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Second Order Renormalization Group Flow on Warped Product Manifolds

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Abstract: In this work we have studied the evolution of a warped product (WP) manifold under second order renormalization group (RG-2) flow. We have shown some conditions for the existence of a solution of RG-2 flow on WP manifolds. Also, we have found a necessary condition for warped function under RG-2 flow. In particular, we study some special WP metric of real line with a manifold. Eventually, by extending conditions to pseudo-Riemannian manifold, we find a PDE for Robertson-Walker (RW) metrics, and show that there is no RG-2 flow for RW metrics.

Keywords: Ricci flow, second order renormalization group flow, warped product manifold.

1. Introduction

The Ricci flow was introduced and studied by Hamilton [8], and has been a topic of interest in both mathematics and physics. The Ricci flow is an evolution equation for Riemannian metrics. In the Ricci flow, one begins with a smooth Riemannian manifold M, equipped with a smooth Riemannian metric g_0 and evolves its metric by the equation

$$\begin{gathered}\tfrac{\partial}{\partial t}g(t) = -2Ric(g(t)),\\ g(0) = g_0,\end{gathered}$$

where $t \in I$, I is an interval, and $Ric(g(t))$ denotes the Ricci curvature of $g(t)$.
Many authors, have tried to extend Ricci flow from different point of views. The Ricci flow is the first-order approximation of renormalization group flow for nonlinear sigma models in quantum field theory. The second order approximation of the renormalization group flow for the nonlinear sigma model of quantum field theory, which we label by RG-2 flow, is specified by

$$\frac{\partial}{\partial t}g = -2Ric - \frac{\alpha}{2}Rm^2, \tag{1}$$

where

$$Rm^2_{ij} = g^{pk}g^{ql}g^{nm}R_{iklm}R_{jpqn}, \tag{2}$$

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denotes the quadratic curvature and α is a positive parameter. Note that for our purposes here, we can assume α to be non-negative. For $\alpha = 0$, the system (1) reduces to the Ricci flow.
RG-2 flow is diffeomorphism invariant but unlike the Ricci flow, it is not a weakly parabolic system. Gimre, Guenther and Isenberg, modifying De-Turck method for RG-2, proved conditions for the short time existence of the second order renormalization group flow in general dimension [4]. Some of the mathematical features of this flow have been studied in recent years, [5], [6], [7] and [14].
The concept of warped product (WP) metrics was first introduced by Bishop and O'Neill [1]. In Riemannian geometry, warped product manifolds have been used to construct new examples. The warped product $B \times_u F$ of two Riemannian manifolds B and F and real warped function $u : B \to \mathbb{R}$, is the product manifold $B \times F$ furnished with the metric $g = g^B + u^2 g^F$.
The Ricci flow on warped product manifolds was studied over the last few years [3], [12], [13], [16] and [17]. In this work, we have studied the property of second order renormalization group flow on warped product manifolds. First, we investigate an existence condition of RG-2 on WP manifolds, and extend the curvature criterion of short-time existence. Then, using WP metric curvature and RG-2 flow, we find some relations for warped function.
Many exact solutions of the Einstein field equations and modified field equations are warped products, for instance, the Robertson-Walker (RW) models are warped products. Robertson and Walker independently showed in the mid-1930s that this is the most general metric possible for describing an expanding, homogeneous and isotropic universe. Hesamifard and Rezaii, studied RG-2 flow on RW metric in spherical coordinates [10]. Using property of RG-2 flow on WP manifolds, we have found a PDE, and we have studied some properties of its solution. We have shown as in [10], that there is no solution of RG-2 flow on RW manifolds.

2. Preliminaries

For any closed Riemannian manifold (M, g_0), and for all sectional curvatures $K_P(g_0)$, at all point $p \in M$ and planes $P \subset T_pM$, if

$$1 + \alpha K_P > 0,$$

then there exists a unique solution $g(t)$ of the initial value problem $\partial_t g = -Ric - \frac{\alpha}{2} Rm^2$, $g(0) = g_0$, on some time interval $[0, T)$ [4].
Let g_k^F be a Riemannian metric on an n-dimensional manifold F with constant curvature k, then $Ric^F = k(n-1)g_k^F$ and $Rm^{2F} = 2k^2(n-1)g_k^F$. If $g^F(t)$ is a solution of the second order renormalization group flow (1), with initial metric $g^F(0) = g_k^F$, then $g^F(t)$ preserves its conformal class,

and we may write $g^F(t) = \phi(t)g^F_k$, where $\phi(t)$ obtained by the following implicit function [5], [6]

$$\phi(t) = -2k(n-1)t + 1 + \frac{\alpha k}{2} Ln\left|\frac{2\phi(t)+\alpha k}{2+\alpha k}\right|.$$

Note that $Ric(\phi g) = Ric(g)$ and $Rm^2(\phi g) = \frac{1}{\phi}Rm^2(g))$.

Let (B, g^B) and (F, g^F) be two (Pseudo-) Riemannian manifolds with dimensions m and n, respectively. Let $M = B \times_u F$ and $g = g^B + u^2 g^F$ where $u : B \to \mathbb{R}$ is smooth positive function. For any point $(x,y) \in M$, and vectors $X^B, Y^B, Z^B ... \in T_x B$ and $X^F, Y^F, Z^F ... \in T_y F$, we have for the Riemannian curvature of warped product manifold (M, g) [15];

$$\begin{aligned}
&R(X^B,Y^B)Z^F = R(X^F,Y^F)Z^B = 0,\\
&R(X^B,Y^B)Z^B = R^B(X^B,Y^B)Z^B,\\
&R(X^F,Y^B)Z^B = \tfrac{1}{u}Hess^B(u)(Y^B,Z^B)X^F,\\
&R(X^B,Y^F)Z^F = ug^F(Y^F,Z^F)\nabla^B_{X^B}(\nabla^B u),\\
&R(X^F,Y^F)Z^F = R^F(X^F,Y^F)Z^F - |\nabla^B u|^2_{g^B}(g^F(X^F,Z^F)Y^F - g^F(Y^F,Z^F)X^F).
\end{aligned} \tag{3}$$

where, R, R^B and R^F are Riemannian curvatures of (M,g), (B,g^B) and (F,g^F), respectively. Also, we have for the Ricci tensor [15]

$$\begin{aligned}
&Ric(X^B.Y^F) = 0,\\
&Ric(X^B,Y^B) = Ric^B(X^B,Y^B) - \tfrac{n}{u}Hess^B(u)(X^B,Y^B),\\
&Ric(X^F,Y^F) = Ric^F(X^F,Y^F) - (u\Delta_{g^B}u + (n-1)|\nabla^B u|^2_{g^B})g^F(X^F,Y^F),
\end{aligned} \tag{4}$$

where, Ric, Ric^B and Ric^F define Ricci tensors of (M,g), (B,g^B) and (g^F,F), respectively.

Generaly, at a point $(x,y) \in M$ and vectors $X,Y \in T_{(x,y)}M$, where $X = X^B + X^F$, $Y = Y^B + Y^F$, $X^B, Y^B \in T_x B$ and $X^F, Y^F \in T_y F$, we have

$$\begin{aligned}
Ric(X,Y) \;=\; & Ric^B(X^B,Y^B) + Ric^F(X^F,Y^F) - \tfrac{n}{u}Hess^B(u)(X^B,Y^B)\\
& -u\Delta_{g^B}ug^F(X^F,Y^F) - (n-1)|\nabla^B u|^2_{g^B}g^F(X^F,Y^F).
\end{aligned}$$

Directly, we can calculate, the Riemannian curvature $R(X,Y,Z,W) = g(R(X,Y)Z,W)$, and have:

$$\begin{aligned}
R(X^B,Y^B,Z^F,W^B) &= R(X^F,Y^F,Z^B,W^F) = R(X^B,Y^B,Z^F,W^F) = 0,\\
R(X^B,Y^B,Z^B,W^B) &= R^B(X^B,Y^B,Z^B,W^B),\\
R(X^B,Y^F,Z^B,W^F) &= -uHess^B(u)(X^B,Z^B)g^F(Y^F,W^F),\\
R(X^F,Y^F,Z^F,W^F) &= u^2R^F(X^F,Y^F,Z^F,W^F)\\
&\quad -u^2|\nabla^B u|^2_{g^B}\Big(g^F(X^F,Z^F)g^F(Y^F,W^F) - g^F(Y^F,Z^F)g^F(X^F,W^F)\Big).
\end{aligned} \tag{5}$$

Let (N,h) be an m-dimensional ($m \geq 2$) Riemannian manifold, and I be an open interval of the real line equipped with the negative of the standard metric. A Lorentzian manifold $(M = I \times_f N, g = -ds^2 + f^2(s)h$ of dimension $m+1$ is a generalized Robertson-Walker space time where $f : I \to \mathbb{R}^+$ is a smooth function [2].

