Representations and $T^*$-extensions of $\delta$-Bihom-Jordan-Lie algebras

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Abstract

The purpose of this article is to study representations of $\delta$-Bihom-Jordan-Lie algebras. In particular, adjoint representations, trivial representations, deformations, $T^*$-extensions of $\delta$-Bihom-Jordan-Lie algebras are studied in detail. Derivations and central extensions of $\delta$-Bihom-Jordan-Lie algebras are also discussed as an application.

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1. Introduction

The notion of Jordan-Lie algebras was introduced in [7], which is closely related to both Lie and Jordan superalgebras. Engel’s theorem of Jordan-Lie algebras was proved, and some properties of Cartan subalgebras of Jordan-Lie algebras were given in [8].

Recently, the definition of $\delta$-hom-Jordan-Lie algebras were introduced in [10], and their representations and $T^*$-extensions were studied in detail.

A Bihom-algebra is an algebra in such a way that the identities defining the structure are twisted by two homomorphisms $\alpha$, $\beta$. This class of algebras was introduced from a categorical approach in [4] as an extension of the class of Hom-algebras. The origin of Hom-structures can be found in the physics literature around 1900, appearing in the study of quasi deformations of Lie algebras of vector fields, in particular q-deformations of Witt and Virasoro algebras in [5]. Since then, many authors have been interested in the study of Hom-algebras, mainly motivated by their applications in mathematical physics (see for instance the recent references [1, 6]). The fundamental for getting the basic notions, motivations, and results on Bihom-algebras is the reference [4].

More applications of the Bihom-Lie algebras, Bihom-algebras, Bihom-Lie superalgebras and Bihom-Lie admissible superalgebras can be found in [3, 9].

The notion of derivations, representations, and $T^*$-extensions of $\delta$-Bihom-Jordan-Lie algebras are not so well developed.

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The paper is organized as follows. In Section 2 we give the definition of \( \delta \)-Bihom-Jordan-Lie algebras, and show that the direct sum of two \( \delta \)-Bihom-Jordan-Lie algebras is still a \( \delta \)-Bihom-Jordan-Lie algebra. A linear map between \( \delta \)-Bihom-Jordan-Lie algebras is a morphism if and only if its graph is a Bihom subalgebra. In Section 3 we study derivations of multiplicative \( \delta \)-Bihom-Jordan-Lie algebras. For any nonnegative integers \( k \) and \( l \), we define \( \alpha^k \beta^l \)-derivations of multiplicative \( \delta \)-Bihom-Jordan-Lie algebras. Considering the direct sum of the space of \( \alpha^k \beta^l \)-derivations, we prove that it is a Lie algebra. In particular, any \( \alpha^k \beta^l \)-derivation gives rise to a derivation extension of the multiplicative \( \delta \)-hom-Jordan-Lie algebra \((L, [\cdot, \cdot]_L, \alpha, \beta)\) (Theorem 3.3). In Section 4 we give the definition of representations of multiplicative \( \delta \)-Bihom-Jordan-Lie algebras. We can obtain the semidirect product \( \delta \)-Bihom-Jordan-Lie algebra \((L \oplus M, [\cdot, \cdot]_L, \alpha + \alpha_M, \beta + \beta_M)\) associated to any representation \( \rho \) on \( M \) of the \( \delta \)-Bihom-Jordan-Lie algebra \((L, [\cdot, \cdot]_L, \alpha, \beta)\). In Section 5 we study trivial representations of multiplicative \( \delta \)-Bihom-Jordan-Lie algebras. We show that central extensions of a multiplicative \( \delta \)-Bihom-Jordan-Lie algebra are controlled by the second cohomology with coefficients in the trivial representation. In Section 6 we study the adjoint representation of a regular \( \delta \)-Bihom-Jordan-Lie algebra \((L, [\cdot, \cdot]_L, \alpha, \beta)\). For any integers \( s, t \), we define the \( \alpha^s \beta^t \)-derivations. We show that a 1-cocycle associated to the \( \alpha^s \beta^t \)-derivation is exactly an \( \alpha^{s+2} \beta^{t-1} \)-derivation of the regular \( \delta \)-Bihom-Jordan-Lie algebra \((L, [\cdot, \cdot]_L, \alpha, \beta)\) in some conditions. We also give the definition of Bihom-Nijenhuis operators of regular \( \delta \)-Bihom-Jordan-Lie algebras. We show that the deformation generated by a Bihom-Nijenhuis operator is trivial. In Section 7 we study \( T^* \)-extensions of \( \delta \)-Bihom-Jordan-Lie algebras, show that \( T^* \)-extensions preserve many properties such as nilpotency, solvability and decomposition in some sense.

2. Definitions and properties of \( \delta \)-Bihom-Jordan-Lie algebras

**Definition 2.1.** ([7]) A \( \delta \)-Jordan Lie algebra is a couple \((L, [\cdot, \cdot]_L)\) consisting of a vector space \( L \) and a bilinear map (bracket) \([\cdot, \cdot]_L : L \times L \to L\) satisfying
\[
[x, y] = -\delta[y, x], \quad \delta = \pm 1, \\
[x, [y, z]] + [y, [z, x]] + [z, [x, y]] = 0, \quad \forall x, y, z \in L.
\]

**Definition 2.2.** ([10]) A \( \delta \)-hom-Jordan Lie algebra is a triple \((L, [\cdot, \cdot]_L, \alpha)\) consisting of a vector space \( L \), a bilinear map (bracket) \([\cdot, \cdot]_L : L \otimes L \to L\) and a linear map \( \alpha : L \to L\) satisfying
\[
[x, y] = -\delta[y, x], \quad \delta = \pm 1, \\
[\alpha(x), [y, z]] + [\alpha(y), [z, x]] + [\alpha(z), [x, y]] = 0, \quad \forall x, y, z \in L.
\]

Especially, for \( \delta = 1 \) one has a hom-Lie algebra and for \( \delta = -1 \) a hom-Jordan Lie algebra.

**Definition 2.3.** ([3]) A Bihom-Lie algebra is a 4-tuple \((L, [\cdot, \cdot]_L, \alpha, \beta)\) consisting of vector space \( L \), a bilinear map \([\cdot, \cdot] : L \times L \to L\) and two homomorphisms \( \alpha, \beta : L \to L\) such that for all elements \( x, y, z \in L \) we have
\[
\alpha \circ \beta = \beta \circ \alpha, \\
[\beta(x), \alpha(y)] = -[\beta(y), \alpha(x)], \\
[\beta^2(x), [\beta(y), \alpha(z)]] + [\beta^2(y), [\beta(z), \alpha(x)]] + [\beta^2(z), [\beta(x), \alpha(y)]] = 0
\]
(Bihom-Jacobi equation).

**Definition 2.4.** A \( \delta \)-Bihom-Jordan Lie algebra is a 4-tuple \((L, [\cdot, \cdot]_L, \alpha, \beta)\) consisting of a vector space \( L \), a bilinear map (bracket) \([\cdot, \cdot]_L : L \otimes L \to L\) and two linear maps
\(\alpha, \beta : L \to L\) satisfying
\[
\alpha \circ \beta = \beta \circ \alpha,
\]
(2.1)
\[
[\beta(x), \alpha(y)] = -\delta[\beta(y), \alpha(x)], \delta = \pm 1,
\]
(2.2)
\[
[\beta^2(x), [\beta(y), \alpha(z)]] + [\beta^2(y), [\beta(z), \alpha(x)]] + [\beta^2(z), [\beta(x), \alpha(y)]] = 0, \forall x, y, z \in L.
\]
(2.3)
Especially, for \(\delta = 1\) one has a Bihom-Lie algebra and for \(\delta = -1\) a Bihom-Jordan Lie algebra.

