One–Dimensional Solute Transport for an Input against the Flow in a Homogeneous Finite Porous Media

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Abstract-A theoretical model comprising advection-dispersion equation with temporal seepage velocity, dispersion coefficient and time dependent pulse type input of uniform nature applied against the flow is studied in a finite porous domain. Input concentration is any continuous smooth function of time acts up to some finite time and then eliminated. Concentration gradient at other boundary is proportional to concentration. Dispersion is proportional to seepage velocity. Interpolation method is applied to reduce the input function into a polynomial. Certain transformations are utilized to reduce the variable coefficient advection-dispersion equation into constant coefficient. The Laplace Transform Technique is applied to get the solution of advection dispersion equation. Two different functions of input are discussed to understand the utility of the present study. Obtained result is demonstrated graphically with the help of numerical example.

Keywords Advection, Dispersion, Porous Medium, Interpolation, Laplace Transformation Technique.

1. Introduction

The advection-dispersion equation (ADE) commonly used for transport of solute in porous media comprises advection and hydrodynamic dispersion where former is controlled by the Darcy's law and later by the combination of mechanical and molecular diffusion which accounts for contaminant arising/spreading stimulated by velocity variations. Advection-dispersion equation may be defined for scale or time or both dependent dispersion/seepage velocity, homogeneous/heterogeneous medium including presence of decay and production of solute depending upon geological formation. Solutions of the ADEs can be obtained analytically or numerically using various methods. Analytical solutions are usually applicable for ideal geometrical conditions while numerical solutions are more flexible comparison to analytical solution to deal with real life problems consisting of non- ideal geometric conditions. On the other hand, analytical solutions is far accurate and dependable comparison to numerical solutions those hardly free from errors. A plenty of analytical solutions comprising various initial boundary and geometrical conditions through one/two/three-dimensional transport problem in porous formation which may be finite /infinite are published in literature up to the date. In the study of solute transport phenomena [1] explained that conductivity is not only responsible for spatial variation in groundwater velocity. An analytical solution was proposed for a solute transport problem with scale dependent dispersion in a heterogeneous porous media [2]. An aspect of solute transport phenomena was established with the fact that hydrodynamic dispersion coefficients are non-linear function of the seepage velocity [3]. Analytical solutions may be obtained by using several techniques according to the nature of the problem. A two dimensional analytical solution was evaluated considering the solute transport for a bounded aquifer by adopting Fourier analysis and Laplace Transform [4]. [5] conceptualized an overlapping control volume method for transient solute transport problems in groundwater to obtain the numerical solution of transport problems. Streamline method was applied in solving the solute transport problem in a single fracture and the method was validated with experimental data [6]. Some analytical solutions were derived for the ADE in cylindrical coordinates by using a Bessel function expansion [7]. Laplace Transformation Technique (LTT) was explored to evaluate analytical solutions of transport problems [8] [9] [10]. First two obtained solutions for pulse type input in one-dimensional transport while last one addressed the problem consisting of space dependent dispersion, velocity and retardation. An analytical result for solute transport problem in a heterogeneous porous medium was proposed to analyze scale dependent dispersion with linear isotherm [11]. By using generalized integral transform technique (GITT) [12] obtained solution for finite one-dimensional domain for dispersion coefficient and velocity which are proportional to non-homogeneous linear expression in position variable while in another study an analytical solution of ADE was evaluated in multi-layer porous media [13]. Two–dimensional solution is obtained for a finite domain porous medium with pulse type input by considering it at the top layer of the outer boundary and at the intermediate portion in the aquifer [14]. Using Green's function an analytical solution of one dimensional porous medium for instantaneous and continuous point source taking dispersion and velocity mutually proportional was developed in groundwater and riverine flow [15]. In numerical solutions, a two-dimensional model based on numerical simulations, equi-concentration lines presented approximate description of time-dependent transport [16]. Finite-volume method has been used for solving the advection and dispersion processes of the virus transport equation describing the movement of virus in one-dimensional unsaturated porous media [17]. The aquifers are always of finite length and therefore the study of solute transport phenomena in finite domain has been much valuable to deal with real world problem. Solute transport phenomena for finite domain was studied to understand the effect of periodic dispersion, velocity along with periodic input [18] [19].

In present paper, interpolation method is used for reducing generalize input into a polynomial for a finite domain study. Laplace transformation Technique is used to get solution of solute transport phenomena with temporal dispersion and for the input which is poised against the flow.

2. Mathematical Formulation and Solution of the Problem

In the present study, solute transport is assumed as one-dimensional on a horizontal plane in a saturated finite length porous medium domain. The governing equation of solute transport in porous media which is a parabolic type partial differential equation is derived on basis of mass conservation and Fick's law of diffusion may be written as follows [20],

$$R\frac{\partial c}{\partial t}=\frac{\partial}{\partial x}\left(D(x,t)\frac{\partial c}{\partial x}-u(x,t)\ c\right) \quad (1)$$