When, (N,h) is a three-dimensional manifold with constant curvature, we call (M,g) a Robertson Walker space time. In terms of spherical coordinates, Robertson-Walker metric can be written in the form

$$g = -ds^2 + f^2(s)\left[\frac{dr^2}{1-kr^2} + r^2(d\theta^2 + \sin^2\theta d\phi^2)\right].$$

3. Short-Time Existence

On a Riemannian manifold, if the sectional curvatures satisfies certain conditions, then the flow is (weakly) parabolic. We extend this curvature criterion for short-time existence for the RG-2 flow to WP manifolds. At first, we calculate the sectional curvature of WP manifolds.

Lemma 1. Let $(M = B \times_u F, g = g^B + u^2 g^F)$ be a WP manifold, then at point $(x,y) \in M$ and any linear independent vectors $X^B, Y^B \in T_xB$ and $X^F, Y^F \in T_yF$, we have

$$\begin{aligned}
|X^B \wedge Y^B|_g^2 &= |X^B \wedge Y^B|_{g^B}^2, \\
|X^F \wedge Y^F|_g^2 &= u^4 |X^F \wedge Y^F|_{g^F}^2, \\
|X^B \wedge Y^F|_g^2 &= u^2 g^B(X^B, X^B) g^F(Y^F, Y^F), \\
|(X^B + X^F) \wedge Y^B|_g^2 &= |X^B \wedge Y^B|_g^2 + |X^F \wedge Y^F|_g^2, \\
|(X^B + X^F) \wedge Y^F|_g^2 &= |X^B \wedge Y^F|_g^2 + |X^F \wedge Y^F|_g^2, \\
|(X^B + X^F) \wedge (Y^B + Y^F)|_g^2 &= |X^B \wedge Y^B| + |X^F \wedge Y^F|_g^2 + |X^B \wedge Y^F|_g^2 + |Y^B \wedge X^F|_g^2 \\
&\quad -2u^2 g^B(X^B, Y^B) g^F(X^F, Y^F).
\end{aligned} \tag{6}$$

where $|X \wedge Y|_g^2 = g(X,X)g(Y,Y) - g^2(X,Y)$ is the area of a parallelogram in $T_{(x,y)}M$ spanned by X and Y.

Proof. The proof is directly obtained from the definition as follows:
We have for linear independent vectors $X^B, Y^B \in T_xB$

$$\begin{aligned}
|X^B \wedge Y^B|_g^2 &= g(X^B, X^B) g(Y^B, Y^B) - g^2(X^B, Y^B) \\
&= g^B(X^B, X^B) g^B(Y^B, Y^B) - (g^B)^2(X^B, Y^B) = |X^B \wedge Y^B|_{g^B}^2.
\end{aligned}$$

For linear independent vectors $X^F, Y^F \in T_xF$, we have

$$\begin{aligned}
|X^F \wedge Y^F|_g^2 &= g(X^F, X^F) g(Y^F, Y^F) - g^2(X^F, Y^F) \\
&= u^2 g^F(X^F, X^F) u^2 g^F(Y^F, Y^F) - \Big(u^2 g^F(X^F, Y^F)\Big)^2 = u^4 |X^F \wedge Y^F|_{g^F}^2.
\end{aligned}$$

Also, for vectors $X^B \in T_xB$ and $Y^F \in T_F$, we have

$$|X^B \wedge Y^F|_g^2 = g(X^B, X^B) g(Y^F, Y^F) - g^2(X^B, Y^F) = u^2 g^B(X^B, X^B) g^F(Y^F, Y^F).$$

Now, for linear independent vectors $X = X^B + X^F \in T_{(x,y)}M$ and $Y^B \in T_xB$, we have

$$\begin{aligned}
|X \wedge Y^B|_g^2 &= g(X^B + X^F, X^B + X^F) g(Y^B, Y^B) - g^2(X^B + X^F, Y^B) \\
&= \Big(g(X^B, X^B) + g(X^F, X^F)\Big) g(Y^B, Y^B) - g^2(X^B, Y^B) \\
&= \Big(g(X^B, X^B) g(Y^B, Y^B) - g^2(X^B, Y^B)\Big) + g(X^F, X^F) g(Y^B, Y^B) \\
&= |X^B \wedge Y^B|_g^2 + |X^F \wedge Y^F|_g^2.
\end{aligned}$$

Moreover, for linear independent vectors $X = X^B + X^F \in T_{(x,y)}M$ and $Y^F \in T_yF$, we have

$$\begin{aligned}
|X \wedge Y^F|_g^2 &= g(X^B + X^F, X^B + X^F)g(Y^F, Y^F) - g^2(X^B + X^F, Y^F) \\
&= \Big(g(X^B, X^B) + g(X^F, X^F)\Big)g(Y^F, Y^F) - g^2(X^F, Y^F) \\
&= \Big(g(X^F, X^F)g(Y^F, Y^F) - g^2(X^F, Y^F)\Big) + g(X^B, X^B)g(Y^F, Y^F) \\
&= |X^F \wedge Y^F|_g^2 + |X^B \wedge Y^F|_g^2.
\end{aligned}$$

Eventually, for linear independent vectors $X = X^B + Y^F, Y = Y^B + Y^F \in T_{(x,y)}M$, we have

$$\begin{aligned}
|X \wedge Y|_g^2 &= g(X^B + X^F, X^B + X^F)g(Y^B + Y^F, Y^B + Y^F) - g^2(X^B + X^F, Y^B + Y^F) \\
&= \Big(g(X^B, X^B) + g(X^F, X^F)\Big)\Big(g(Y^B, Y^B) + g(Y^F, Y^F)\Big) \\
&\quad - \Big(g(X^B, Y^B) + g(X^F, Y^F)\Big)^2 \\
&= \Big(g(X^B, X^B)g(Y^B, Y^B) - g^2(X^B, Y^B)\Big) \\
&\quad + \Big(g(X^F, X^F)g(Y^F, Y^F) - g^2(X^F, Y^F)\Big) \\
&\quad \Big(g(X^B, X^B)g(Y^F, Y^F)\Big) + \Big(g(X^F, X^F)g(Y^B, Y^B)\Big) \\
&\quad - 2g(X^B, Y^B)g(X^F, Y^F) \\
&= |X^B \wedge Y^B|_g^2 + |X^F \wedge Y^F|_g^2 + |X^B \wedge Y^F|_g^2 + |X^F \wedge Y^B|_g^2 \\
&\quad - 2u^2 g^B(X^B, Y^B)g^F(X^F, Y^F).
\end{aligned}$$