**Definition 2.5.** 1) A \(\delta\)-Bihom-Jordan Lie algebra \((L, [\cdot, \cdot]_L, \alpha, \beta)\) is multiplicative if \(\alpha\) and \(\beta\) are algebra morphisms, i.e., for any \(x, y \in L\), we have
\[
\alpha([x, y])_L = [\alpha(x), \alpha(y)]_L \quad \text{and} \quad \beta([x, y])_L = [\beta(x), \beta(y)]_L.
\]
2) A \(\delta\)-Bihom-Jordan Lie algebra \((L, [\cdot, \cdot]_L, \alpha, \beta)\) is regular if \(\alpha\) and \(\beta\) are algebra automorphisms.
3) A subvector space \(\eta \subset L\) is a Bihom subalgebra of \((L, [\cdot, \cdot]_L, \alpha, \beta)\) if \(\alpha(\eta) \subset \eta\), \(\beta(\eta) \subset \eta\) and
\[
[x, y]_L \in \eta, \quad \forall x, y \in \eta.
\]
4) A subvector space \(\eta \subset L\) is a Bihom ideal of \((L, [\cdot, \cdot]_L, \alpha, \beta)\) if \(\alpha(\eta) \subset \eta, \beta(\eta) \subset \eta\) and
\[
[x, y]_L \in \eta, \quad \forall x \in \eta, y \in L.
\]

**Definition 2.6.** A \(\delta\)-Bihom associative algebra is a triple \((L, \alpha, \beta)\) consisting of a vector space \(L\), a bilinear map on \(L\), and two linear commuting maps \(\alpha, \beta : L \to L\) satisfying
\[
\alpha(x)(yz) = \delta(xy)\beta(z), \quad \forall x, y, z \in L.
\]
(2.4)

**Proposition 2.7.** Let \((L, \alpha, \beta)\) be a multiplicative \(\delta\)-Bihom associative algebra. Define a bilinear map (bracket) \([\cdot, \cdot]_L : L \times L \to L\) satisfying
\[
[x, y]_L = xy - \delta\alpha^{-1}(\beta(y))\beta^{-1}(\alpha(x)), \forall x, y \in L.
\]
(2.5)

Then \((L, [\cdot, \cdot]_L, \alpha, \beta)\) is a \(\delta\)-Bihom-Jordan-Lie algebra.

**Proof.** First we check that the bracket product \([\cdot, \cdot]\) is compatible with the structure maps \(\alpha\) and \(\beta\). For any \(x, y \in L\), we have
\[
[\alpha(x), \alpha(y)] = \alpha(x)\alpha(y) - \delta(\alpha^{-1}\beta(\alpha(y)))\alpha\beta^{-1}(\alpha(x)) = \alpha(x)\alpha(y) - \delta(\beta(y)\alpha^2\beta^{-1}(x)) = \alpha([x, y]).
\]
Similarly, one can prove that \(\beta([x, y]) = [\beta(x), \beta(y)]\).

And
\[
[\beta(x), \alpha(y)] = \beta(x)\alpha(y) - \delta(\alpha^{-1}\beta(\alpha(y)))\alpha\beta^{-1}(\beta(x)) = \beta(x)\alpha(y) - \delta(\beta(y)\alpha(x)) = -\delta[\beta(y), \alpha(x)].
\]

Now we prove the Bihom-Jacobi condition. For any elements \(x, y \in L\), we have
\[
[\beta^2(x), [\beta(y), \alpha(z)]] = [\beta^2(x), \beta(y)\alpha(z) - \delta\alpha^{-1}\beta(\alpha(z))\alpha\beta^{-1}(\beta(y))] = [\beta^2(x), \beta(y)\alpha(z)] - \delta[\beta^2(x), \beta(z)\alpha(y)] = (\beta^2(x)[\beta(y)\alpha(z)] - \delta(\alpha^{-1}(\beta^2(y))\beta(z)\alpha(\beta(x)))) - \delta(\beta^2(x)\beta(z)\alpha(y)) = \delta(\alpha^{-1}(\beta^2(y))\beta(z)\alpha(\beta(x))).
\]
Similarly, we have

\[ \begin{align*}
[\beta^2(y), [\beta(z), \alpha(x)]] &= \left( \beta^2(y)(\beta(z)\alpha(x)) - \delta(\alpha^{-1}(\beta^2(z))\beta(x))\alpha(\beta(y)) \right) \\
- \delta &\left( \beta^2(y)(\beta(x)\alpha(z)) - \delta(\alpha^{-1}(\beta^2(x))\beta(z))\alpha(\beta(y)) \right)
\end{align*} \]

\[ \begin{align*}
[\beta^2(z), [\beta(x), \alpha(y)]] &= \left( \beta^2(z)(\beta(x)\alpha(y)) - \delta(\alpha^{-1}(\beta^2(x))\beta(y))\alpha(\beta(z)) \right) \\
- \delta &\left( \beta^2(z)(\beta(y)\alpha(x)) - \delta(\alpha^{-1}(\beta^2(y))\beta(x))\alpha(\beta(z)) \right)
\end{align*} \]

Note that

\[ \begin{align*}
\beta^2(x)(\beta(y)\alpha(z)) &= \delta(\alpha^{-1}(\beta^2(x))\beta(y))\alpha(\beta(z)), \\
\beta^2(y)(\beta(x)\alpha(z)) &= \delta(\alpha^{-1}(\beta^2(y))\beta(x))\alpha(\beta(z)), \\
\beta^2(x)(\beta(z)\alpha(y)) &= \delta(\alpha^{-1}(\beta^2(x))\beta(z))\alpha(\beta(y)), \\
\beta^2(y)(\beta(z)\alpha(x)) &= \delta(\alpha^{-1}(\beta^2(y))\beta(z))\alpha(\beta(x)), \\
\beta^2(z)(\beta(x)\alpha(y)) &= \delta(\alpha^{-1}(\beta^2(z))\beta(x))\alpha(\beta(y)), \\
\beta^2(z)(\beta(y)\alpha(x)) &= \delta(\alpha^{-1}(\beta^2(z))\beta(y))\alpha(\beta(x)).
\end{align*} \]

Then we obtain \( [\beta^2(x), [\beta(y), \alpha(z)]] + [\beta^2(y), [\beta(z), \alpha(x)]] + [\beta^2(z), [\beta(x), \alpha(y)]] = 0. \) \( \square \)

**Proposition 2.8.** Given two \( \delta \)-Bihom-Jordan-Lie algebras \( (L, [\cdot, \cdot]_L, \alpha_1, \beta_1) \) and \( (L', [\cdot, \cdot]_{L'}, \alpha_2, \beta_2) \), there is a \( \delta \)-Bihom-Jordan-Lie algebra \( (L \oplus L', [\cdot, \cdot]_{L \oplus L'}, \alpha_1 + \alpha_2, \beta_1 + \beta_2) \), where the bilinear map \([\cdot, \cdot]_{L \oplus L'} : L \oplus L' \times L \oplus L' \to L \oplus L'\) is given by

\[ [u_1 + v_1, u_2 + v_2]_{L \oplus L'} = [u_1, v_1]_L + [u_2, v_2]_{L'}, \forall u_1, u_2, v_1, v_2 \in L,\ L', \]

and the two linear maps \( \alpha_1 + \alpha_2, \beta_1 + \beta_2 : L \oplus L' \to L \oplus L' \) defined by

\[ \begin{align*}
(\alpha_1 + \alpha_2)(u_1 + v_1) &= \alpha_1(u_1) + \alpha_2(v_1), \\
(\beta_1 + \beta_2)(u_1 + v_1) &= \beta_1(u_1) + \beta_2(v_1).
\end{align*} \]

**Proof.** For any \( u_1, u_2, u_3, v_1, v_2, v_3 \in L \) and \( v_1, v_2, v_3 \in L' \) we have:

\[ \begin{align*}
[([\beta_1 + \beta_2](u_1 + v_1), (\alpha_1 + \alpha_2)(u_2 + v_2)]_{L \oplus L'} &= [[\beta_1(u_1), \alpha_1(u_2)]_L + [\beta_2(u_1), \alpha_2(u_2)]_{L'}, -\delta[\beta_1(u_2), \alpha_1(u_1)]_L - \delta[\beta_2(v_2), \alpha_2(v_1)]_{L'}] \\
&= -\delta([\beta_1(u_2), \alpha_1(u_1)]_L + [\beta_2(v_2), \alpha_2(v_1)]_{L'}) \\
&= -\delta((\beta_1 + \beta_2)(u_2 + v_2), (\alpha_1 + \alpha_2)(u_1 + v_1)]_{L \oplus L'} \\
&= (\alpha_1 + \alpha_2)(\beta_1(u_1) + \beta_2(v_1)) = \alpha_1 \circ \beta_1(u_1) + \alpha_2 \circ \beta_2(v_1) \\
&= \beta_1 \circ \alpha_1(u_1) + \beta_2 \circ \alpha_2(v_1) \\
&= (\beta_1 + \beta_2)(\alpha_1 + \alpha_2)(u_1 + v_1).
\end{align*} \]