where $c\left[ML^{-3}\right]$,$u\left[LT^{-1}\right]$, $D\left[L^2T^{-1}\right]$and R are representing solute concentration, seepage velocity, dispersion coefficient and retardation factor respectively at position $x[L]$ and at time $t[T]$. The dispersion coefficient, combined effect of molecular diffusion and mixing in the axial direction, is assumed to be proportional to seepage velocity u (Yim and Mohsen,1992). The dispersion coefficient and seepage velocity both are time dependent and defined as $u(x,t)=u_0(1+mt)^{-1}$, $D=D_0(1+mt)^{-1}$. Retardation R is R_0 and where $D_0\left[L^2T^{-1}\right], u_0\left[LT^{-1}\right]$ and R_0 are constants. Here unsteady parameter $m\left[T^{-1}\right]$ regulates the dependency of the dispersion and seepage velocity on time .Therefore, Eq.(1) may be re-written as;

$$R_0\frac{\partial c}{\partial t}=\frac{\partial}{\partial x}\left(D_0(1+mt)^{-1}\frac{\partial c}{\partial x}-u_0(1+mt)^{-1}c\right) \quad (2)$$

The geological formation which is assumed to be of finite length along horizontal direction is initially polluted and its dependency may be defined as sine hyperbolic function of space. Input at one end is considered against the flow and may be defined as any continuous smooth function of time while concentration gradient at other end of finite domain is supposed proportional to concentration. In order to formulate the proposed problem mathematically, the initial and boundary condition may be written as;

$$c(x,t)=c_i\sinh(\alpha x);\, L_1\le x\le L,\ t=0 \quad (3)$$

$$c(x,t)=\begin{cases} c_0F(m't);\, 0<t\le t_0 & (4a)\\ & x=L\\ 0\,;\, t>t_0 & (4b)\end{cases}$$

$$\frac{\partial c(x,t)}{\partial x}=\frac{u_0}{2D_0}c\ ;\text{as } x=L_1 \quad (5)$$

where, c_0 and c_i are the reference and resident concentrations respectively. $m'[T^{-1}]$ is an unsteady parameter and $\alpha[L^{-1}]$ is a constant on which initial concentration depends. Dimensionless $F(m't)(\ge 0)$is continuous , smooth (differentiable) and bounded function in a time domain $\left[0,t_0\right]$. Weirstrass approximation theorem says that any continuous function on a bounded interval can be uniformly approximated by polynomial function. Since $F(m't)$ is continuous and bounded in domain $\left[0,t_0\right]$, following Weirstrass approximation theorem, $F(m't)$may be written as interpolation polynomial in t of degree n. Hence Eqs. (3-5) may be written as;

$$c(x,t)=c_i\sinh(\alpha x);\, L_1\le x\le L,\ t=0 \quad (6)$$

$$c(x,t)=\begin{cases} c_0G_n(t);\, 0<t\le t_0 & (7a)\\ & x=L\\ 0\,;\, t>t_0 & (7b)\end{cases}$$

$$\frac{\partial c(x,t)}{\partial x}=\frac{u_0}{2D_0}c\ ;\text{at } x=L_1\ ,\ t\ge 0 \quad (8)$$

where, dimensionless quantity $G_n(t)$ which is interpolation polynomial may be defined as;

$$G_n(t) = a_0 + a_1 t + a_2 t^2 + \dots\dots\dots + a_n t^n \qquad (9)$$

and a_i's are constants. For reducing the Eq.(2) into a constant coefficient, a new time variable T is introduced with following transformation [21]

$$T = \int_0^t (1+mt)^{-1} dt \quad \text{or} \quad T = \frac{1}{m}\log(1+mt)$$

and $t = \frac{1}{m}\left(e^{mT} - 1\right)$ (10)

With transformation Eq. (10), Eq. (2) and Eqs. (6-8) are reduced into following form:

$$R_0 \frac{\partial c}{\partial T} = D_0 \frac{\partial^2 c}{\partial x^2} - u_0 \frac{\partial c}{\partial x} \qquad (11)$$

$$c(x,T) = c_i \sinh(\alpha x)\,;\, L_1 \le x \le L,\ T = 0 \quad (12)$$

$$c(x,T) = \begin{cases} H(T);\, 0 < T \le T_0 \quad (13a) \\ \qquad x = L \\ 0\,;\, T > T_0 \quad (13b) \end{cases}$$

$$\frac{\partial c(x,T)}{\partial x} = \frac{u_0}{2D_0} c\ ;\text{at}\ x = L_1\ ,\quad T \ge 0 \qquad (14)$$

where,

$$H(T) = c_0 G_n\left(\frac{e^{mT} - 1}{m}\right) = b_0 + b_1 e^{mT} + b_2 e^{2mT} + \dots.. + b_n e^{nmT}$$

& $T_0 = \frac{1}{m}\log(1 + mt_0)$ and b_i's are having dimension $\left[ML^{-3}\right]$.