■

Proposition 1. Let $M = B \times_u F$ be a warped product manifold, then at any point $(x, y) \in M$, the sectional curvature of warped product metric $g = g^B + u^2 g^F$ is calculated as follows. In fact, for linear independent vectors $X^B, Y^B \in T_xB$ and $X^F, Y^F \in T_yF$, we have

$$\begin{aligned}
K(X^B, Y^B) &= K^B(X^B, Y^B), \\
K(X^B, Y^F) &= -\frac{Hess^B(u)(X^B, X^B)}{ug^B(X^B, X^B)}, \\
K(X^F, Y^F) &= \frac{1}{u^2}\Big(K^F(X^F, Y^F) - |\nabla^B u|_{g^B}^2\Big),
\end{aligned} \tag{7}$$

where K, K^B and K^F are sectional curvatures of (M, g), (B, g^B) and (F, g^F), respectively. Also, we have

$$\begin{aligned}
K(X, Y^B) &= \frac{K(X^B, Y^B)|X^B \wedge Y^B|_g^2 + K(X^F, Y^B)|X^F \wedge Y^B|_g^2}{|X^B \wedge Y^B|_g^2 + |X^F \wedge Y^B|_g^2} \\
K(X, Y^F) &= \frac{K(X^B, Y^F)|X^B \wedge Y^F|_g^2 + K(X^F, Y^F)|X^F \wedge Y^F|_g^2}{|X^B \wedge Y^F|_g^2 + |X^F \wedge Y^F|_g^2} \\
K(X, Y) &= \frac{1}{|X \wedge Y|_g^2}\Big(K(X^B, Y^B)|X^B \wedge Y^B|_g^2 + K(X^F, Y^F)|X^F \wedge Y^F|_g^2 \\
&\quad + K(X^B, Y^F)|X^B \wedge Y^F|_g^2 + K(X^F, Y^B)|X^F \wedge Y^B|_g^2 - 2R(X^B, Y^F, Y^B, X^F)\Big),
\end{aligned} \tag{8}$$

where $X = X^B + X^F, Y = Y^B + Y^F \in T_{(x,y)}M$.

Proof. By (5), (6) and definition of sectional curvature, for a plane in T_xB, generated by vectors X^B and Y^B, we have

$$K(X^B, Y^B) = \frac{R(X^B, Y^B, X^B, Y^B)}{|X^B \wedge Y^B|_g^2} = \frac{R^B(X^B, Y^B, X^B, Y^B)}{|X^B \wedge Y^B|_{g^B}^2} = K^B(X^B, Y^B).$$

Also, for a plane in T_yF, generated by vectors X^F and Y^F, we have

$$\begin{aligned} K(X^F,Y^F) &= \frac{R(X^F,Y^F,X^F,Y^F)}{|X^F\wedge Y^F|_g^2} = \frac{u^2R^F(X^F,Y^F,X^F,Y^F)-u^2|\nabla^B u|^2_{g^B}|X^F\wedge Y^F|^2_{g^F}}{u^4|X^F\wedge Y^F|^2_{g^F}} \\ &= \frac{1}{u^2}K^F(X^F,Y^F)-\frac{|\nabla^B u|^2_{g^B}}{u^2}. \end{aligned}$$

For a plane in $T_{(x,y)}M$, generated by vectors X^B and Y^F, we have

$$\begin{aligned} K(X^B,Y^F) &= \frac{R(X^B,Y^F,X^B,Y^F)}{|X^B\wedge Y^F|_g^2} = \frac{-uHess^B(u)(X^B,X^B)g^F(Y^F,Y^F)}{u^2g^B(X^B,X^B)g^F(Y^F,Y^F)} \\ &= -\frac{Hess^B(u)(X^B,X^B)}{ug^B(X^B,X^B)}. \end{aligned}$$

Now, from (5) we have

$$R(X^B+X^F,Y^B,X^B+X^F,Y^B) = R(X^B,Y^B,X^B,Y^B)+R(X^F,Y^B,X^F,Y^B), \tag{9}$$

$$R(X^B+X^F,Y^F,X^B+X^F,Y^F) = R(X^B,Y^F,X^B,Y^F)+R(X^F,Y^F,X^F,Y^F), \tag{10}$$

$$\begin{aligned} R(X^B+X^F,Y^B+Y^F,X^B+X^F,Y^B+Y^F) &= R(X^B,Y^B,X^B,Y^B)+R(X^F,Y^F,X^F,Y^F) \\ &\quad +R(X^F,Y^B,X^F,Y^B)+R(X^B,Y^F,X^B,Y^F) \\ &\quad -2R(X^B,Y^F,Y^B,X^F). \end{aligned} \tag{11}$$

From (6) and (9) we have

$$\begin{aligned} K(X^B+X^F,Y^B) &= \frac{R(X^B+X^F,Y^B,X^B+X^F,Y^B)}{|X^B+X^F\wedge Y^B|_g^2} \\ &= \frac{R(X^B,Y^B,X^B,Y^B)+R(X^F,Y^B,X^F,Y^B)}{|X^B\wedge Y^B|_g^2+|X^F\wedge Y^B|_g^2} \\ &= \frac{K(X^B,Y^B)|X^B\wedge Y^B|_g^2+K(X^F,Y^B)|X^F\wedge Y^B|_g^2}{|X^B\wedge Y^B|_g^2+|X^F\wedge Y^B|_g^2}. \end{aligned}$$

Also, from (6) and (10) we have

$$\begin{aligned} K(X^B+X^F,Y^F) &= \frac{R(X^B+X^F,Y^F,X^B+X^F,Y^F)}{|X^B+X^F\wedge Y^F|_g^2} \\ &= \frac{R(X^B,Y^F,X^B,Y^F)+R(X^F,Y^F,X^F,Y^F)}{|X^B\wedge Y^F|_g^2+|X^F\wedge Y^F|_g^2} \\ &= \frac{K(X^B,Y^F)|X^B\wedge Y^F|_g^2+K(X^F,Y^F)|X^F\wedge Y^F|_g^2}{|X^B\wedge Y^F|_g^2+|X^F\wedge Y^F|_g^2}. \end{aligned}$$

At last, let $X=X^B+X^F$ and $Y=Y^B+Y^F$, so from (5) and (11) we have

$$\begin{aligned} K(X,Y) &= \frac{R(X,Y,X,Y)}{|X\wedge Y|_g^2} \\ &= \frac{1}{|X\wedge Y|_g^2}\Big(R(X^B,Y^B,X^B,Y^B)+R(X^F,Y^F,X^F,Y^F) \\ &\quad +R(X^F,Y^B,X^F,Y^B)+R(X^B,Y^F,X^B,Y^F)-2R(X^B,Y^F,Y^B,X^F)\Big) \\ &= \frac{1}{|X\wedge Y|_g^2}\Big(K(X^B,Y^B)|X^B\wedge Y^B|_g^2+K(X^F,Y^F)|X^F\wedge Y^F|_g^2+K(X^B,Y^F)|X^B\wedge Y^F|_g^2 \\ &\quad +K(X^F,Y^B)|X^F\wedge Y^B|_g^2+2uHess^B(u)(X^B,Y^B)g^F(Y^F,X^F)\Big). \end{aligned}$$

■

By extending short-time existence of RG-2 flow we have the following theorem.

Theorem 1. Let $(M = B \times_{u_0} F, g_0 = g_0^B + u_0^2 g_0^F)$ be a closed Riemannian manifold, which satisfies the following conditions:
i) For all points $x \in B$, and planes $P_1 \subset T_x B$, the sectional curvature $K_{P_1}^B$ of (B, g_0^B) satisfies:

$$1 + \alpha K_{P_1}^B > 0.$$

ii) For all points $y \in F$, and planes $P_2 \subset T_y F$, the sectional curvature $K_{P_2}^F$ of (F, g_0^F) satisfies:

$$\alpha K_{P_2}^F \geq \sup_{x \in B} \Big(\alpha |\nabla^B u_0|^2_{g_0^B} - u_0^2 \Big).$$

iii) The tensor $u_0 g_0^B - \alpha Hess^B(u_0)$ is positive definite.
iv) For all point $x \in B$, and linear independent vectors $X^B, Y^B \in T_x B$, we have

$$\alpha Hess^B(u_0)(X^B, Y^B) = u_0 g_0^B(X^B, Y^B).$$

Then there exist a unique solution $g(t)$ of second order renormalization group flow (1), with initial metric $g(0) = g_0 = g_0^B + u_0^2 g_0^F$, on some interval $[0, T)$.

