descending families of Lowen fuzzy topology on $X$ such that $\tau_r = \bigcap_{s \leq r} \tau_s$ and $\tau^*_r = \bigcap_{s \leq r} \tau^*_s$, $\forall r \in (0, 1]$. Since for each $r \in (0, 1], \tau_r \subset \tau^*_r$, then the ordered pair $(\tau_r, \tau^*_r)$ is an inclusive
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Lowen fuzzy bitopology on $X$ and $(X, \tau_r, \tau^*_r)$ will be called an $r$-level inclusive fuzzy bitopological space.

Proposition 3.6. Let $(X, T, T^*)$ be a Lowen fuzzy bitopological space. Define for each $r \in (0, 1]$, two mappings $T^r, (T^*)^r: \mathbb{I}^X \to \mathbb{I}$ by

$$T^r(\mu) = \begin{cases} 1 & \mu = c_X \\ r & \mu \in T, \mu \neq c_X, \\ 0 & \text{otherwise} \end{cases}$$

and

$$(T^*)^r(\mu) = \begin{cases} 0 & \mu = c_X \\ 1 - r & \mu \in T^*, \mu \neq c_X, \\ 1 & \text{otherwise} \end{cases}$$

Then $(T^r, (T^*)^r)$ is an intuitionistic gradation of openness on $X$ such that $(T^r)_r = T$ and $((I^rJ^r)_r = I^r$.

Proof. First notice that $T^r(c_X) = 1$ and $(T^*)^r(c_X) = 0$ and clearly $T^r(\mu) + (T^*)^r(\mu) \leq 1$ for all $\mu \in \mathbb{I}^X$. The rest of the proof is similar to [13, Theorem 2.18].

The ordered pair $(T^r, (T^*)^r)$ is called $r$-th intuitionistic gradation of openness on $X$ and $(X, T^r, (T^*)^r)$ is called $r$-th graded intuitionistic fuzzy topological space.

Definition 3.7. Let $(\tau, \tau^*)$ be an intuitionistic gradation of openness on $X$ and $Y \subseteq X$. Define two mappings $\tau_Y, \tau^*_Y: \mathbb{I}^Y \to \mathbb{I}$ by $\tau_Y(\mu) = \bigvee\{\tau(\eta) : \eta \in \mathbb{I}^X, \eta|_Y = \mu\}$ and $\tau^*_Y(\mu) = \bigwedge\{\tau^*(\eta) : \eta \in \mathbb{I}^X, \eta|_Y = \mu\}$

for all $\mu \in \mathbb{I}^Y$. Then $(\tau_Y, \tau^*_Y)$ is an intuitionistic gradation of openness on $Y$. We say that $(\tau_Y, \tau^*_Y)$ is a subgradation of $(\tau, \tau^*)$. It is obvious that $\tau_Y(\mu) \geq \tau(\mu_X), \tau^*_Y(\mu) \leq \tau^*(\mu_X)$.

Definition 3.8. Let $(X, \tau, \tau^*)$ and $(Y, \sigma, \sigma^*)$ be two intuitionistic fuzzy topological spaces and $f: X \to Y$ be a mapping. Then $f$ is called a gradation preserving map (strongly gradation preserving map) if for each $\mu \in \mathbb{I}^Y$, $\sigma(\mu) \leq \tau(f^{-1}(\mu))$ and $\sigma^*(\mu) \geq \tau^*(f^{-1}(\mu))$.

For example, let $(\tau, \tau^*)$ and $(\sigma, \sigma^*)$ are two intuitionistic fuzzy topologies on a given set $X$ such that $(\tau, \tau^*)$ is finer than $(\sigma, \sigma^*)$. Thus, the identity map $i: (X, \tau, \tau^*) \to (X, \sigma, \sigma^*)$ is a gradation preserving map.
Definition 3.9. Two intuitionistic gradations of openness \((\tau, \tau^*)\) and \((\sigma, \sigma^*)\) on \(X\) are called equal and it is denoted by \((\sigma, \sigma^*) \approx (\tau, \tau^*)\) if the identity map be a strongly gradation preserving map from \((X, \tau, \tau^*)\) to \((X, \sigma, \sigma^*)\). In this case \(\sigma_r = \tau_r\) and \(\sigma^*_r = \tau^*_r\) for all \(r \in I\).