A new transformation is introduced so as to convective term in advection-dispersion equation Eq. (11) may be removed. The transformation follows as [19]:

$$c(x,T) = k(x,T)\exp\left[\frac{u_0}{2D_0}x - \frac{u_0^{\ 2}}{4D_0 R_0}T\right] \quad (15)$$

Therefore Eqs. (11-14) are reduced into:

$$R_0 \frac{\partial k}{\partial T} = D_0 \frac{\partial^2 k}{\partial x^2} \quad (16)$$

$$k(x,T) = \frac{1}{2} c_i \begin{bmatrix} \exp\{(\alpha - \beta)x\} - \\ \exp\{-(\alpha + \beta)x\} \end{bmatrix};\, L_1 \le x \le L,\ T = 0 \quad (17)$$

$$k(x,T) = \begin{cases} \{H(T)\}\exp\left(-\beta x + \eta^2 T\right);\, 0 < T \le T_0 \quad (18a) \\ \qquad x = L \\ 0\,;\, T > T_0 \quad (18b) \end{cases}$$

$$\frac{\partial k(x,T)}{\partial x} = 0\ ;\text{at}\ x = L_1, T \ge 0 \quad (19)$$

where $\beta = \frac{u_0}{2D_0}$ and $\eta = \sqrt{\frac{u_0^{\ 2}}{4D_0 R_0}}$

Applying Laplace transformation on Eqs. (16–19) to reduce Eq. (16) into ordinary differential equation.

$$\frac{d^2 \bar{k}}{dx^2} - \frac{pR}{D_0}\bar{k} = -\frac{R_0}{D_0}\left[\frac{1}{2}c_i \begin{bmatrix} \exp\{(\alpha - \beta)x\} - \\ \exp\{-(\alpha + \beta)x\} \end{bmatrix}\right] \qquad (20)$$

$$\bar{k}(x,p) = \exp(-\beta L)\sum_{r=0}^{n} b_r \frac{\left[1 - \exp - \left\{p - \left(\eta^2 + rm\right)\right\}T_0\right]}{\left\{p - \left(\eta^2 + rm\right)\right\}};\quad x = L \quad (21)$$

$$\frac{d\bar{k}}{dx} = 0\ ;\text{as}\ x = L_1, T > 0 \quad (22)$$

The general solution of Eq. (20) may be written as;

$$\bar{k}(x,p) = c_1 \cosh(Mx) + c_2 \sinh(Mx) + \frac{c_i}{2} \times \left[\frac{\exp\{(\alpha - \beta)x\}}{\left\{p - \frac{D_0}{R_0}(\alpha - \beta)^2\right\}} - \frac{\exp\{-(\alpha + \beta)x\}}{\left\{p - \frac{D_0}{R_0}(\alpha + \beta)^2\right\}}\right] \quad (23)$$

where, $M = \sqrt{\frac{R_0 p}{D_0}}$

Solution of the present problem may be obtained by using Eq.(21) and Eq.(22)in Eq.(23). So coefficient may be written as;

$$c_1 = \exp(-\beta L)\sum_{r=0}^{n} b_r \frac{\left[1 - \exp - \left\{p - \left(\eta^2 + rm\right)\right\}T_0\right]}{\left\{p - \left(\eta^2 + rm\right)\right\}} \times \frac{\cosh(ML_1)}{\cosh(ML - ML_1)} - \frac{c_i}{2}\left[\frac{1}{\left\{p - \frac{D_0}{R_0}(\alpha - \beta)^2\right\}} \times \left\{\frac{\exp\{(\alpha - \beta)L\}\cosh ML_1}{\cosh(ML - ML_1)} - \frac{(\alpha - \beta)\exp\{(\alpha - \beta)L_1\}\sinh ML}{M\cosh(ML - ML_1)}\right\} - \frac{1}{\left\{p - \frac{D_0}{R_0}(\alpha + \beta)^2\right\}} \times \left\{\frac{\exp\{-(\alpha + \beta)L\}\cosh ML_1}{\cosh(ML - ML_1)} - \frac{-(\alpha + \beta)\exp\{-(\alpha + \beta)L_1\}\sinh ML}{M\cosh(ML - ML_1)}\right\}\right] \qquad (24)$$

and

$$c_2 = -\exp(-\beta L)\sum_{r=0}^{n} b_r \frac{\left[1-\exp-\left\{p-\left(\eta^2+rm\right)\right\}T_0\right]}{\left\{p-\left(\eta^2+rm\right)\right\}}\frac{\sinh(ML_1)}{\cosh(ML-ML_1)} - \frac{c_i}{2}\left[\frac{1}{\left\{p-\frac{D_0}{R_0}(\alpha-\beta)^2\right\}}\times\left\{\frac{(\alpha-\beta)\exp\{(\alpha-\beta)L_1\}\cosh ML}{M\cosh(ML-ML_1)} - \frac{\exp\{(\alpha-\beta)L\}\sinh ML_1}{\cosh(ML-ML_1)}\right\} - \frac{1}{\left\{p-\frac{D_0}{R_0}(\alpha+\beta)^2\right\}}\times\left\{\frac{-(\alpha+\beta)\exp\{-(\alpha+\beta)L_1\}\cosh ML}{M\cosh(ML-ML_1)} - \frac{\exp\{-(\alpha+\beta)L\}\sinh ML_1}{\cosh(ML-ML_1)}\right\}\right] \quad (25)$$