Then, we have \( (\alpha_1 + \alpha_2) \circ (\beta_1 + \beta_2) = (\beta_1 + \beta_2) \circ (\alpha_1 + \alpha_2). \)

By a direct computation, we have

\[ \begin{align*}
\circ_{(u_1 + v_1), (u_2 + v_2), (u_3 + v_3)} &\left( [\beta_1 + \beta_2](u_1 + v_1), [\beta_1 + \beta_2](u_2 + v_2), (\alpha_1 + \alpha_2)(u_3 + v_3)]_{L \oplus L'} \right)_{L \oplus L'} \\
= \circ_{(u_1 + v_1), (u_2 + v_2), (u_3 + v_3)} &\left( [\beta_1(u_1), \alpha_1(u_2)]_L + [\beta_2(u_2), \alpha_2(u_1)]_{L'}, [\beta_1(u_2), \alpha_1(u_1)]_L + [\beta_2(v_2), \alpha_2(v_1)]_{L'} \right)_{L \oplus L'} \\
= \circ_{u_1, u_2, u_3, v_1, v_2, v_3} &\left( [\beta_1(u_1), [\beta_1(u_2), \alpha_1(u_3)]_L]_{L'} + \circ_{v_1, v_2, v_3} [\beta_2(v_1), [\beta_1(u_2), \alpha_1(u_3)]_L]_{L'} \right) \\\n= 0,
\end{align*} \]

where \( \circ_{x,y,z} \) denotes summation over the cyclic permutation on \( x, y, z. \) \( \square \)
**Definition 2.9.** Let \((L, [\cdot, \cdot], \alpha_1, \beta_1)\) and \((L', [\cdot, \cdot], \alpha_2, \beta_2)\) be two \(\delta\)-Bihom-Jordan-Lie algebras. A linear map \(\phi : L \to L'\) is said to be a morphism of \(\delta\)-Bihom-Jordan-Lie algebras if
\[
\phi[u,v]_L = [\phi(u), \phi(v)]_{L'}, \forall u,v \in L, \tag{2.6}
\]
\[
\phi \circ \alpha_1 = \beta_1 \circ \phi, \tag{2.7}
\]
\[
\phi \circ \alpha_2 = \beta_2 \circ \phi. \tag{2.8}
\]
Denote by \(\mathcal{G}_\phi \subset L \oplus L'\) the graph of a linear map \(\phi : L \to L'\). **Proposition 2.10.** A map \(\phi : (L, [\cdot, \cdot], \alpha_1, \beta_1) \to (L', [\cdot, \cdot], \alpha_2, \beta_2)\) is a morphism of \(\delta\)-Bihom-Jordan-Lie algebras if and only if the graph \(\mathcal{G}_\phi \subset L \oplus L'\) is a Bihom subalgebra of \((L \oplus L', [\cdot, \cdot]_{L \oplus L'}, \alpha_1 + \alpha_2, \beta_1 + \beta_2)\).

**Proof.** Let \(\phi : (L, [\cdot, \cdot], \alpha_1, \beta_1) \to (L', [\cdot, \cdot], \alpha_2, \beta_2)\) be a morphism of \(\delta\)-Bihom-Jordan-Lie algebras, then for any \(u,v \in L\), we have
\[
[u + \phi(u), v + \phi(v)]_{L \oplus L'} = [u,v]_L + [\phi(u), \phi(v)]_{L'} = [u,v]_L + \phi[u,v]_L.
\]
Then the graph \(\mathcal{G}_\phi\) is closed under the bracket operation \([\cdot, \cdot]_{L \oplus L'}\). So, we obtain
\[
(\alpha_1 + \alpha_2)(u + \phi(u)) = \alpha_1(u) + \alpha_2 \circ \phi(u) = \alpha_1(u) + \phi(\alpha_2(u)),
\]
and
\[
(\beta_1 + \beta_2)(u + \phi(u)) = \beta_1(u) + \beta_2 \circ \phi(u) = \beta_1(u) + \phi(\beta_2(u)),
\]
which implies that \((\alpha_1 + \alpha_2)(\mathcal{G}_\phi) \subset \mathcal{G}_\phi\) and \((\beta_1 + \beta_2)(\mathcal{G}_\phi) \subset \mathcal{G}_\phi\). Then \(\mathcal{G}_\phi\) is a Bihom subalgebra of \((L \oplus L', [\cdot, \cdot]_{L \oplus L'}, \alpha_1 + \alpha_2, \beta_1 + \beta_2)\).

Now, suppose that the graph \(\mathcal{G}_\phi \subset L \oplus L'\) is a Bihom subalgebra of \((L \oplus L', [\cdot, \cdot]_{L \oplus L'}, \alpha_1 + \alpha_2, \beta_1 + \beta_2)\), then we have
\[
[u + \phi(u), v + \phi(v)]_{L \oplus L'} = [u,v]_L + [\phi(u), \phi(v)]_{L'} \in \mathcal{G}_\phi,
\]
which implies that
\[
[\phi(u), \phi(v)]_{L'} = \phi[u,v]_L.
\]
Furthermore, \((\alpha_1 + \alpha_2)(\mathcal{G}_\phi) \subset \mathcal{G}_\phi\) and \((\beta_1 + \beta_2)(\mathcal{G}_\phi) \subset \mathcal{G}_\phi\) implies
\[
(\alpha_1 + \alpha_2)(u + \phi(u)) = \alpha_1(u) + \alpha_2 \circ \phi(u) \in \mathcal{G}_\phi\) and \((\beta_1 + \beta_2)(u + \phi(u)) = \beta_1(u) + \beta_2 \circ \phi(u) \in \mathcal{G}_\phi\).
\]
Which is equivalent to the condition \(\alpha_1 \circ \phi(u) = \phi \circ \beta_1(u)\), and \(\alpha_2 \circ \phi(u) = \phi \circ \beta_2(u)\) i.e.
\[
\alpha_1 \circ \phi = \phi \circ \beta_1
\]
and
\[
\alpha_2 \circ \phi = \phi \circ \beta_2.
\]
Therefore, \(\phi\) is a morphism of \(\delta\)-Bihom-Jordan-Lie algebras.

**Example 2.11.** Let \((L, [\cdot, \cdot])\) be a \(\delta\)-Jordan-Lie algebra and \(\alpha, \beta : L \to L\) two commuting linear maps such that \(\alpha([x,y]) = [\alpha(x), \alpha(y)]\) and \(\beta([x,y]) = [\beta(x), \beta(y)]\), for all \(x, y \in L\). Then \((L, [\cdot, \cdot], \alpha, \beta)\), where \([x,y]_L = [\alpha(x), \beta(y)]\), is a \(\delta\)-Bihom-Jordan-Lie algebra. Moreover, suppose that \((L', [\cdot, \cdot])\) is another \(\delta\)-Jordan-Lie algebra and \(\alpha', \beta' : L' \to L'\) are two algebra endomorphisms. If \(f : L \to L'\) is a \(\delta\)-Jordan-Lie algebra homomorphism that satisfies \(f \circ \alpha = \alpha' \circ f\) and \(f \circ \beta = \beta' \circ f\), then \(f : (L, [\cdot, \cdot], \alpha, \beta) \to (L', [\cdot, \cdot], \alpha', \beta')\) is also a homomorphism of \(\delta\)-Bihom-Jordan-Lie algebras.