It is worth noting that the constant functions between Chang fuzzy topological spaces are not necessarily continuous. Therefore, the function \(f\) in [13, Theorem 4.3] cannot be constant unlike the next Proposition.

Proposition 3.10. Let \((X, \tau, \tau^*)\) and \((Y, \sigma, \sigma^*)\) be two intuitionistic fuzzy topological spaces and \(f: X \rightarrow Y\) be a mapping. Then \(f\) is a gradation preserving map if and only if

\[
f: (X, \tau_r, \tau^*_r) \rightarrow (Y, \sigma_r, \sigma^*_r)
\]

is continuous for all \(r \in (0, 1]\).

Proof. This is straightforward.

Definition 3.11. Let \((X, \tau, \tau^*)\) be an intuitionistic fuzzy topological space. Then \((\tau, \tau^*)\) is called an intuitionistic zero gradation if the inclusive fuzzy bitopological space \((X, \tau_r, \tau^*_r)\) is zero-dimensional for all \(r \in (0, 1]\).

Example 3.12. Let \(X = \mathbb{R}\) define the mappings \(\tau, \tau^*: \mathbb{R} \rightarrow I\) by

\[
(\mu) = \begin{cases} 
1 & \mu = c_X \\
0 & \text{otherwise}
\end{cases}
\quad \text{and} \quad
(\tau^*(\mu)) = \begin{cases} 
0 & \mu = c_X \\
1 & \text{otherwise}
\end{cases}
\]

Since \(\tau_r = \tau^*_r = [c_X: c \in I]\), it is ready to see that \(\text{ind}(X, \tau_r) = \text{ind}(X, \tau^*_r) = 0\), then \((\tau, \tau^*)\) is an intuitionistic zero gradation.

Example 3.13. Let \((I, \tau, \tau^*)\) be an intuitionistic fuzzy topological space, where \(\tau\) and \(\tau^*\) are the same as Example 3.2. Then \((\tau, \tau^*)\) is not an intuitionistic zero gradation because the Lowen fuzzy topological spaces \((I, \tau_{\emptyset^4})\) and \((I, \tau^*_{\emptyset^4})\) are not zero-dimensional.

Theorem 3.14. Let \((X, \tau, \tau^*)\) be an intuitionistic fuzzy topological space. Let \((\tau, \tau^*)\) be an intuitionistic zero gradation and \(Y \subset X\). Then the subgradation \((\tau_Y, \tau^*_Y)\) of \((\tau, \tau^*)\) is an intuitionistic zero gradation.

Proof. We show that the Lowen fuzzy topological spaces \((Y, (\tau_Y)_r)\) and \((Y, (\tau^*_Y)_r)\) are zero-dimensional for all \(r \in (0, 1]\). To prove zero dimensionality of \((Y, (\tau^*_Y)_r)\) take an arbitrary fuzzy point \(y_\alpha\) in \(Y\) and fuzzy open set \(\mu\) in \((\tau^*_Y)_r\) containing \(y_\alpha\). Note that \(\tau^*_Y(\mu) \leq 1 - r\). Now consider the fuzzy point \(y_\alpha\) as a fuzzy point in \(X\). There exists an open fuzzy set \(\mu'\) in \(\tau^*_r\) such that \(\tau^*_r(\mu') \leq 1 - r\) and \(\mu'|_Y = \mu\). Since \((X, \tau^*_r)\) is zero-dimensional, there exists a fuzzy open set \(\eta'\) such that \(F^r(\eta') = 0_X\).
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\[ y_{\alpha} \in \eta' \leq \mu \text{ and } \tau^*(\eta') \leq 1 - r \text{. Put } \eta = \eta' | Y, \text{ then } y_{\alpha} \in \eta \leq \mu \text{. Similarly, } (Y, (\tau Y)_r) \text{ is zero-dimensional.} \]

Proposition 3.15. The intersection of a family of intuitionistic zero gradations on a given set \( X \) is an intuitionistic zero gradation.