Therefore putting the value of c_1 and c_2 from Eq. (24&25) , Eq.(23)may be written as:

$$\bar{k}(x,p) = \left[\exp(-\beta L)\sum_{r=0}^{n} b_r \frac{\left[1-\exp-\left\{p-\left(\eta^2+rm\right)\right\}T_0\right]}{\left\{p-\left(\eta^2+rm\right)\right\}}\frac{\cosh(ML_1)}{\cosh(ML-ML_1)} - \frac{c_i}{2}\left[\frac{1}{\left\{p-\frac{D_0}{R_0}(\alpha-\beta)^2\right\}}\times\left\{\frac{\exp\{(\alpha-\beta)L\}\cosh ML_1}{\cosh(ML-ML_1)} - \frac{(\alpha-\beta)\exp\{(\alpha-\beta)L_1\}\sinh ML}{M\cosh(ML-ML_1)}\right\} - \frac{1}{\left\{p-\frac{D_0}{R_0}(\alpha+\beta)^2\right\}}\times\left\{\frac{\exp\{-(\alpha+\beta)L\}\cosh ML_1}{\cosh(ML-ML_1)} - \frac{-(\alpha+\beta)\exp\{-(\alpha+\beta)L_1\}\sinh ML}{M\cosh(ML-ML_1)}\right\}\right]\right]\times \cosh(Mx) + \left[-\exp(-\beta L)\sum_{r=0}^{n} b_r \frac{\left[1-\exp-\left\{p-\left(\eta^2+rm\right)\right\}T_0\right]}{\left\{p-\left(\eta^2+rm\right)\right\}}\frac{\sinh(ML_1)}{\cosh(ML-ML_1)} - \frac{c_i}{2}\left[\frac{1}{\left\{p-\frac{D_0}{R_0}(\alpha-\beta)^2\right\}}\times\left\{\frac{(\alpha-\beta)\exp\{(\alpha-\beta)L_1\}\cosh ML}{M\cosh(ML-ML_1)} - \frac{\exp\{(\alpha-\beta)L\}\sinh ML_1}{\cosh(ML-ML_1)}\right\} - \frac{1}{\left\{p-\frac{D_0}{R_0}(\alpha+\beta)^2\right\}}\times\left\{\frac{-(\alpha+\beta)\exp\{-(\alpha+\beta)L_1\}\cosh ML}{M\cosh(ML-ML_1)} - \frac{\exp\{-(\alpha+\beta)L\}\sinh ML_1}{\cosh(ML-ML_1)}\right\}\right]\right]\times \sinh(Mx) + \frac{c_i}{2}\left[\frac{\exp\{(\alpha-\beta)x\}}{\left\{p-\frac{D_0}{R_0}(\alpha-\beta)^2\right\}} - \frac{\exp\{-(\alpha+\beta)x\}}{\left\{p-\frac{D_0}{R_0}(\alpha+\beta)^2\right\}}\right] \quad (26)$$

Or

$$\bar{k}(x,p) = \left[\left\{\exp(-\beta L)\sum_{r=0}^{n} b_r \frac{\left[1-\exp-\left\{p-\left(\eta^2+rm\right)\right\}T_0\right]}{\left\{p-\left(\eta^2+rm\right)\right\}}\right\}\frac{\cosh(Mx-ML_1)}{\cosh(ML-ML_1)} - \frac{c_i}{2}\left[\frac{1}{\left\{p-\frac{D_0}{R_0}(\alpha-\beta)^2\right\}}\left\{\frac{\exp\{(\alpha-\beta)L\}\cosh(Mx-ML_1)}{\cosh(ML-ML_1)} + \frac{(\alpha-\beta)\exp\{(\alpha-\beta)L_1\}\sinh(Mx-ML)}{M\cosh(ML-ML_1)}\right\} - \frac{1}{\left\{p-\frac{D_0}{R_0}(\alpha+\beta)^2\right\}}\left\{\frac{\exp\{-(\alpha+\beta)L\}\cosh(Mx-ML_1)}{\cosh(ML-ML_1)} + \frac{-(\alpha+\beta)\exp\{-(\alpha+\beta)L_1\}\sinh(Mx-ML)}{M\cosh(ML-ML_1)}\right\}\right]\right] + \frac{c_i}{2}\left[\frac{\exp\{(\alpha-\beta)x\}}{\left\{p-\frac{D_0}{R_0}(\alpha-\beta)^2\right\}} - \frac{\exp\{-(\alpha+\beta)x\}}{\left\{p-\frac{D_0}{R_0}(\alpha+\beta)^2\right\}}\right] \quad (27)$$

Taking inverse Laplace transform of equation (27), the solution of advection-dispersion equation may be obtained as;

$$c(x,T) = \left[\exp(-\beta L)\sum_{r=0}^{n} b_r F_r(x,T) + \frac{c_i}{2}\left\{I_\alpha(x,T) - I_{-\alpha}(x,T)\right\}\right]\times\exp\left[\frac{u_0}{2D_0}x - \frac{u_0^{\,2}}{4D_0R_0}T\right] \quad ;\ 0 \le T \le T_0 \quad (28a)$$

and

$$c(x,T) = \left[\exp(-\beta L)\sum_{r=0}^{n} b_r\left[F_r(x,T) - F_r(x,T-T_0)\exp\left\{\eta_r^{\,2}T_0\right\}\right] + \frac{c_i}{2}\left\{I_\alpha(x,T) - I_{-\alpha}(x,T)\right\}\right]\times\exp\left[\frac{u_0}{2D_0}x - \frac{u_0^{\,2}}{4D_0R_0}T\right] \quad ;\ T_0 \le T \quad (28b)$$