Proof. If $1 + \alpha K_{P_1}^B(g_0^B) > 0$ for any $x \in B$ and planes $P_1 \subset T_x B \subset T_{(x,y)} M$, by (7) we have $1 + \alpha K_{P_1}(g_0) > 0$.
Also, if $\alpha K_{P_2}^F > \alpha |\nabla^B u_0|^2_{g_0^B} - u_0^2$, for any $x \in B$, $y \in F$ and planes $P_2 \subset T_x F \subset T_{(x,y)} M$, we have

$$1 + \frac{\alpha}{u_0^2} \Big(K_{P_2}^F(g_0^F) - |\nabla^B u_0|^2_{g_0^B} \Big) > 0,$$

so, by (7), we have $1 + \alpha K_{P_2}(g_0) > 0$.
Let $g_0^B - \frac{\alpha}{u_0} Hess^B(u_0)$ be positive definite, this means $(g_0^B - \frac{\alpha}{u_0} Hess^B(u_0))(X^B, X^B) > 0$, for all point $x \in B$ and all vectors $X^B \in T_x B$. So, for any point $y \in F$ and $Y^F \in T_y F$, we have

$$1 - \alpha \frac{Hess^B(u_0)(X^B, X^B) g_0^F(Y^F, Y^F)}{u_0 g_0^B(X^B, X^B) g_0^F(Y^F, Y^F)} > 0.$$

Then, from (7), at the point $(x, y) \in M = B \times F$, and planes $P_3 \subset T_{(x,y)} M$ spanned by $X^B \in T_x B$ and $Y^F \in T_y F$, we have $1 + \alpha K_{P_3}(g_0) > 0$.
Now, from (i) we have

$$\alpha K^B(X^B, Y^B) |X^B \wedge Y^B| > -|X^B \wedge Y^B|, \tag{12}$$

and from (ii), we have

$$\frac{|X^F \wedge Y^F|^2_g}{u^2} \alpha \Big(K^F(X^F, Y^F) - |\nabla^B u|^2_{g_0} \Big) > -|X^F \wedge Y^F|^2_g, \tag{13}$$

also, from (iii), we have

$$-\alpha u Hess^B(u)(X^B, X^B) g^F(Y^F, Y^F) > -|X^B \wedge Y^F|^2_g, \tag{14}$$

$$-\alpha u Hess^B(u)(Y^B, Y^B) g^F(X^F, X^F) > -|Y^B \wedge X^F|^2_g. \tag{15}$$

Let $X = X^B + X^F$ an $Y = Y^B + Y^F$. By adding two sides of inequalities (12) and (15) we have

$$\alpha\Big(K(X^B,Y^B)|X^B \wedge Y^B|_g^2 + K(X^F,Y^B)|X^F \wedge Y^B|_g^2\Big) < -|X^B \wedge Y^B|_g^2 - |X^F \wedge Y^B|_g^2. \quad (16)$$

So, from (8) and (16) we have $1+\alpha K_{P_4} > 0$, where $P_4 \subset T_{(x,y)}M$ spanned by vectors X and Y^B.
Now, By adding two sides of inequalities (12) and (14) we have

$$\alpha\Big(K(X^B,Y^F)|X^B \wedge Y^F|_g^2 + K(X^F,Y^F)|X^F \wedge Y^F|_g^2\Big) < -|X^B \wedge Y^F|_g^2 - |X^F \wedge Y^F|_g^2. \quad (17)$$

So, from (8) and (17), we have $1+\alpha K_{p_5} > 0$, where $P_5 \subset T_{(x,y)}M$ spanned by vectors X and Y^F.
Finally, by adding two sides of inequalities (12), (13), (14) and (15) we have

$$\begin{aligned} \alpha K(X,Y)|X \wedge Y|_g^2 \quad > \quad & -|X^B \wedge Y^B| - |X^F \wedge Y^F|_g^2 - |X^B \wedge Y^F|_g^2 \\ & -|Y^B \wedge X^F|_g^2 + 2\alpha u Hess^B(u)(X^B,Y^B)g^F(X^F,Y^F). \end{aligned} \quad (18)$$

We have from (6) and (18)

$$\begin{aligned} \alpha K(X,Y)|X \wedge Y|_g^2 \quad > \quad & -|X \wedge Y|_g^2 - 2u^2 g^B(X^B,Y^B)g^F(X^F,Y^F) \\ & +2\alpha u Hess^B(u)(X^B,Y^B)g^F(X^F,Y^F). \end{aligned} \quad (19)$$

Then, from (19) and (iv), we have $1+\alpha K_{P_6} > 0$, where $P_6 \subset T(x,y)M$, is spanned by vectors X and Y.
Therefore, at any point $(x,y) \in M = B \times F$ and all plane $P \subset T_{(x,y)}(B \times F)$, we have

$$1+\alpha K_P(g_0) > 0,$$

then from [4], there exist a unique solution $g(t)$ of (1) on $M = B \times_{u_0} F$ with $g_0 = g_0^B + u_0^2 g_0^F$. ■

Remark 3.1. The first condition of the short-time existence for the RG-2 flow on $(B \times_u F, g_0^B + u^2 g_0^F)$, satisfies the condition of existence of RG-2 flow for (B, g_0^B), too.

Remark 3.2. From (iii) and (iv), we know $\alpha \Delta^B u_0 \le m u_0$, where m is the dimension of B.

Remark 3.3. Let S^2 be endowed with metric $g_{S^2} = d\theta^2 + \sin^2\phi d\phi^2$ and (F, g_0^F) be a closed Riemannian manifold with positive sectional curvature bigger than 1. If in local coordinates $u_0(\theta,\phi) = \cos\theta\sin\phi$, where $-\frac{\pi}{2} < \theta < \frac{\pi}{2}$ and $0 < \phi < \pi$, then there exist a unique solution of RG-2 flow, with initial metric $g(0) = g_{S^2} + u_0^2 g_0^F$, on some interval $[0,T)$.

4. Evolution Of Warped Product Metrics

Now, we try to find a family of WP metrics $g(t) = g^B(t) + u^2(x,t)g^F(t)$ on some interval $[0,T)$, with $u : B \times [0,T) \to R^+$, that satisfy the RG-2 flow on product manifold $M = B \times F$. First, we calculate quadratic curvature Rm^2 of any WP manifold.
In local coordinates, we use $i, j, k, \ldots$ for the indices of coordinates of B, and $\beta, \gamma, \lambda, \ldots$ for indices of coordinates of F. Also, $A, B, C, \ldots$ show the indices of coordinates of B and F, generally.

Proposition 2. Let $M = B \times_u F$ be a warped product manifold, then at any point $(x,y) \in M$, for $X^B, Y^B \in T_xB$ and $X^F, Y^F \in T_yF$, the quadratic curvatures of warped product metric $g = g^B + u^2 g^F$ are as follows

$$\begin{array}{rcl} Rm^2(X^B,Y^F) & = & 0, \\ Rm^2(X^B,Y^B) & = & Rm^{2B}(X^B,Y^B) + \dfrac{2n}{u^2} g^B(Hess^B(u)(X^B,.), Hess^B(u)(Y^B,.)), \\ Rm^2(X^F,Y^F) & = & \frac{1}{u^2} Rm^{2F}(X^F,Y^F) - \frac{4}{u^2} |\nabla^B u|^2_{g^B} Ric^F(X^F,Y^F) \\ & & +2\frac{n-1}{u^2} |\nabla^B u|^4_{g^B} g^F(X^F,Y^F) + 2|Hess^B(u)|^2_{g^B} g^F(X^F,Y^F). \end{array} \tag{20}$$

where Ric^F denotes the Ricci tensor of (F, g^F), and Rm^{2B} and Rm^{2F} are quadratic curvatures of (B, g^B) and (F, g^F), respectively.