**Proof.** It is easy to show that \((L, [\cdot, \cdot], \alpha, \beta)\) satisfies \([\beta(x), \alpha(y)]_L = [\alpha(x), \beta(y)] = \alpha\beta([x,y]) = \alpha\beta(-[y,x]) = -\delta([\alpha(y), \alpha(x)]) = -\delta([\alpha(x), \alpha(y)]) =\)
\[
\delta([\beta^2(x), \beta(y), \alpha(z)])_L + [\beta^2(y), \beta(y), \alpha(x)]_L + [\beta^2(z), \beta^2(x), \beta(y)]_L =
\]
\[
[\alpha\beta^2(x), \beta([\alpha(y), \alpha(z)])_L + [\alpha\beta^2(y), \beta([\alpha(z), \alpha(x)])_L = [\alpha\beta^2(z), \beta([\alpha(x), \beta(y)])_L =
\]
\[
\alpha\beta^2([x, y, z]) + [y, z, x] + [z, x, y] = 0.
\]
Then $(L, [\cdot, \cdot]_L, \alpha, \beta)$ is a $\delta$-Bihom-Jordan-Lie algebra.

The second assertion follows from

$$f([x, y]_L) = f([\alpha(x), \beta(y)]) = [f(\alpha(x)), f(\beta(y))] = [\alpha(f(x)), \beta(f(y))] = [f(x), f(y)]_L.$$ 

Then $f : (L, [\cdot, \cdot]_L, \alpha, \beta) \rightarrow (L', [\cdot, \cdot]_{L'}, \alpha', \beta')$ is also a homomorphism of $\delta$-Bihom-Jordan-Lie algebras. 

**Example 2.12.** A three dimensional linear space $L$ has a basis

$$e_1 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix},\ e_2 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix},\ e_3 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$ 

Then $(L, [\cdot, \cdot])$ is a $\delta$-Lie algebra with respect to the product:

$$\begin{pmatrix} 0 & a & b \\ 0 & 0 & c \end{pmatrix},\ \begin{pmatrix} 0 & a' & b' \\ 0 & 0 & c' \end{pmatrix} = \begin{pmatrix} 0 & 0 & ac' - b'c \\ 0 & 0 & 0 \end{pmatrix}.$$ 

If we define two algebra endomorphisms $\alpha$ and $\beta$ by

$$\alpha(e_1) = \delta e_1,\ \alpha(e_2) = e_3,\ \alpha(e_3) = e_2,$$

and

$$\beta(e_1) = \delta e_1,\ \beta(e_2) = e_3,\ \beta(e_3) = e_2.$$

Then $(L, \alpha \otimes \beta([\cdot, \cdot]_L) = [\alpha(\cdot), \beta(\cdot)], \alpha, \beta)$ is a $\delta$-Bihom-Jordan-Lie algebra.

**3. Derivations of $\delta$-Bihom-Jordan-Lie algebras**

In this section, we will study derivations of $\delta$-Bihom-Jordan-Lie algebras. Let $(L, [\cdot, \cdot]_L, \alpha, \beta)$ be a multiplicative $\delta$-Bihom-Jordan-Lie algebra. For any nonnegative integers $k, l$, denote by $\alpha^k$ the $k$-times composition of $\alpha$ and $\beta^l$ the $l$-times composition of $\beta$, i.e.

$$\alpha^k = \underbrace{\alpha \circ \cdots \circ \alpha}_{(k\text{-times})},\ \beta^l = \underbrace{\beta \circ \cdots \circ \beta}_{(l\text{-times})}.$$ 

Since the maps $\alpha, \beta$ commute, we denote by

$$\alpha^k \beta^l = \underbrace{\alpha \circ \cdots \circ \alpha}_{(k\text{-times})} \circ \underbrace{\beta \circ \cdots \circ \beta}_{(l\text{-times})}.$$ 

In particular, $\alpha^0 \beta^0 = 1d, \alpha^1 \beta^1 = \alpha \beta, \alpha^{-k} \beta^{-l}$ is the inverse of $\alpha^k \beta^l$. If $(L, [\cdot, \cdot]_L, \alpha, \beta)$ is a regular $\delta$-Bihom-Jordan-Lie algebra, we denote by $\alpha^{-k}$ the $k$-times composition of $\alpha^{-1}$, the inverse of $\alpha$.

**Definition 3.1.** For any nonnegative integers $k, l$, a linear map $D : L \rightarrow L$ is called an $\alpha^k \beta^l$-derivation of the multiplicative $\delta$-Bihom-Jordan-Lie algebra $(L, [\cdot, \cdot]_L, \alpha, \beta)$, if

$$[D, \alpha] = 0, \ \ i.e. \ \ D \circ \alpha = \alpha \circ D,$$ 

$$[D, \beta] = 0, \ \ i.e. \ \ D \circ \beta = \beta \circ D, \quad (3.1)$$ 

and

$$D[u, v]_L = \delta^k ([D(u), \alpha^k \beta^l(v)]_L + [\alpha^k \beta^l(u), D(v)]_L), \ \forall u, v \in L.$$ 

For a regular $\delta$-Bihom-Jordan-Lie algebra, $\alpha^{-k} \beta^{-l}$-derivations can be defined similarly.

Note first that if $\alpha$ and $\beta$ are bijective, the skew-symmetry condition (2.3) implies

$$[u, v] = -\delta^l [\alpha^{-1} \beta(v), \alpha^{-1} \beta^{-1}(u)]_L, \ \forall u, v \in L.$$ 

Denote by $\text{Der}_{\alpha \times \beta}(L)$ is the set of $\alpha^k \beta^l$-derivations of the multiplicative $\delta$-Bihom-Jordan-Lie algebra $(L, [\cdot, \cdot]_L, \alpha, \beta)$. For any $u \in L$ satisfying $\alpha(u) = u$, and $\beta(u) = u,$
define $D_{k,l}(u) : L \to L$ by

$$D_{k,l}(u)(v) = -\delta[\alpha^k \beta^l(v), u]_L, \delta^k = 1, \quad \forall v \in L.$$ 

By Equation (3.4),

$$D_{k,l}(u)(v) = -\delta[\alpha^k \beta^l(v), u]_L = \delta[\alpha^{-1} \beta(u), \alpha \beta^{-1} (\alpha^k \beta^l(v))]_L = \delta[u, \alpha^{k+1} \beta^{l-1}(v)]_L.$$ 

Then $D_{k,l}(u)$ is an $\alpha^{k+1} \beta^l$-derivation. We call an inner $\alpha^{k+1} \beta^l$-derivation. In fact, we have

$$D_{k,l}(u)(\alpha(v)) = -\delta[\alpha^{k+1} \beta^l(v), u]_L = -\alpha(\delta[\alpha^k \beta^l(v), u]_L) = \alpha \circ D_{k,l}(u)(v).$$

$$D_{k,l}(u)(\beta(v)) = -\delta[\alpha^k \beta^{l+1}(v), u]_L = -\beta(\delta[\alpha^k \beta^l(v), u]_L) = \beta \circ D_{k,l}(u)(v).$$

On the other hand, we have

$$D_{k,l}(u)([v, w]_L) = -\delta[\alpha^k \beta^l([v, w]_L), u]_L = -\delta[[\alpha^k \beta^l(v), \alpha^k \beta^l(w)]_L, \beta^2(u)]_L = \delta[\beta^2(u), [\alpha^k \beta^l(v), \alpha^k \beta^l(w)]_L] = -\delta[[\alpha^{k+1} \beta^l(v), [\alpha^k \beta^l(w), \alpha(u)]_L] + [\alpha^k \beta^l(v), [\beta(u), \alpha^{k+2} \beta^{l-2}(v)]_L]_L)$$

$$= -\delta[[\alpha^{k+1} \beta^l(v), [\alpha^k \beta^l(w), \alpha(u)]_L] - \delta[\alpha^{k+1} \beta^l(v), [\beta(u), \alpha^{k+2} \beta^{l-2}(v)]_L]_L$$

$$= -\delta^{k+1}[\alpha^{k+1} \beta^l(v), D_{k,l}(u)(w)]_L + [D_{k,l}(u)(v), \alpha^{k+1} \beta^l(v)]_L.$$

Therefore, $D_{k,l}(u)$ is an $\alpha^{k+1} \beta^l$-derivation. Denote by $\text{Inn}_{\alpha^k \beta^l}(L)$ the set of inner $\alpha^k \beta^l$-derivations, i.e.