Proof. Let \( \{ (\tau_i, \tau_i^*) \} \) be an arbitrary family of intuitionistic zero gradations on \( X \). One can readily check that \( \bigcap_i (\tau_i, \tau_i^*) = (\bigwedge_i \tau_i, \bigvee_i \tau_i^*) \) is an intuitionistic fuzzy topology on \( X \). Because for all \( r \in (0, 1] \) the \( r \)-level Lowen fuzzy topologies \( (\bigwedge i \tau_i)_r = \{ \mu \in \mathbb{P}^X : \bigwedge_i \tau_i (\mu) \geq r \} \) and

\[ (\bigvee_i \tau_i^*_r) = \{ \mu \in \mathbb{P}^X : \bigvee_i \tau_i^*_r (\mu) \leq 1 - r \} \]

are subspaces of \( (\tau_i)_r \) and \( (\tau_i^*)_r \), respectively, we conclude that \( \bigcap_i (\tau_i, \tau_i^*) \) is an intuitionistic zero gradation by Remark 2.9.

Note that the converse of the Proposition 3.15 is not necessarily hold. For example, the intersection of two intuitionistic gradations in Example 3.12 and Example 3.13.

3.2. Is an intuitionistic zero gradation.

Theorem 3.16. Let \( f : (X, \tau, \tau^*) \rightarrow (Y, \sigma, \sigma^*) \) be a bijective strongly gradation preserving map. If \( (\tau, \tau^*) \) is an intuitionistic zero gradation, \( (\sigma, \sigma^*) \) will be an intuitionistic zero gradation.

Proof. First, we show that Lowen space \( (Y, \sigma^*_r) \) is zero-dimensional for all \( r \in (0, 1] \). Take a fuzzy point \( y_{\alpha} \) in \( Y \) and let \( \mu \) be a fuzzy set containing \( y_{\alpha} \) which \( \sigma^*_r (\mu) \leq 1 - r \), for all \( r \in (0, 1] \). \( f^{-1}(y_{\alpha}) \) is the fuzzy point \( x_{\alpha} \) in \( X \) where \( f^{-1}(y) = x \) and \( f^{-1}(\mu) \) is a fuzzy open set in \( X \) respect to \( \tau^*_r \). Note that \( \tau^*_r (f^{-1}(\mu)) = \sigma^*_r (\mu) \leq 1 - r \). Thereis a clopen fuzzy set \( \eta' \)

\[ (X, \tau^*_r) \text{ Put } \eta = f(\eta') \text{in } \tau^* \text{ such that } x_{\alpha} \in \eta \leq f^{-1}(\mu) \text{ by zero dimensionality of } \]

Thus \( y_{\alpha} \in \eta \leq \mu \) and \( \eta \in \sigma^* \) because \( \sigma^* (\eta) = \tau^* (f^{-1}(\eta)) = \tau^* (\eta') \leq 1 - r \). Similarly, \( (Y, \sigma^*_r) \) is zero-dimensional.

Remark 3.17. The strongly condition for the gradation preserving map \( f \) in the Theorem 3.16 cannot be removed. Consider the identity map \( i : (X, \tau, \tau^*) \rightarrow (X, \tau, \tau^*) \), where \( X = I \),

the intuitionistic gradation \( (\tau, \tau^*) \) is the same as Example 3.2 and \( (I, \tau^*) \) is defined by the rule \( \tau(\mu) = 1 \), \( \tau^*(\mu) = 0 \), \( \forall \mu \in \mathbb{P}^X \). Because \( \tau(\eta_1) \neq \tau(\eta_2) \), then the identity map \( i \) is not a strongly gradation preserving map.

Note that \( (I, \tau^*) \) is an intuitionistic zero gradation and \( (\tau, \tau^*) \) is not an intuitionistic zero gradation.

Proposition 3.18. Let \( (\tau, \tau^*) \) be an intuitionistic zero gradation on \( X \) and \( (\sigma, \sigma^*) \approx \)
(τ, τ*), then (σ, σ*) is an intuitionistic zero gradation.

Proof. The proof is obvious since \{στ : r ∈ [0, 1]\} = \{ττ : r ∈ [0, 1]\} and \{στ* : r ∈ [0, 1]\}.

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