Proof. By curvature tensor properties in local coordinates;

$$\begin{array}{c} R_{ABCD} = -R_{BACD}, \\ R_{ABCD} = R_{CDAB} = R_{BADC}, \\ R_{ABCC} = R_{AACD} = 0. \end{array} \tag{21}$$

Also, by using (5), we have in local coordinates;

$$\begin{array}{c} R_{\beta ijk} = R_{\beta\gamma ij} = R_{\beta\gamma\theta i} = 0, \\ R_{ijkl} = R^B_{ijkl}, \\ R_{i\beta j\gamma} = -uHess^B(u)_{ij} g^F_{\beta\gamma}, \\ R_{\beta\gamma\theta\lambda} = u^2 R^F_{\beta\gamma\theta\lambda} - u^2 |\nabla^B u|^2_{g^B} (g^F_{\beta\theta} g^F_{\gamma\lambda} - g^F_{\gamma\theta} g^F_{\beta\lambda}). \end{array} \tag{22}$$

Then, from (2), (21) and (22) we have

$$\begin{array}{rcl} Rm^2_{i\beta} & = & g^{CD} g^{EF} g^{GH} R_{iCEG} R_{\beta DFH} \\ & = & g^{jk} g^{lp} gqm R_{ijlq} R_{\beta kpm} + g^{jk} g^{lp} g^{\mu\lambda} R_{ijl\mu} R_{\beta kp\gamma} + g^{jk} g^{\mu\gamma} g^{lm} R_{ij\mu l} R_{\beta k\gamma m} \\ & & + g^{jk} g^{\mu\gamma} g\lambda\eta R_{ij\mu\lambda} R_{\beta k\gamma\eta} + g^{\mu\gamma} g^{jk} g^{lm} R_{i\mu jl} R_{\beta\gamma km} + g^{\mu\gamma} g^{jk} g^{\lambda\eta} R_{i\mu j\lambda} R_{\beta\gamma k\eta} \\ & & + g^{\mu\gamma} g^{\lambda\eta} g^{jk} R_{i\mu\lambda j} R_{\beta\gamma\eta k} + g^{\mu\gamma} g^{\lambda\eta} g^{\theta\zeta} R_{i\mu\lambda\theta} R_{\beta\gamma\eta\zeta} = 0. \end{array}$$

All terms in above equation are zero. Similarly,

$$\begin{array}{rcl} Rm^2_{ij} & = & g^{CD} g^{EF} g^{GH} R_{iCEG} R_{jDFH} \\ & = & g^{kl} g^{mn} g^{pq} R_{ikmp} R_{jlnq} + g^{\mu\beta} g^{kl} g^{\gamma\lambda} R_{i\mu k\gamma} R_{j\beta l\lambda} + g^{\mu\beta} g^{\gamma\lambda} g^{kl} R_{i\mu\gamma k} R_{j\beta\lambda l} \\ & = & g^{B;kl} g^{B;mn} g^{B;pq} R^B_{ikmp} R^B_{jlnq} + 2(\frac{1}{u^2} g^{F;\mu\beta}) g^{B;kl} (\frac{1}{u^2} g^{F;\gamma\lambda}) \\ & & \times(-uHess^B(u)_{ik} g^{F;\mu\gamma})(-uHess^B(u)_j l g^F_{\beta\lambda}) \\ & = & Rm^{2B}_{ij} + \dfrac{2n}{u^2} g^{B;kl} Hess^B(u)_{ik} Hess^B(u)_{jl}. \end{array}$$

Also,

$$\begin{aligned}
Rm^2_{\beta\gamma} &= g^{CD}g^{EF}g^{GH}R_{\beta CEG}R_{\gamma DFH} \\
&= g^{\mu\lambda}g^{\eta\zeta}g^{\delta\theta}R_{\beta\mu\eta\delta}R_{\gamma\lambda\zeta\theta}+g^{ij}g^{kl}g^{\mu\lambda}R_{\beta ik\mu}R_{\gamma\lambda\zeta\theta}+g^{ij}g^{\mu\lambda}g^{kl}R_{\beta i\mu k}R_{\gamma j\lambda l} \\
&= (\tfrac{1}{u^2}g^{F;\mu\lambda})(\tfrac{1}{u^2}g^{F;\eta\zeta})(\tfrac{1}{u^2}g^{F;\delta\theta})\Big(u^2R^F_{\beta\mu\eta\delta}-u^2|\nabla^B u|^2_{g^B}(g^F_{\beta\eta}g^F_{\mu\delta}-g^F_{\mu\eta}g^F_{\beta\delta})\Big) \\
&\quad \times\Big(u^2R^F_{\gamma\lambda\zeta\theta}-u^2|\nabla^B u|^2_{g^B}(g^F_{\gamma\zeta}g^F_{\lambda\theta}-g^F_{\lambda\zeta}g^F_{\gamma\theta})\Big) \\
&\quad +2g^{F;ij}g^{F;kl}(\tfrac{1}{u^2}g^{F;\mu\lambda})(-uHess^B(u)_{ik}g^F_{\beta\mu})(-uHess^B(u)_{jl}g^F_{\gamma\lambda}) \\
&= \tfrac{1}{u^2}\Big(g^{F;\mu\lambda}g^{F;\eta\zeta}g^{F;\delta\theta}R^F_{\beta\mu\eta\delta}R^F_{\gamma\lambda\zeta\theta}+2|\nabla^B u|^4_{g^B}(n-1)g^F_{\beta\gamma}-4g^{F;\mu\delta}R^F_{\beta\mu\gamma\delta}|\nabla^B u|^2_{g^B}\Big) \\
&\quad +2g^{B;ij}g^{B;kl}Hess^B(u)_{ik}Hess^B(u)_{jl}g^F_{\beta\gamma} \\
&= \tfrac{1}{u^2}Rm^{2F}_{\beta\gamma}+2\tfrac{n-1}{u^2}|\nabla^B u|^4_{g^B}g^F_{\beta\gamma}-\tfrac{4}{u^2}Ric^F_{\beta\gamma}|\nabla^B u|^2_{g^B}+2|Hess^B(u)|^2_{g^B}g^F_{\beta\gamma},
\end{aligned}$$

where $|Hess^B(u)|^2_{g^B}=g^{B;ij}g^{B;kl}Hess^B(u)_{ik}Hess^B(u)_{jl}$. ■

We know from remark 3.1, the conditions for short-time existence of RG-2 flow on $B\times_u F$ with initial WP metric $g_0=g^B_0+u^2g^F_0$, we show that at least there is a solution of RG-2 flow on B with initial metric g^B_0. So, for finding RG-2 flow $g(t)$ on $M=B\times F$, we assume there is RG-2 flow $g^B(t)$ and $g^F(t)$ on manifolds B and F, with initial metric g^B_0 and g^F_0, respectively.

Proposition 3. Let $(B,g^B(t))$ and $(F,g^F(t))$ be the RG-2 flows, i.e. $\partial_t g^B=-2Ric^B-\frac{\alpha}{2}Rm^{2B}$ and $\partial_t g^F=-2Ric^F-\frac{\alpha}{2}Rm^{2F}$. Let $u:B\times[0,T)\to R^+$ be a function satisfying

$$uHess^B(u)(X^B,Y^B) = \frac{\alpha}{2}g^B(Hess^B(u)(X^B,.),Hess^B(u)(Y^B,.)), \tag{23}$$

$$\begin{aligned}
\partial_t u g^F(X^F,Y^F) &= \frac{n-1}{u^3}|\nabla^B u|^2_{g^B}\Big(u^2-\frac{\alpha}{2}|\nabla^B u|^2_{g^B}\Big)g^F(X^F,Y^F) \\
&\quad +\Big(\frac{\alpha}{u^3}|\nabla^B u|^4_{g^B}+u-\frac{1}{u}\Big)Ric^F(X^F,Y^F) \\
&\quad +\frac{\alpha}{4}\frac{u^4-1}{u^3}Rm^{2F}(X^F,Y^F).
\end{aligned} \tag{24}$$

for any vectors $X^B,Y^B\in TB$ and $X^F,Y^F\in TF$. Then, there is solution of RG-2 flow on $M=B\times_u F$ with the metric $g(t)=g^B(t)+u(t)^2g^F(t)$.