$$\text{Inn}_{\alpha^k \beta^l}(L) = \{-\delta[\alpha^{k-1} \beta^l(\cdot), u]_L | u \in L, \alpha(u) = u, \beta(u) = u, \delta^k = 1\}. \quad (3.5)$$

For any $D \in \text{Der}_{\alpha^k \beta^l}(L)$ and $D' \in \text{Der}_{\alpha^k \beta^l}(L)$, define their commutator $[D, D']$ as usual:

$$[D, D'] = D \circ D' - D' \circ D.$$  

(3.6)

Lemma 3.2. For any $D \in \text{Der}_{\alpha^k \beta^l}(L)$ and $D' \in \text{Der}_{\alpha^k \beta^l}(L)$, we have

$$[D, D'] \in \text{Der}_{\alpha^{k+s} \beta^{l+t}}(L).$$

Proof. For any $u, v \in L$, we have

$$[D, D']([u, v]_L) = D \circ D'([u, v]_L) - D' \circ D([u, v]_L) = \delta^s D([D'(u), \alpha^s \beta^l(v)]_L + [\alpha^s \beta^l(u), D'(v)]_L)$$

$$- \delta^s D'([D(u), \alpha^s \beta^l(v)]_L + [\alpha^s \beta^l(u), D(v)]_L)$$

$$= \delta^s D([D'(u), \alpha^s \beta^l(v)]_L + s D([\alpha^s \beta^l(u), D'(v)]_L)$$

$$- \delta^s D'([D(u), \alpha^s \beta^l(v)]_L - \delta^k D'([\alpha^k \beta^l(u), D(v)]_L)$$

$$= \delta^{k+s}([D \circ D'(u), \alpha^{k+s} \beta^{l+t}(v)]_L + [\alpha^k \beta^l \circ D'(v)]_L + [\alpha^{k+s} \circ D(u), D \circ D'(v)]_L$$

$$-[D' \circ D(u), \alpha^{k+s}(v)]_L - [\alpha^s \circ D(u), D' \circ \alpha^k(v)]_L$$

$$-[D' \circ \alpha^k(u), \alpha^s \circ D(v)]_L - [\alpha^{k+s} \beta^{l+t}(u), D' \circ D(v)]_L.$$
Since any two of maps $D, D', \alpha, \beta$ commute, we have
\[
D \circ \alpha^s = \alpha^s \circ D, \quad D' \circ \alpha^k = \alpha^k \circ D',
D \circ \beta^t = \beta^t \circ D, \quad D' \circ \beta^l = \beta^l \circ D'.
\]
Therefore, we have
\[
[D, D'](u,v)_L = \delta^{k+s}([D \circ D'(u) - D' \circ D(u), \alpha^{k+s} \beta^{l+t}(v)]_L
+ [\alpha^{k+s} \beta^{l+t}(u), D \circ D'(u) - D' \circ D(v)]_L)
= \delta^{k+s}([D, D'](u), \alpha^{k+s} \beta^{l+t}(v)]_L + [\alpha^{k+s} \beta^{l+t}(u), [D, D'](v)]_L).
\]
Furthermore, it is straightforward to see that
\[
[D, D'] \circ \alpha = D \circ D' \circ \alpha - D \circ D' \circ \alpha = \alpha \circ D \circ D' - \alpha \circ D \circ D'
= \alpha \circ [D, D'],
\]
and
\[
[D, D'] \circ \beta = D \circ D' \circ \beta - D \circ D' \circ \beta = \beta \circ D \circ D' - \beta \circ D \circ D'
= \beta \circ [D, D'].
\]
Therefore, $[D, D'] \in \text{Der}_{\alpha^{k+s} \beta^{l+t}}(L)$. \hfill \Box

For any integer $k, l$, denote by $\text{Der}(L) = \bigoplus_{k \geq 0, l \geq 0} \text{Der}_{\alpha^{k} \beta^{l}}(L)$. Obviously, $\text{Der}(L)$ is a Lie algebra, in which the Lie bracket is given by equation (3.6).

In the end, we consider the derivation extension of the regular $\delta$-Bihom-Jordan-Lie algebra $(L, [\cdot, \cdot]_L, \alpha, \beta)$ and give an application of the $\alpha^0 \beta^1$-derivation $\text{Der}_{\alpha^{k} \beta^{l}}(L)$.

For any linear map $D, \alpha, \beta : L \to L$, where $\alpha$ and $\beta$ are inverse, consider the vector space $L \oplus RD$. Define a skew-symmetric bilinear bracket operation $[\cdot, \cdot]_D$ on $L \oplus RD$ by
\[
[u, v]_D = [u, v]_L, [D, u]_D = -\delta[\alpha^{-1} \beta(u), \alpha \beta^{-1} D]_D = D(u), \forall u, v \in L.
\]
Define two linear maps by $\alpha_D(u, D) = (\alpha(u), D)$, and $\beta_D(u, D) = (\beta(u), D)$.

And the linear maps $\alpha, \beta$ involved in the definition of the bracket operation $[\cdot, \cdot]_D$ are required to be multiplicative, that is
\[
\alpha \circ [D, u]_D = [\alpha \circ D, \alpha(u)]_D, \quad \beta \circ [D, u]_D = [\beta \circ D, \beta(u)]_D.
\]
Then, we have
\[
[u, D]_D = -\delta[\alpha^{-1} \beta D, \alpha \beta^{-1} (u)]_D
= -\delta\alpha^{-1} \beta [D, \alpha^2 \beta^{-2} (u)]_D
= -\delta\alpha^{-1} \beta D (\alpha^2 \beta^{-2} (u))
= -\delta \alpha \beta^{-1} D(u).
\]

**Theorem 3.3.** With the above notations, $(L \oplus RD, [\cdot, \cdot]_D, \alpha_D, \beta_D)$ is a multiplicative $\delta$-Bihom-Jordan-Lie algebra if and only if $D$ is an $\alpha^0 \beta^1$-derivation of the multiplicative $\delta$-Bihom-Jordan-Lie algebra $(L, [\cdot, \cdot]_L, \alpha, \beta)$.

**Proof.** For any $u, v \in L, m, n \in R$, we have
\[
\alpha_D \circ \beta_D(u, mD) = \alpha_D(\beta(u), mD) = (\alpha \circ \beta(u), mD),
\]
and
\[
\beta_D \circ \alpha_D(u, mD) = \beta_D(\alpha(u), mD) = (\beta \circ \alpha(u), mD).
\]
Hence, we have

\[ \alpha_D \circ \beta_D = \beta_D \circ \alpha_D \quad \iff \alpha \circ \beta = \beta \circ \alpha. \]

On the other hand,

\[
\alpha_D[(u, mD), (v, nD)]_D = \alpha_D([u, v]_L + [u, nD]_D + [mD, v]_D)
\]

\[
= \alpha_D([u, v]_L - \delta nD \circ \alpha \beta^{-1}(u) + mD(v))
\]

\[
= \alpha([u, v]_L) - \delta n \alpha \circ D \circ \alpha \beta^{-1}(u) + m\alpha \circ D(v)),
\]

\[
[\alpha_D(u, mD), \alpha_D(v, nD)]_D = [\alpha(u, mD), \alpha(v, nD)]_D
\]

if and only if

\[
D \circ \alpha = \alpha \circ D, \quad D \circ \beta = \beta \circ D.
\]

Similarly

\[
\beta_D[(u, mD), (v, nD)]_D = [\beta_D(u, mD), \beta_D(v, nD)]_D
\]

if and only if

\[
D \circ \alpha = \alpha \circ D, \quad D \circ \beta = \beta \circ D.
\]