where,

$$I_\phi(x,T) = \exp\{\phi x - \beta x + q_\phi T\} - \exp(\phi L - \beta L)\times\left[e^{q_\phi T}\frac{\cosh\left\{(x-L_1)\sqrt{q_\phi R_0/D_0}\right\}}{\cosh\left\{(L-L_1)\sqrt{q_\phi R_0/D_0}\right\}} - 2\pi D_0(L-L_1)^2\sum_{s=0}^{\infty}\left\{\frac{(-1)^s(s+1/2)\cos\left\{(s+1/2)\pi\frac{(x-L_1)}{(L-L_1)}\right\}}{\left\{(s+1/2)^2\pi^2D_0 + q_\phi R_0(L-L_1)^2\right\}}\times exp\left\{\frac{-(s+1/2)^2\pi^2D_0}{R_0(L-L_1)^2}T\right\}\right\}\right] - (\phi-\beta)\exp(\phi L_1 - \beta L_1)\times\left[e^{q_\phi T}\frac{\sinh(x-L)\sqrt{q_\phi R_0/D_0}}{\cosh(L-L_1)\sqrt{q_\phi R_0/D_0}} - 2D_0(L-L_1)\times\sum_{s=0}^{\infty}\left\{\frac{(-1)^s\sin\left\{(s+1/2)\pi\frac{(x-L)}{(L-L_1)}\right\}}{\left\{(s+1/2)^2\pi^2D_0 + q_\phi R_0(L-L_1)^2\right\}}\times exp\left\{\frac{-(s+1/2)^2\pi^2D_0}{R_0(L-L_1)^2}T\right\}\right\}\right]$$

$$F_r(x,T) = \exp\left\{\eta_r^{\,2}T\right\}\frac{\cosh\left\{(x-L_1)\eta_r\sqrt{\frac{R_0}{D_0}}\right\}}{\cosh\left\{(L-L_1)\eta_r\sqrt{\frac{R}{D_0}}\right\}} - 2\pi D_0(L-L_1)^2\times\sum_{s=0}^{\infty}\left[\frac{(-1)^s(s+1/2)\cos\left\{(s+1/2)\pi\frac{(x-L_1)}{(L-L_1)}\right\}}{\left\{(s+1/2)^2\pi^2D_0 + \eta_r^{\,2}R_0(L-L_1)^2\right\}}\times\exp\left\{\frac{-(s+1/2)^2\pi^2D_0}{R_0(L-L_1)^2}T\right\}\right]$$

$\eta_r = \sqrt{\eta^2 + rm}$ and $q_\phi = \dfrac{D_0(\phi-\beta)^2}{R_0}$

3. Result and Discussion

The obtained solution Eqs. (28a, b) is discussed for two form of input functions, namely $F(m't) = c_0\{l+\sin(m't)\}$ and $F(m't) = c_0\exp(m't)$ in a finite domain $0 \le x(m) \le 4$ along longitudinal direction for a chosen set of data taken from the experimental and theoretical published literatures are illustrated graphically. For both cases, the concentration values c/c_0 are evaluated assuming reference concentrations as $c_0 = 1.0$, $c_i = 0.01$. Source is applied up to time $t_0(day) = 9$. Other values of common parameters seepage velocity, dispersion coefficient are taken as $u_0 = 1.32(m\ day^{-1})$, $D_0 = 1.66(m^2 day^{-1})$, and $R_0 = 1.45$ respectively. The range of groundwater velocity is taken from $2m/day$ to $2m/year$ depending upon geometrical conditions of porous medium [22].

3.1. Case I-When input is of the form $F(m't) = c_0\{l+\sin(m't)\}$

For the present case, the value of frequency $m'(day^{-1})$, $m(day^{-1})$ and parameter l are taken 0.8, 0.8 and 2 respectively. Concentration value in the time domain $0 \le t(day) \le 9$ are computed at times $t(day) = 2,5$ and 8 while for $t(day) > 9(= t_0)$ the same is computed at times $t(day) = 10,12$ and 14.

Figure 1(a) and 1(b) demonstrate the concentration pattern of solute transport for line graph at times $t(day) = 2,5,8$ and surface plot (c-x-t) respectively. As time elapses, concentration at $x(m) = 4$ fluctuates due to sinusoidal nature of input. Input concentration is shown fluctuating nearly between 1 and 3 value of c/c_0 which is in good agreement with periodic nature input. Surface plot helps us gauge the concentration pattern for any combination time and space. It may be observed that concentration reduces to its lowest on proceeding from $x(m) = 4$ to $x(m) = 0$ for each time in the time domain $(0 \le t(day) \le 9)$.

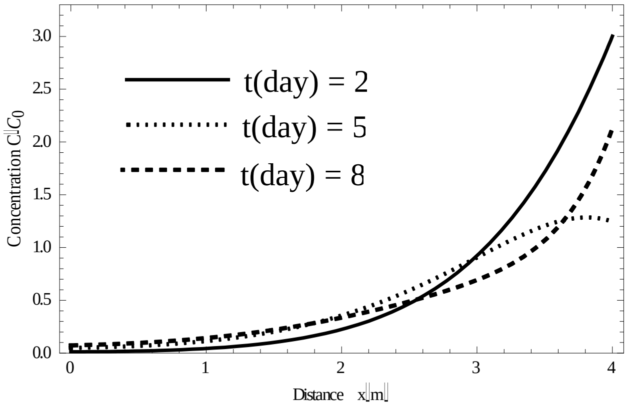

Fig.1(a). Distribution of dimensionless concentration for various time $(0 \le t \le t_0)$