Proof. From (23), we have

$$Hess^B(u)_{ij}=\frac{\alpha}{2}\frac{1}{u}g^{B;kl}Hess^B(u)_{ik}Hess^B(u)_{jl} \tag{25}$$

Also, we have

$$\partial_t g_{ij}=\partial_t g^B_{ij}=-2Ric^B_{ij}-\frac{\alpha}{2}Rm^{2B}_{ij}. \tag{26}$$

Therefore by (4), (20), (25) and (26), we have $\partial_t g_{ij}=-2Ric_{ij}-\dfrac{\alpha}{2}Rm^2_{ij}$.

Since the $\Delta_{g^B}u=g^{B;ij}Hess^B(u)_{ij}$, by taking trace of the two sides of equality (25), we have

$$\Delta_{g^B}u=\frac{\alpha}{2}\frac{1}{u}|Hess^B(u)|^2_{g^B}. \tag{27}$$

From (24) and (27), we have

$$
\begin{aligned}
2u\partial_t u g^F &= 2(u^2-2)Ric^F_{\beta\gamma} + \frac{\alpha}{2}(u^2-\frac{1}{u^2})Rm^{2F}_{\beta\gamma} \\
&+2\Big(u\Delta_{g^B}u + (n-1)|\nabla^B u|^2_{g^B}\Big)g^F_{\beta\gamma} - \alpha\frac{n-1}{u^2}|\nabla^B u|^4_{g^B}g^F_{\beta\gamma} \\
&+\frac{2\alpha}{u^2}|\nabla^B u|^4_{g^B}Ric^F_{\beta\gamma} - \alpha|Hess^B(u)|^2_{g^B}g^F_{\beta\gamma}.
\end{aligned} \tag{28}
$$

Also,

$$
\begin{aligned}
\partial_t g_{\beta\gamma} &= \partial_t(u^2 g^F_{\beta\gamma}) = 2u\partial_t u g^F_{\beta\gamma} + u^2\partial g^F_{\beta\gamma} \\
&= 2u\partial_t u g^F_{\beta\gamma} - 2u^2 Ric^F_{\beta\gamma} - u^2\frac{\alpha}{2}Rm^{2F}_{\beta\gamma}.
\end{aligned} \tag{29}
$$

From (4), (20), (28) and (29), we have $\partial_t g_{\beta\gamma} = -Ric_{\beta\gamma} - \frac{\alpha}{2}Rm^2_{\beta\gamma}$. Also, from (4) and (20), $\partial_t g_{i\beta} = -Ric_{i\beta} - \frac{\alpha}{2}Rm^2_{i\beta}$ is obvious ∎

Corollary 1. With the same assumption as in proposition (3), a necessary conditions for solution of the second order renormalization group flow is

$$
u\Delta_{g^B}u = \frac{\alpha}{2}|Hess^B(u)|^2_{g^B}, \tag{30}
$$

$$
\begin{aligned}
\partial_t u &= \frac{1}{n}\Big(\frac{\alpha}{u^3}|\nabla^B u|^4_{g^B} + \frac{u^2-1}{u}\Big)scal^F + \frac{\alpha}{4n}\frac{u^4-1}{u^3}|R^F|^2_{g^B} \\
&+\frac{n-1}{u^3}|\nabla^B u|^2_{g^B}\Big(u^2 - \frac{\alpha}{2}|\nabla^B u|^2_{g^B}\Big).
\end{aligned} \tag{31}
$$

where $scal^F$ and R^F denote the scaler curvature and Riemannian curvature of $(F, g^F(t))$.

Proof. The equation (30) will proved by taking trace of (23). We have

$$
tr_{g^F}Rm^{2F} = g^{F;\beta\gamma}Rm^{2F}_{\beta\gamma} = g^{F;\beta\gamma}g^{F;\zeta\lambda}g^{F;\eta\mu}g^{F;\varepsilon\delta}R^F_{\beta\zeta\eta\varepsilon}R^F_{\gamma\lambda\mu\delta} = |R^F|^2_{g^F}. \tag{32}
$$

By taking trace of the two sides of equality (24), the equality (31) can be concluded. ∎

Remark 4.1. Note, that the eqality (23), for Ricci flow reduces to $Hess^B(u)(X^B, Y^B) = 0$ [12].

Remark 4.2. Any part of (31) depends on parameter t and point $x \in B$, and is independent of $y \in F$, except the first two, they depend on point $y \in F$, as well. So,

$$
\Big(\frac{u^2-1}{u} + \frac{\alpha}{u^3}|\nabla^B u|^4_{g^B}\Big)scal^F + \frac{\alpha}{4}\frac{u^4-1}{u^3}|R^F|^2_{g^F}
$$

is independent of y.

For the Ricci flow, the necessary condition reduces to [12]

$$
\partial_t u = \Delta_{g^B}u + \frac{n-1}{u}|\nabla^B u|^2_{g^B} + \frac{u^2-1}{nu}scal^F.
$$

As it is seen, all part are independent of $q \in F$, except the last one. So, $scal^F$ is independent of q, this shows Ricci flow $(F, g^F(t))$ has constant scaler curvature. Comparing with Ricci flow, this

fact for second order renormalization group flow is totally different.
We have for the 2-dimensional Riemannian manifold (F, g^F), $Ric^F = \frac{1}{2} scal^F g^F$ and $Rm^{2F} = \frac{1}{2}(scal^F)^2 g^F$ [14]. So, we can simplify corollary (1) to the following corollary.

Corollary 2. Let (B, g_0^B) be a Riemannian manifold and (F, g_0^F) be a 2-dimensional Riemannian manifold, and $g^B(t)$ and $g^F(t)$ be solution of RG-2 flow on B and F with initial value g_0^B and g_0^F, respectively. Obviously, the WP metric $g(t) = g^B(t) + u^2(t) g^F(t)$ on $M = B \times F$ is RG-2 flow if

$$\begin{aligned} \partial_t u &= \frac{1}{2}\Big(\frac{\alpha}{u^3}|\nabla^B u|^4_{g^B} + \frac{u^2-1}{u}\Big) scal^F + \frac{\alpha}{8}\frac{u^4-1}{u^3}(scal^F)^2 \\ &\quad + \frac{1}{u^3}|\nabla^B u|^2_{g^B}\Big(u^2 - \frac{\alpha}{2}|\nabla^B u|^2_{g^B}\Big). \end{aligned}$$

Now, important question is whether there is a solution of PDE system (23) and (24)? We know, that it can not be solved, by standard methods for solving a PDE. Therefore, we assume that manifold F has constant sectional curvature, and we will get a simpler differential equation, which does not depend on the points of manifold F.

Proposition 4. Let (B, g_0^B) be a Riemannian manifold and $g^B(t)$ be a solution of RG-2 on B with initial value $g^B(0) = g_0^B$. Moreover, let (F, g_k^F) be a Riemannian manifold with constant curvature k. Then the WP metric $g(t) = g^B(t) + u^2 g_k^F$ on $M = B \times F$ is an RG-2 flow, if the warped function u satisfies

$$u\Delta_{g^B} u = \frac{\alpha}{2}|Hess^B(u)|^2_{g^B}, \tag{33}$$

$$\partial_t u = \frac{n-1}{u}\Big(|\nabla^B u|^2_{g^B} - k\Big) - \frac{\alpha}{2}\frac{n-1}{u^3}\Big(|\nabla^B u|^2_{g^B} - k\Big)^2 \tag{34}$$

Proof. Analogous to the proposition 3 and corollary 1, from (33) and $\partial_t g^B_{ij} = -2Ric^B_{ij} - \frac{\alpha}{2} Rm^{2B}_{ij}$, we conclude $\partial_t g_{ij} = -2Ric_{ij} - \frac{\alpha}{2} Rm^2_{ij}$.
We have for constant curvature manifold g_k^F,

$$\begin{aligned} \hat{Ric} &= k(n-1) g_k^F, \\ Rm^{2F} &= 2k^2(n-1) g_k^F. \end{aligned} \tag{35}$$