Next, we have

\[
[\beta_D(v, nD), \alpha_D(u, mD)]_D = [(\beta(v), nD), (\alpha(u), mD)]_D
\]

\[
= [\beta(v), \alpha(u)]_L + [\beta(v), mD]_D + [nD, \alpha(u)]_D
\]

\[
= [\beta(v), \alpha(u)]_L - \delta m \alpha \beta^{-1} \circ D \circ (\beta(v)) + nD(\alpha(u))
\]

\[
= -\delta([\beta(u), \alpha(v)]_L + m\alpha \beta^{-1} \circ D \circ (\beta(v)) - \delta nD(\alpha(u))),
\]

\[
[\beta_D(u, mD), \alpha_D(v, nD)]_D = [(\beta(u), mD), (\alpha(v), nD)]_D
\]

\[
= [\beta(u), \alpha(v)]_L + [\beta(u), nD]_D + [mD, \alpha(v)]_D
\]

\[
= [\beta(u), \alpha(v)]_L - \delta n \alpha \beta^{-1} \circ D \circ (\beta(u)) + mD(\alpha(v)),
\]

thus

\[
[\beta_D(v, nD), \alpha_D(u, mD)]_D = -\delta[\beta_D(u, mD), \alpha_D(v, nD)]_D
\]

if and only if

\[
D \circ \alpha = \alpha \circ D, \quad D \circ \beta = \beta \circ D.
\]
On the other hand, we have

\[
\begin{align*}
&[\beta^2_D(u, mD), [\beta_D(v, nD), \alpha(w, lD)]_D]_D + [\beta^2_D(v, nD), [\beta_D(w, lD), \alpha_D(u, mD)]_D]_D \\
&+ [\beta^2_D(w, ID), [\beta_D(u, mD), \alpha_D(v, nD)]_D]_D \\
&= [(\beta^2(u, mD), [\beta(v, nD), \alpha(w, lD)]_D]_D + [(\beta^2(v, nD), [\beta(w, lD), \alpha(u, mD)]_D]_D \\
&+ [(\beta^2(w, lD), [\beta(u, mD), \alpha(v, nD)]_D]_D \\
&= [(\beta^2(u, mD), [\beta(v, nD), \alpha(w)] - \delta \alpha \circ D(v) + nD \circ \alpha(w)]_D \\
&+ [(\beta^2(v, nD), [\beta(w, \alpha(u)] - \delta \alpha \circ D(w) + mD \circ \alpha(u)]_D \\
&+ [(\beta^2(w, lD), [\beta(u, \alpha(v)] - \delta \alpha \circ D(u) + mD \circ \alpha(v)]_D \\
&= [(\beta^2(u, [\beta(v, \alpha(w)] - \delta \alpha \circ D(v) + nD \circ \alpha(w)]_D + [(\beta^2(v, nD) \circ \alpha(u)]_D \\
&+ [\beta^2(w, [\beta(u, \alpha(v)] - \delta \alpha \circ D(u) + mD \circ \alpha(v)]_D \\
&+ [\beta^2(u, [\beta(v, \alpha(w)] - \delta \alpha \circ D(v) + mD \circ \alpha(w)]_D + mnD \circ \alpha(w)] \\
&+ [\beta^2(v, [\beta(w, \alpha(u)] - \delta \alpha \circ D(w) + mD \circ \alpha(w)]_D + \beta^2(w, lD \circ \alpha(u)]_D \\
&+ [\beta^2(w, [\beta(u, \alpha(v)] - \delta \alpha \circ D(u) + \beta^2(w, mD \circ \alpha(v)]_D + [\beta^2(u, lD \circ \alpha(v)]_D \\
&+ [\beta^2(v, nD \circ \alpha(u)]_D + [\beta^2(w, lD \circ \alpha(v)]_D \\
&+ [\beta^2(u, nD \circ \alpha(w)]_D - \delta \alpha \circ D^2(u) + lmD \circ \alpha(v). \\
\end{align*}
\]

If \( D \) is an \( \alpha^0 \beta^1 \)-derivation of the multiplicative \( \delta \)-Bihom-Jordan-Lie algebra \((L, [, ]_L, \alpha, \beta)\), then

\[
\begin{align*}
[mD, [\beta(v), \alpha(w)]]_D &= mD[\beta(v), \alpha(w)] \\
&= \delta [mD \circ \beta(v), \alpha^0 \beta^1(\alpha(w))] + [\alpha^0 \beta^2(v), mD \circ \alpha(w)] \\
&= -\delta [\alpha^0 \beta^2(v), mD \circ \alpha(w)] + [\alpha^0 \beta^2(v), mD \circ \alpha(w)] \\
&= -\delta [\beta^2(v), mD \circ \alpha(v)] + [\beta^2(v), mD \circ \alpha(v)].
\end{align*}
\]

Similarly

\[
[nD, [\beta(w), \alpha(u)]]_D = -\delta [\beta^2(u), nD \circ \alpha(w)] + [\beta^2(w), nD \circ \alpha(u)].
\]

And

\[
[LD, [\beta(u), \alpha(v)]]_D = -\delta [\beta^2(v), lD \circ \alpha(u)] + [\beta^2(v), lD \circ \alpha(u)].
\]

Therefore, the \( \delta \)-Bihom-Jacobi identity is satisfied if and only if \( D \) is an \( \alpha^0 \beta^1 \)-derivation of \((L, [, ]_L, \alpha, \beta)\). Thus \((L \oplus RD, [, ]_D, \alpha_D, \beta_D)\) is a multiplicative \( \delta \)-Bihom-Jordan-Lie algebra if and only if \( D \) is an \( \alpha^0 \beta^1 \)-derivation of \((L, [, ]_L, \alpha, \beta)\).

\[\square\]

4. Representations of \( \delta \)-Bihom-Jordan-Lie algebras

In this section we study representations of \( \delta \)-Bihom-Jordan-Lie algebras and give the corresponding coboundary operators. We can also construct the semidirect product of \( \delta \)-Bihom-Jordan-Lie algebras. Let \( A \in \text{End}(V) \) be an arbitrary linear transformation from \( V \) to \( V \).
Thus we obtain a well-defined map

\[ d^k_p : C^k_{(\alpha, \alpha M)}(L, M) \to C^{k+1}_{(\alpha, \alpha M)}(L, M) \]

with \( k = 1, 2 \).

**Proposition 4.3.** With the above notations, we have \( d^2_p \circ d^1_p = 0 \).
Proof. By straightforward computations, we have
\[ d^2_{\rho} \circ d^1_{\rho} f(u_1, u_2, u_3) \]
\[ = \rho(\alpha(\beta(u_1))) d^1_{\rho} f(u_2, u_3) - \delta \rho(\alpha(\beta(u_2))) d^1_{\rho} f(u_1, u_3) + \rho(\alpha(\beta(u_3))) d^1_{\rho} f(u_1, u_2) \]
\[ - d^1_{\rho} f([\alpha^{-1}\beta(u_1), u_2], [\beta(u_2), u_3])_L + \delta d^1_{\rho} f([\alpha^{-1}\beta(u_1), u_3], \beta(u_2)) \]
\[ - d^1_{\rho} f([\alpha(-1\beta(u_1), u_3], \beta(u_2)) \]
\[ = \rho(\alpha(\beta(u_1))) (\rho(\alpha(\beta(u_2))) f(u_3) - \delta \rho(\alpha(\beta(u_3))) f(u_2) - \delta f([\alpha^{-1}\beta(u_2), u_3])_L) \]
\[ - \delta \rho(\alpha(\beta(u_2))) (\rho(\alpha(\beta(u_1))) f(u_3) - \delta \rho(\alpha(\beta(u_3))) f(u_1) - \delta f([\alpha^{-1}\beta(u_1), u_3]))_L \]
\[ + \rho(\alpha(\beta(u_3))) (\rho(\alpha(\beta(u_2))) f(u_3) - \delta \rho(\alpha(\beta(u_2))) f(u_1) - \delta f([\alpha^{-1}\beta(u_1), u_2]))_L \]
\[ - \rho(\alpha(\beta(u_2))) f([\alpha^{-1}\beta(u_1), u_2])_L) + \delta \rho(\alpha(\beta(u_3))) f(u_1) \]
\[ + \delta f([\alpha^{-1}\beta([\alpha^{-1}\beta(u_1), u_2]]_L, \beta(u_3)_L) \]
\[ - \delta \rho(\alpha([\alpha^{-1}\beta(u_1), u_3])_L) f(\beta(u_2)) + \rho(\beta(u_2)) f(u_1) \]
\[ + f([\alpha^{-1}\beta([\alpha^{-1}\beta(u_1), u_3]]_L, \beta(u_2)_L) \]
\[ + \delta \rho(\alpha([\alpha^{-1}\beta(u_2), u_3])_L) f(\beta(u_1)) - \rho(\beta(u_1)) f(u_2) \]
\[ - f([\alpha^{-1}\beta([\alpha^{-1}\beta(u_2), u_3]]_L, \beta(u_1)_L) \]
\[ = 0. \]