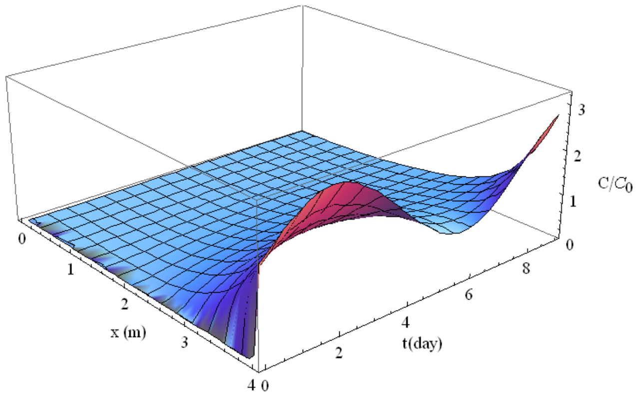

Fig.1(b). Surface plot of Distribution of dimensionless concentration for time $(0 \le t \le t_0)$.

Figure 2(a) and 2(b) are drawn to study the concentration pattern in absence of solute at times $t(day) = 10,12,14$ and surface plot (c-x-t) respectively. On elimination of the source acting against the flow at far end boundary of the domain i.e., $x(m) = 4$ the concentration at this end is measured zero. From Fig. 2(a) it is depicted that maximum concentration is lower is for higher time and higher for lower. Since there is no source of contaminant for time $t(day) > 9$, from the Fig. 2(b) it may be observed that concentration gradually attenuates with time throughout the domain.

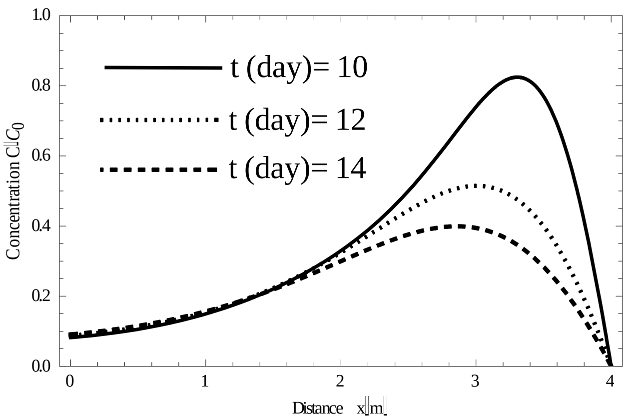

Fig. 2(a). Distribution of dimensionless concentration for various time $(t_0 < t)$

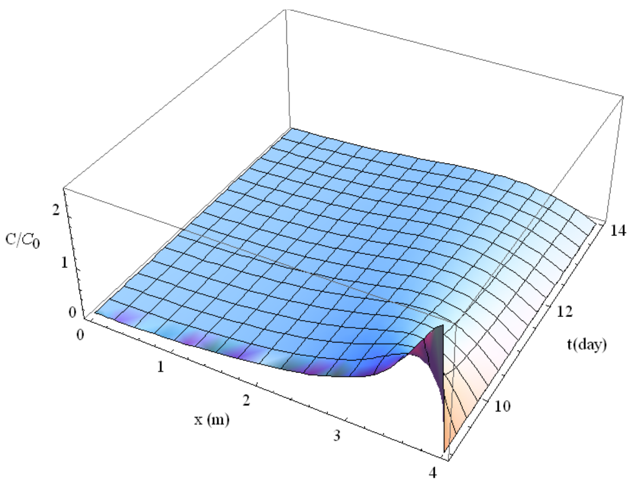

Fig. 2(b). Surface plot of Distribution of dimensionless concentration for time $(t_0 < t)$

3.2. Case II-When input is of the form $F(m't) = c_0 \exp(m't)$

For the present case, the value of unsteady parameters is $m'(day^{-1}) = 0.2$ and $m(day^{-1}) = 0.1$. Concentration values in the time domain $0 \le t(day) \le 9$ are computed at times $t(day) = 2{,}5$ and 8 while for $t > 9 (= t_0)$ the same is computed at times $t(day) = 10{,}12$ and 14.

Figure 3(a) and 3(b) explore the concentration pattern of solute transport with line graph at times $t(day) = 2{,}5{,}8$ and surface plot (c-x-t) respectively in presence of input source. With increase of time, concentration at $x(m) = 4$ increases because input is increasing exponentially with time. Figure 3(a) exhibits that rate of concentration increment is rapid from time $5(day)$ to $8(day)$ throughout the domain in comparison to same duration from time $2(day)$ to $5(day)$. From Fig. 3(b) concentration pattern for any combination time and space can be analyzed when the source is present. It is recorded that concentration c / c_0 is nearly 1 at time $t(day) = 0$ at far end i.e., $x(m) = 4$ which is in accordance of our input for this case.