As a result of (4), (20) and (35);

$$\begin{aligned} Ric_{\beta\gamma} &= \Big(k(n-1) - u\Delta_{g^B} u - (n-1)|\nabla^B u|^2_{g^B}\Big) g^F_{\beta\gamma} \\ Rm^2_{\beta\gamma} &= \Big(2k^2\tfrac{n-1}{u^2} - 4k\tfrac{n-1}{u^2}|\nabla^B u|^2_{g^B} + 2\tfrac{n-1}{u^2}|\nabla^B u|^4_{g^B} + 2|Hess^B(u)|^2_{g^B}\Big) g^F_{\beta\gamma} \\ &= 2\tfrac{n-1}{u^2}\Big(|\nabla^B u|^2_{g^B} - k\Big)^2 g^F_{\beta\gamma} + 2|Hess^B(u)|^2_{g^B} g^F_{\beta\gamma} \end{aligned} \tag{36}$$

Also, we have

$$\partial_t g_{\beta\gamma} = \partial_t (u^2 g^F_{\beta\gamma}) = 2u\partial_t u g^F_{\beta\gamma} \tag{37}$$

From (33) and (34) we have

$$u\partial_t u = (n-1)\Big(|\nabla^B u|^2_{g^B} - k\Big) - \frac{\alpha}{2}\frac{n-1}{u^2}\Big(|\nabla^B u|^2_{g^B} - k\Big)^2 + u\Delta_{g^B} u - \frac{\alpha}{2}|Hess^B(u)|^2_{g^B} \tag{38}$$

From (36), (37) and (38), we have $\partial_t g_{\beta\gamma} = -2Ric_{\beta\gamma} - \frac{\alpha}{2} Rm^2_{\beta\gamma}$. ∎

Corollary 3. The function u in proposition 4, satisfies the following inequality

$$u(x,t) \leq U(x)\exp(\frac{n-1}{2\alpha}t), \tag{39}$$

where U is a positive real function on B, i.e. $U : B \to R^+$.

Proof. From (34), we have

$$\begin{aligned} &\alpha^2\Big(|\nabla^B u|^2_{g^B} - k\Big)^2 - 2\alpha u^2\Big(|\nabla^B u|^2_{g^B} - k\Big) = -\frac{2\alpha u^3}{n-1}\partial_t u \\ \Rightarrow\quad &\Big(\alpha|\nabla^B u|^2_{g^B} - k\alpha - u^2\Big)^2 = u^4 - \frac{2\alpha u^3}{n-1}\partial_t u = \frac{u^3}{n-1}\Big((n-1)u - 2\alpha\partial_t u\Big). \end{aligned}$$

So $(n-1)u - 2\alpha\partial_t u \geq 0$, that means function u is a sub-solution of the following

$$\frac{\partial_t u}{u} \leq \frac{n-1}{2\alpha}.$$

∎

Now, as an example of the gained equation, we study the case $(B, g^B) = (R, ds^2)$.

Corollary 4. Let (F, g^F_k) be a n-dimensional manifold, with constant curvature k. If the function $u(s,t)$ satisfies the following PDEs:

$$uu_{ss} = \frac{\alpha}{2}(u_{ss})^2 \tag{40}$$

$$u_t = \frac{n-1}{u}\Big((u_s)^2 - k\Big) - \frac{\alpha}{2}\frac{n-1}{u^3}\Big((u_s)^2 - k\Big)^2, \tag{41}$$

or, equivalently,

$$u^3(u - \frac{2\alpha}{n-1}u_t) = \Big(\alpha(u_s)^2 - k\alpha - u^2\Big)^2, \tag{42}$$

the warped product metric $g(t) = ds^2 + u^2(s,t)g^F_k$ on $M = I \times_u F$ is a solution of RG-2 flow.

Remark 4.3. From (40), u is the solution of ODE $u_{ss} = 0$ or $\frac{\alpha}{2}u_{ss} = u$. So,the warped function u is one of the following:

i) $u(s,t) = A(t)s + B(t)$,

ii) $u(s,t) = A(t)\exp(as) + B(t)\exp(-as)$,

where A and B are real functions, and $\alpha a^2 = 2$.

Corollary 5. Let the warped product metric $g(t) = ds^2 + u^2(s,t)g^F_k$ is a fixed point of RG-2 flow. Then, the warped function u has one of the following forms:

i) $u(s) = bs + c$, for $k = b^2$,

ii) $u(s) = \frac{b}{a}\sinh(as + c)$, for $k = b^2$,

iii) $u(s) = \frac{b}{a}\cosh(as + c)$, for $k = -b^2$,

iv) $u(s) = \exp(as + c)$, for $k = 0$,

where $\alpha a^2 = 2$, $b \in \mathbb{R}$ and c is a constant parametr.

Proof. If $u_t = 0$, then from (41)

$$(u_s)^2 = k. \tag{43}$$

Or

$$(u_s)^2 = \frac{2}{\alpha}u^2 + k. \tag{44}$$

Let $k = b^2$, then from (43), $u(s) = bs + c$, this proves (i).
Also, from (44), $u_s = \sqrt{a^2u^2 + b^2}$, where $\alpha a^2 = 2$. The solution of the above equation is as follows

$$Ln(\sqrt{a^2u^2 + b^2} + au) - Lnb = as + c.$$

So

$$u(s) = \frac{b}{a}\sinh(as + c),$$

wich is (ii).
Let $k = -b^2$, then from (44), $u_s = \sqrt{a^2u^2 - b^2}$, where $\alpha a^2 = 2$. Also, The solution of above equation is as follow

$$Ln(\sqrt{a^2u^2 - b^2} + au) - Lnb = as + c.$$

So

$$u(s) = \frac{b}{a}\cosh(as + c).$$

wich proves (iii).
Now. let $k = 0$, then from (44), $u_s = au$, where $\alpha a^2 = 2$. So,

$$u(s) = \exp(as + c).$$

i.e. (v) ■

Corollary 6. There is no RG-2 flow on WP manifolds $M = R \times_u F$ with WP metric $g(t) = ds^2 + u^2(s,t)g_k^F$, except fixed point of RG-2 flow, or (non-warped) product metric $g(t) = ds^2 + u^2(t)g_k^F$.

Proof. From part (i) of remark 4.3, let $u(s,t) = A(t)s + B(t)$. we have, $u_s = A$ and $u_t = A^{'}s + B^{'}$. So

$$\begin{aligned}
u^3(u - \frac{2\alpha}{n-1}u_t) &= (As + B)^3\Big(As + B - \frac{2\alpha}{n-1}(A^{'}s + B^{'})\Big) \\
&= \Big(A^4 - \frac{2\alpha}{n-1}A^3A^{'}\Big)s^4 + \Big(4A^3B - \frac{2\alpha}{n-1}A^2(AB^{'} + 3BA^{'})\Big)s^3 \\
&\quad \Big(6A^2B^2 - \frac{6\alpha}{n-1}AB(BA^{'} + AB^{'})\Big)s^2 \\
&\quad + \Big(4AB^3 - \frac{2\alpha}{n-1}B^2(3AB^{'} + BA^{'})\Big)s + B^3\Big(B - \frac{2\alpha}{n-1}B^{'}\Big).
\end{aligned} \tag{45}$$

Similarly

$$\begin{aligned}
\Big(\alpha(u_s)^2-k\alpha-u^2\Big)^2 &= \Big(\alpha A^2-k\alpha-(As+B)^2\Big)^2\\
&= A^4s^4+4A^3Bs^3+\Big(4A^2B^2+2A^2\alpha(kB^2-A^2)\Big)s^2 \\
&\quad +4AB\Big(B^2+k\alpha-\alpha A^2\Big)s+\Big(B^2+k\alpha-\alpha A^2\Big)^2.
\end{aligned} \tag{46}$$

From (45), (46) and by comparing the coefficients of $s^4, s^3, \ldots$ at two sides of (42), we have

$$\begin{aligned}
&AA^{'}=0,\\
&A(AB^{'}+3BA^{'})=0,\\
&2A^2\alpha(kB^2-A^2)=2A^2B^2-\frac{6\alpha}{n-1}AB(BA^{'}+AB^{'}),\\
&2(n-1)AB(k-A^2)+B^2(3AB^{'}+BA^{'})=0,\\
&\alpha(k-A^2)^2+2kB^2-2A^2B^2+\frac{2}{n-1}B^3B^{'}=0.
\end{aligned} \tag{47}$$