Then \( d^2_{\rho} \circ d^1_{\rho} f(u_1, u_2, u_3) = 0. \)

Associated to the representation \( \rho, \) we obtain the complex \( (C^k_{(\alpha,\alpha_M)}(L, M), d_{\rho}). \) Denote the set of closed \( k\)-Bihom-cochains by \( Z^k_{\alpha,\beta}(L; \rho) \) and the set of exact \( k\)-Bihom-cochains by \( B^k_{\alpha,\beta}(L, \rho), \) \( k = 1, 2. \)

Denote the corresponding cohomology by
\[ H^k_{\alpha,\beta}(L, \rho) = Z^k_{\alpha,\beta}(L; \rho)/B^k_{\alpha,\beta}(L, \rho), \]
where
\[ Z^k_{\alpha,\beta}(L; \rho) = \{ f \in C^k_{(\alpha,\alpha_M)}(L, M) \mid d^k_{\rho} f = 0 \}, \]
\[ B^k_{\alpha,\beta}(L, \rho) = \{ d^k_{\rho} g \mid g \in C^{k-1}_{(\alpha,\alpha_M)}(L, M) \}. \]

In the case of Lie algebras, we can form semidirect products when given representations. Similarly, we have

**Proposition 4.4.** Let \( (L, [\cdot, \cdot]_L, \alpha, \beta) \) be a multiplicative \( \delta \)-Bihom-Jordan-Lie algebra and \( (M, \rho, \alpha_M, \beta_M) \) a representation of \( L. \) Assume that the maps \( \alpha_M \) and \( \beta_M \) are bijective. Then \( L \ltimes M = (L \oplus M, [\cdot, \cdot]_L, \alpha \oplus \alpha_M, \beta \oplus \beta_M) \) is a \( \delta \)-Bihom-Jordan-Lie algebra, where \( \alpha \oplus \alpha_M, \beta \oplus \beta_M : L \oplus M \to L \oplus M \) are defined by \( (\alpha \oplus \alpha_M)(u + x) = \alpha(u) + \alpha_M(x) \) and
$(\beta \oplus \beta_M)(u + x) = \beta(u) + \beta_M(x)$, for all $u, v \in L$ and $x, y \in M$, the bracket $[\cdot, \cdot]_\rho$ is defined by
\[
[u + v, u + y]_\rho = [u, v]_L + \delta \rho(u)(y) - \rho(\alpha^{-1}\beta(v))(\alpha_M \beta_M^{-1}(x)).
\] 
We call $L \ltimes M$ the semidirect product of the multiplicative $\delta$-Bihom-Jordan-Lie algebra $(L, [\cdot, \cdot]_L, \alpha, \beta)$ and $M$.

**Proof.** First we show that $[\cdot, \cdot]_\rho$ satisfies antisymmetry,
\[
[\beta, \alpha_{\alpha_M}(u + x)]_\rho = -\delta([\beta(u), \alpha_{\alpha_M}(u) + \alpha_M(x)]) - \rho(\alpha^{-1}\beta(v))(\alpha_M \beta_M^{-1}(\beta_M(y)))
\] 
Next we show that $(\alpha \oplus \alpha_M)$ and $(\beta \oplus \beta_M)$ are algebra morphisms. On the one hand, we have
\[
(\alpha \oplus \alpha_M)[u + v, u + y]_\rho = (\alpha \oplus \alpha_M)[u + x, y, v + y]_\rho.
\] 
Furthermore,
\[
[(\beta \oplus \beta_M)^2(u + x), [\beta \oplus \beta_M](v + y), (\alpha \oplus \alpha_M)(w + z)]_\rho
\] 
Similarly,
\[
[(\beta \oplus \beta_M)^2(v + y), [\beta \oplus \beta_M](w + z), (\alpha \oplus \alpha_M)(u + x)]_\rho
\] 
And
\[
[(\beta \oplus \beta_M)^2(w + z), [\beta \oplus \beta_M](u + x), (\alpha \oplus \alpha_M)(v + y)]_\rho
\] 
By (4.3), the $\delta$-Bihom-Jacobi identity is satisfied. Thus, $(L \oplus M, [\cdot, \cdot]_\rho, \alpha \oplus \alpha_M, \beta \oplus \beta_M)$ is a multiplicative $\delta$-Bihom-Jordan-Lie algebra.
5. The trivial representation of $\delta$-Bihom-Jordan-Lie algebras

In this section, we study the trivial representation of multiplicative $\delta$-hom-Jordan-Lie algebras. As an application, we show that the central extension of a multiplicative $\delta$-Bihom-Jordan-Lie algebra $(L, [\cdot, \cdot]_L, \alpha, \beta)$ is controlled by the second cohomology of $L$ with coefficients in the trivial representation.

Now let $M = \mathbb{R}$, then we have $\text{End}(M) = \mathbb{R}$. Any $\alpha_M, \beta_M \in \text{End}(M)$ is exactly two real numbers, which we denote by $r_1, r_2$ respectively. Let $\rho : L \to \text{End}(M) = \mathbb{R}$ be the zero map. Obviously, $\rho$ is a representation of the multiplicative $\delta$-Bihom-Jordan-Lie algebra $(L, [\cdot, \cdot]_L, \alpha, \beta)$ with respect to any $r_1, r_2 \in \mathbb{R}$. We will always assume that $r_1 = r_2 = 1$. We call this representation the trivial representation of the multiplicative $\delta$-Bihom-Jordan-Lie algebra $(L, [\cdot, \cdot]_L, \alpha, \beta)$.

Associated to the trivial representation, the set of $k$-cochains on $L$, which we denote by $C^k(V) = \wedge^k L^*$, is the set of skew-symmetric $k$-linear maps from $V \times \cdots \times V$ to $\mathbb{R}$. The set of $k$-Bihom-cochains is given by

$$C^k_{\alpha, \beta}(L) = \{ f \in C^k(L)| f \circ \alpha = f, f \circ \beta = f \}.$$  

The corresponding coboundary operator $d_T : C^k_{\alpha, \beta}(L) \to C^{k+1}_{\alpha, \beta}(L)(k = 1, 2)$ is given by

$$d^1_T f(u_1, u_2) = -\delta f([\alpha^{-1}\beta(u_1), u_2], L),$$  

$$d^2_T f(u_1, u_2, u_3) = -f([\alpha^{-1}\beta(u_1), u_2], L, \beta(u_3)) + \delta f([\alpha^{-1}\beta(u_1), u_3], L, \beta(u_2))$$

$$-f([\alpha^{-1}\beta(u_2), u_3], L, \beta(u_1)).$$

Denote $Z^k_{\alpha, \beta}(L)$ and $B^k_{\alpha, \beta}(L)(k = 1, 2)$ similarly.

In the following we consider central extensions of the multiplicative $\delta$-Bihom-Jordan-Lie algebra $(L, [\cdot, \cdot]_L, \alpha, \beta)$. Obviously, $\mathbb{R}, 0, 1, 1$ is an abelian multiplicative $\delta$-Bihom-Jordan-Lie algebra with the trivial bracket and the identity morphism. Let $\theta \in C^2_{\alpha, \beta}(L)$, we have $\theta \circ \alpha = \theta, \theta \circ \beta = \theta$ and $\theta(u, v) = -\delta \theta(v, u), \forall u, v \in L$. We consider the direct sum $\mathfrak{g} = L \oplus \mathbb{R}$ with the following bracket

$$[u + s, v + t]_\theta = [u, v]_L + \theta(\alpha\beta^{-1}(u), v), \quad \forall u, v \in L, s, t \in \mathbb{R}.$$  

(5.2)

Define $\alpha_\mathfrak{g}, \beta_\mathfrak{g} : \mathfrak{g} \to \mathfrak{g}$ by $\alpha_\mathfrak{g}(u + s) = \alpha(u) + s$, and $\beta_\mathfrak{g}(u + s) = \beta(u) + s$.