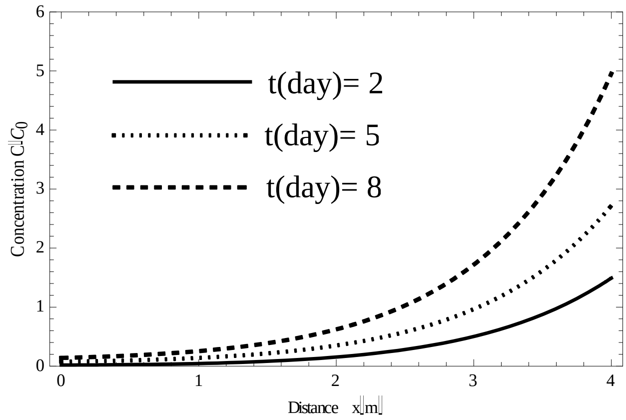

Fig. 3(a). Distribution of dimensionless concentration for various time $(0 \le t \le t_0)$

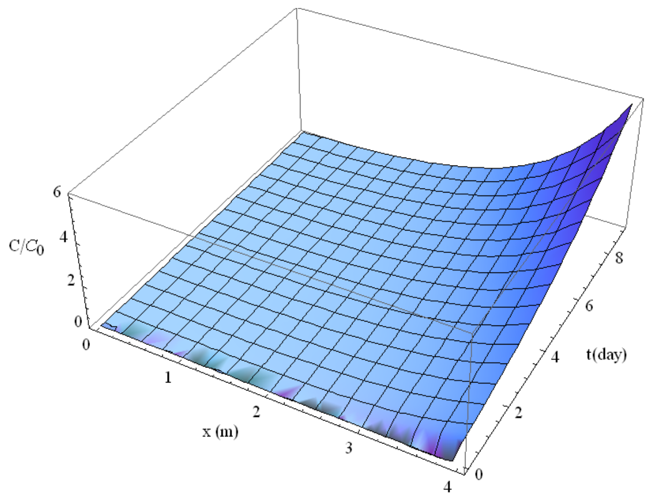

Fig. 3(b). Surface plot ofDistribution of dimensionless concentration for time $(0 \le t \le t_0)$.

Figure 4(a) and 4(b) are plotted to examine concentration pattern in absence of source with concentration-space graph at times $t(day) = 10{,}12{,}14$ and c-x-t graph respectively. Like Fig. 2(b) the solute concentration at the far end reduce to zero as the source eliminated. From the Fig. 4(a) it recorded concentration peak drops sharp with increase of time. It is revealed fromFig. 4(b) that concentration attenuated very fast as the source is eliminated.

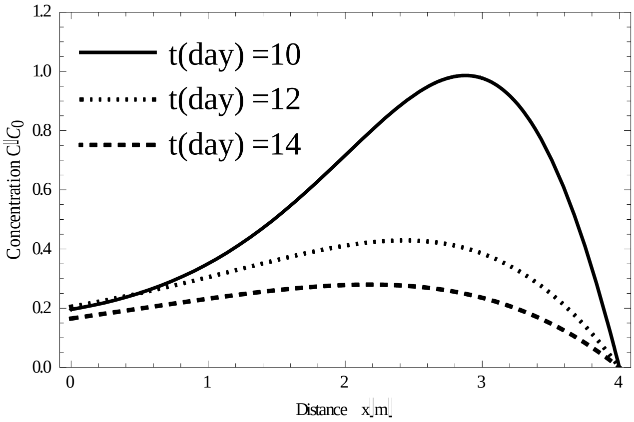

Fig. 4(a). Distribution of dimensionless concentration for various time $(t_0 < t)$

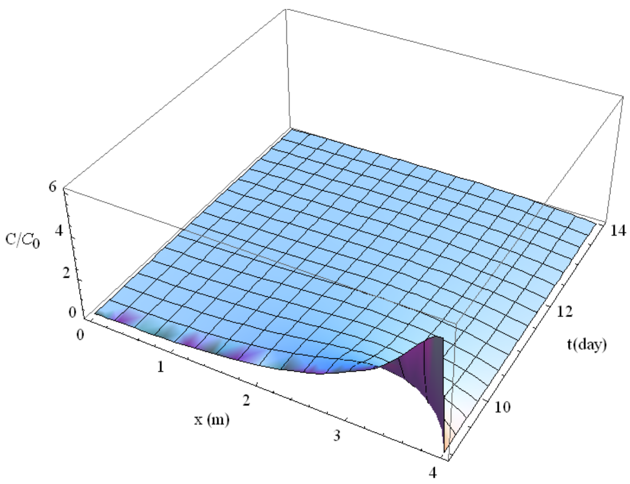

Fig. 4(b). Surface plot of distribution of dimensionless concentration for $(t > t_0)$

4. Verification of Solution

Consider the case where input for $F(m't) = 1/(1 + m't)$ and $m' = m$ the solution (A7&A8) is obtained the way given in appendix. For a chosen set of data taken from the experimental and theoretical published literatures described as $c_0 = 1.0$, $c_i = 0.01$ $u_0 = 1.32(m\ day^{-1})$ and $D_0 = 1.66(m^2 day^{-1})$, $R = 1.45, \alpha = 0.032$ and $m = m' = 0.2$ respectively and with time of elimination of source $t_0(day) = 9$,concentration- space graphs Figs. 5(a)and 5(b) are plotted from solution Eqs. (24&24b) and Eqs. (A7andA8) in appendix in domain $0 \le x(m) \le 4$ times $t(day) = 1{,}5 and 9$ in presence of source, while at times $t(day) = 10{,}12$ and 14 in absence of source.
The Figs. 5(a) and 5(b) are drawn to compare the solution Eqs. (28a&28b) and solution Eqs. (A7andA8) in presence and absence of source respectively. The Figs. 5(a) and 5(b) show good agreement in the both of the solutions. In further above result verifies the solution Eqs.(28aand28b) at some extent.