From, first condition of (47), we have $A=0$ or $A^{'}=0$. Let $A=0$, so

$$\alpha k^2+2kB^2+\frac{2}{n-1}B^3B^{'}=0. \tag{48}$$

For $k=0$, B is constant.
For $k=-b^2$, we have $B=\frac{b}{a}$, where $\alpha a^2=2$. Or, B is the following implicit function

$$B^2=\frac{\alpha}{2}kLn(\alpha k+2B^2)+2k(1-n)t+c, \tag{49}$$

where c is a constant dependent on u_0.
For $k=b^2$, B is not constant, and B is the implicit function (49).
Now, Let $A^{'}=0$ and $A\neq 0$, from the second condition of (47), we have $B^{'}=0$. So, from (47)

$$\begin{aligned}
&(\alpha k-1)B^2=A^2,\\
&B(k-A^2)=0,\\
&\alpha(k-A^2)^2+2kB^2-2A^2B^2=0.
\end{aligned} \tag{50}$$

Since the $A\neq 0$ and from the first condition of (50), we have $B\neq 0$ and $\alpha k\neq 1$, so from second condition we have $k=A^2$. It is the same as part (i) of corollary 5.
From part (ii) of remark 4.3, let $u(s,t)=A(t)\exp(as)+B(t)\exp(-as)$. we have, $u_s=aA\exp(as)-$

$aB\exp(-as)$ and $u_t = A^{'}\exp(as)+B^{'}\exp(-as)$, where $\alpha a^2 = 2$. So

$$\begin{aligned} u^3(u-\frac{2\alpha}{n-1}u_t) &= \Big(A\exp(as)+B\exp(-as)\Big) \\ &\quad \times\Big(A\exp(as)+B\exp(-as)-\frac{2\alpha}{n-1}(A^{'}\exp(as)+B^{'}\exp(-as)\Big) \\ &= A^3\Big(A-\frac{2\alpha}{n-1}A^{'}\Big)\exp(4as)+B^3\Big(B-\frac{2\alpha}{n-1}B^{'}\Big)\exp(-4as) \\ &\quad \Big(4A^3B-\frac{2\alpha}{n-1}A^2(AB^{'}+3BA^{'})\Big)\exp(2as) \\ &\quad +\Big(4AB^3-\frac{2\alpha}{n-1}B^2(BA^{'}+3AB^{'})\Big)\exp(-2as) \\ &\quad \Big(3A^2B(B-\frac{2\alpha}{n-1}B^{'})+3AB^2(A-\frac{2\alpha}{n-1}A^{'})\Big). \end{aligned} \tag{51}$$

Moreover,

$$\begin{aligned} (\alpha(u_s)^2-k\alpha-u^2)^2 &= \Big(\alpha(aA\exp(as)-aB\exp(-as))^2-k\alpha-(A\exp(as) \\ &\quad +B\exp(-as))^2\Big)^2 \\ &= A^4\exp(4as)+B^4\exp(-4as)+(38A^2B^2+K^2\alpha^2+12k\alpha AB) \\ &\quad -2A^2(6AB+k\alpha)\exp(2as)-2B^2(6AB+k\alpha)\exp(-2as). \end{aligned} \tag{52}$$

From (51), (52) and by comparing two sides of (42), we have

$$\begin{aligned} &AA^{'}=0, \\ &BB^{'}=0, \\ &16A^3B+2k\alpha A^2-\frac{2\alpha}{n-1}A^2(AB^{'}+3BA^{'})=0, \\ &16AB^3+2k\alpha B^2-\frac{2\alpha}{n-1}B^2(BA^{'}+3AB^{'})=0, \\ &32A^2B^2+\frac{6\alpha}{n-1}AB(AB^{'}+BA^{'})+k^2\alpha^2+12k\alpha AB=0. \end{aligned} \tag{53}$$

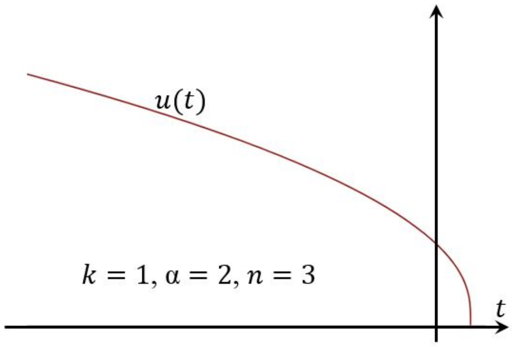
From the first condition of (53), $A=0$ or $A^{'}=0$, and from second condition of that, $B=0$ or $B^{'}=0$. By some conclusion we have either $A=B=k=0$, or $A,B\neq 0$. If $A,B\neq 0$ and $A^{'}=B^{'}=0$, from third condition of (53), we have $8AB=-k\alpha$. It is the same as parts (ii) and (iii) of corollary 5. ■

Corollary 7. Let (N,h) be a n-dimensional non-flat manifolds with constant curvature k. Let $u(t)$ is the following implicit function:

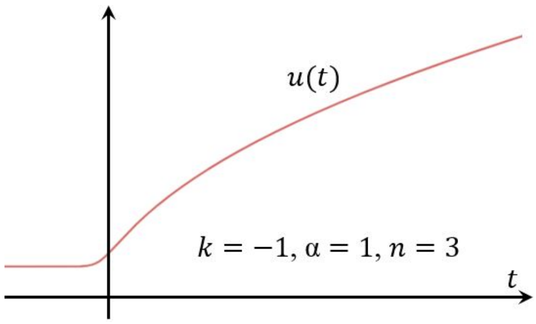
$$u^2(t)=\frac{\alpha}{2}kLn\Big(\frac{\alpha k+2u^2(t)}{\alpha k+2}\Big)+2k(1-n)t+1, \tag{54}$$

the metric family $g(t)=ds^2+u^2(t)h$ is RG-2 flow, with $u(0)=1$.

In special case, if $\alpha=2$, $n=3$ and $k=1$, the function $u(t)$ is as the following:



it shows that the solution is ancient, and pinch at time $t_0 = \frac{1}{4}(1 - Ln2)$. Also, if $\alpha = 1$, $n = 3$ and $k = -1$, the function $u(t)$ is as the following:



5. Evolution Of RobertsonWalker Metrics

One of the important pseudo-Riemannian metrics in general relativity is Robertson-Walker metric. We shall consider a RobertsonWalker spacetime as a warped product metric.
The Properties (4) and (5) are established for pseudo-Riemannian manifolds, too. So, the propositions 2 and 3 apply to warped product manifolds of pseudo-Riemannian manifolds.

Proposition 5. Let (N,h) be a 3-dimensional manifold, with constant curvature k, the Robertson-Walker metric $g(t) = -ds^2 + f^2(s,t)h$ is a solution of (1), if f satisfies the following PDEs:

$$f f_{ss} = -\frac{\alpha}{2}(f_{ss})^2 \tag{55}$$

$$f_t = -\frac{2}{f}\Big((f_s)^2 + k\Big) - \frac{\alpha}{f^3}\Big((f_s)^2 + k\Big)^2. \tag{56}$$

Proof. Let $(B, g^B) := (I, -ds^2)$ and $(F, g^F) := (N,h)$. We have $\nabla^B f = -f_{ss}$ and $Hess^B(f) = f_{ss}$
Then,

$$|\nabla f|^2_{g^B} = g^B(f_s\partial_s, f_s\partial_s) = -(f_s)^2,$$

and

$$|Hess^B(f)|_{g^B} = (f_{ss})^2.$$

From proposition (4), desired outcome is obtained. ■

From (55), is either $f(s,t) = A(t)s + B(s)$, or $f(s,t) = A(t)\sin(as) + B(t)\cos(as)$. Therefore, the following corollary from [10] is a consequence of corollary (6).

Corollary 8. There is no RG-2 flow on Robertson-Walker manifolds with WP metric $g(t) = ds^2 + f^2(s,t)h$.

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