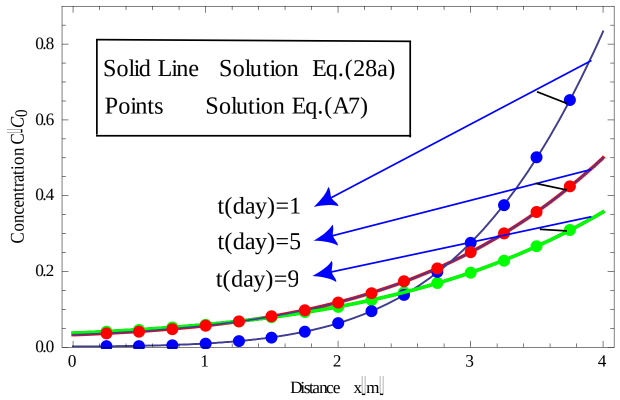

Fig. 5(a). Distribution of dimensionless concentration for various times $(0 \le t \le t_0)$

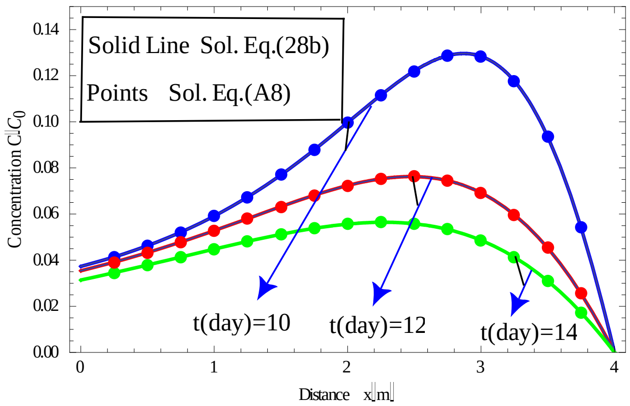

Fig. 5(b). Distribution of dimensionless concentration for various time $(t_0 < t)$

5. Conclusion

Solution of finite domain theoretical model comprising temporal dispersion and seepage velocity with any time dependent smoothly varying input of pulse type applied against the flow has been obtained. Such solution is well applicable for real world finite domain aquifers for predicting contaminant variations in presence and in absence i.e. in other word during rehabilitation process when source is eliminated. The generalization of the input source added the worth of solution. Laplace Transformation Technique and interpolation method are applied to obtained the solution. The Accuracy of solution enhances with optimum selection of interpolation polynomial method.

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Appendix:

The input boundary conditions with $F(m't) = 1/(1+m't)$ may be described as;
Assuming $m' = m$ in particular, we have following initial and boundary condition:

$$c(x,t) = c_i \sinh(\alpha x);\, L_1 \le x \le L,\, t = 0 \quad \text{(A1)}$$

$$c(x,t) = \begin{cases} c_0/(1+mt);\, 0 < t \le t_0 & \text{(A2a)} \\ \quad ;\, x = L & \\ 0\,;\, t > t_0, & \text{(A2b)} \end{cases}$$

$$\frac{\partial c(x,t)}{\partial x} = \frac{u_0}{2D_0} c \text{ at } x = L_1,\, t \ge 0 \quad \text{(A3)}$$

With transformation Eq. (10) may written as:

$$c(x,T) = c_i \sinh(\alpha x);\, L_1 \le x \le L,\, T = 0 \quad \text{(A4)}$$

$$\begin{cases} c_0 \exp(-mT)\,;\, 0 < T \le T_0 \, \text{(A5a)} \end{cases}$$

$$c(x,T) = \qquad x = L$$

$$0 \;;\; T > T_0 \qquad \text{(A5b)}$$

$$\frac{\partial c(x,T)}{\partial x} = \frac{u_0}{2D_0} c \text{ at } x = L_1, T \geq 0 \text{(A6)}$$

Now, performing same steps as Eq.(15) to Eq.(27) then taking inverse laplace transform , solution may be written as:

$$c(x,T) = \left[\exp(-\beta L) F_{-1}(x,T) + \frac{c_i}{2}\{I_\alpha(x,T) - I_{-\alpha}(x,T)\}\right]$$

$$\times \exp\left[\frac{u_0}{2D_0} x - \frac{u_0^{\,2}}{4D_0 R_0} T\right]$$

$$;\; 0 < T \leq T_0 \text{ (A7)}$$

$$c(x,T) = \left[\exp(-\beta L)\begin{bmatrix} F_{-1}(x,T) - F_{-1}(x, T - T_0) \times \\ \exp\{(\eta^2 - m)T_0\} \end{bmatrix} + \right.$$

and

$$\left.\frac{c_i}{2}\{I_\alpha(x,T) - I_{-\alpha}(x,T)\}\right] \times \exp\left[\frac{u_0}{2D_0} x - \frac{u_0^{\,2}}{4D_0 R_0} T\right]$$

$$;\; T_0 \leq T \text{ (A8)}$$

where,

$$\eta_{-1} = \sqrt{\eta^2 - m}$$

***