Theorem 5.1. With the above notations, the 4-tuple $(\mathfrak{g}, [\cdot, \cdot]_\mathfrak{g}, \alpha_\mathfrak{g}, \beta_\mathfrak{g})$ is a multiplicative $\delta$-Bihom-Jordan-Lie algebra if and only if $\theta \in C^2_{\alpha, \beta}(L)$ is a 2-cocycle associated to the trivial representation, i.e.

$$d_T \theta = 0.$$

We call the multiplicative $\delta$-Bihom-Jordan-Lie algebra $(\mathfrak{g}, [\cdot, \cdot]_\mathfrak{g}, \alpha_\mathfrak{g}, \beta_\mathfrak{g})$ the central extension of $(L, [\cdot, \cdot]_L, \alpha, \beta)$ by the abelian $\delta$-Bihom-Jordan-Lie algebra $(\mathbb{R}, 0, 1, 1)$.

Proof. Obviously, since $\alpha \circ \beta = \beta \circ \alpha$, we have $\alpha_\mathfrak{g} \circ \beta_\mathfrak{g} = \beta_\mathfrak{g} \circ \alpha_\mathfrak{g}$. Then we show that $\alpha_\mathfrak{g}$ is an algebra morphism with the respect to the bracket $[\cdot, \cdot]_\theta$. On one hand, we have

$$[\alpha_\mathfrak{g}(u + s), \alpha_\mathfrak{g}(v + t)]_\theta = [\alpha(u) + s, \alpha(v) + t]_\theta = [\alpha(u), \alpha(v)]_L + \theta(\alpha\beta^{-1}(\alpha(u)), \alpha(v)).$$

On the other hand, we have

$$[\alpha_\mathfrak{g}(u + s), \alpha_\mathfrak{g}(v + t)]_\theta = [\alpha(u + s), \alpha(v + t)]_\theta = [\alpha(u) + s, \alpha(v) + t]_\theta = [\alpha(u), \alpha(v)]_L + \theta(\alpha\beta^{-1}(\alpha(u)), \alpha(v)).$$

Since $\alpha$ is an algebra morphism and $\theta(\alpha\beta^{-1}(\alpha(u)), \alpha(v)) = \theta\circ\alpha(\alpha\beta^{-1}(u), v) = \theta(\alpha\beta^{-1}(u), v)$. Then $\alpha_\mathfrak{g}$ is an algebra morphism.

Similarly, we have $\beta_\mathfrak{g}$ is also an algebra morphism.
Furthermore, we have
\[
[\beta_{g}(u + s), \alpha_{g}(v + t)]_{\theta} = [\beta(u) + s, \alpha(v) + t]_{\theta} = [\beta(u), \alpha(v)]_{L} + \theta(\alpha\beta^{-1}(\beta(u)), \alpha(v)) = [\beta(u), \alpha(v)]_{L} + \theta(\alpha(u), \alpha(v)) = [\beta(u), \alpha(v)]_{L} + \theta(u, v)
\]
and
\[
[\beta_{g}(v + t), \alpha_{g}(u + s)]_{\theta} = [\beta(v) + t, \alpha(u) + s]_{\theta} = [\beta(v), \alpha(u)]_{L} + \theta(\alpha\beta^{-1}(\beta(v)), \alpha(u)) = [\beta(v), \alpha(u)]_{L} + \theta(\alpha(v), \alpha(u)) = [\beta(v), \alpha(u)]_{L} + \theta(v, u) = -\delta([\beta(u), \alpha(v)]_{L} + \theta(u, v)).
\]
Then \([\beta_{g}(u + s), \alpha_{g}(v + t)]_{\theta} = -\delta[\beta_{g}(v + t), \alpha_{g}(u + s)]_{\theta} \).

By direct computations, we have
\[
[\beta_{g}^{2}(u + s), [\beta_{g}(v + t), \alpha_{g}(w + r)]_{\theta} + [\beta_{g}^{2}(v + t), \beta_{g}(w + r), \alpha_{g}(u + s)]_{\theta} = [\beta_{g}^{2}(u + s), [\beta_{g}(v + t), \alpha_{g}(w + r)]_{\theta} + [\beta_{g}^{2}(v + t), \beta_{g}(w + r), \alpha_{g}(u + s)]_{\theta} = [\beta^{2}(u) + s, [\beta(v) + t, \alpha(w) + r]_{\theta} + [\beta^{2}(v) + t, [\beta(w) + r, \alpha(u) + s]_{\theta}
+ [\beta^{2}(w) + r, [\beta(u) + s, \alpha(v) + t]_{\theta}
 = [\beta^{2}(u) + s, [\beta(v) + t, \alpha(w) + r]_{\theta} + [\beta^{2}(v) + t, [\beta(w) + r, \alpha(u) + s]_{\theta}
+ [\beta^{2}(w) + r, [\beta(u) + s, \alpha(v) + t]_{\theta}
 = [\beta^{2}(u) + s, [\beta(v) + t, \alpha(w) + r]_{\theta} + [\beta^{2}(v) + t, [\beta(w) + r, \alpha(u) + s]_{\theta}
+ [\beta^{2}(w) + r, [\beta(u) + s, \alpha(v) + t]_{\theta}
 = [\beta^{2}(u) + s, [\beta(v) + t, \alpha(w) + r]_{\theta} + [\beta^{2}(v) + t, [\beta(w) + r, \alpha(u) + s]_{\theta}
+ [\beta^{2}(w) + r, [\beta(u) + s, \alpha(v) + t]_{\theta}
 = [\beta^{2}(u) + s, [\beta(v) + t, \alpha(w) + r]_{\theta} + [\beta^{2}(v) + t, [\beta(w) + r, \alpha(u) + s]_{\theta}
+ [\beta^{2}(w) + r, [\beta(u) + s, \alpha(v) + t]_{\theta}
Thus by the Bihom-Jacobi identity of \( L \), \([\cdot, \cdot]_{\theta}\) satisfies the \( \delta \)-Bihom-Jacobi identity if and only if
\[
\theta(\alpha\beta(u)), [\beta(v), \alpha(w)]_{L} + \theta(\alpha\beta(v)), [\beta(w), \alpha(u)]_{L} + \theta(\alpha\beta(w)), [\beta(u), \alpha(v)]_{L} = 0.
\]
Namely,
\[
\theta(\beta(u), [\alpha^{-1}\beta(v), w]_{L}) + \theta(\beta(v), [\alpha^{-1}\beta(w), u]_{L}) + \theta(\beta(w), [\alpha^{-1}\beta(u), v]_{L}) = 0.
\]
On the other hand,
\[
d_{T}\theta(u, v, w) = \delta(\theta([\alpha^{-1}\beta(u), v]_{L}, \beta(w)) + \theta([\alpha^{-1}\beta(w), u]_{L}, \beta(v)) - \delta(\theta([\alpha^{-1}\beta(v), w]_{L}, \beta(u)))
\]
\[
= -\theta([\alpha^{-1}\beta(u), v]_{L}, \beta(w)) + \theta([\alpha^{-1}\beta(w), u]_{L}, \beta(v)) + \theta([\alpha^{-1}\beta(v), w]_{L}, \beta(u)))
\]
\[
= \delta([\beta_{g}^{2}(u + s), [\beta_{g}(v + t), \alpha_{g}(w + r)]_{\theta} + [\beta_{g}^{2}(v + t), [\beta_{g}(w + r), \alpha_{g}(u + s)]_{\theta}
\]
\[
= 0.
\]
Then the 4-tuple \((g, [\cdot, \cdot]_{\theta}, \alpha_{g}, \beta_{g})\) is a multiplicative \( \delta \)-Bihom-Jordan-Lie algebra if and only if \( \theta \in C_{\alpha, \beta}(L) \) satisfies \( d_{T}\theta = 0 \).
Proposition 5.2. For \( \theta_1, \theta_2 \in \mathbb{Z}^2(V) \), if \( \delta(\theta_1 - \theta_2) \) is exact, the corresponding two central extensions \( (g, [\cdot, \cdot]_{\theta_1}, \alpha_\theta, \beta_\theta) \) and \( (g, [\cdot, \cdot]_{\theta_2}, \alpha_\theta, \beta_\theta) \) are isomorphic.

**Proof.** Assume that \( \theta_1 - \theta_2 = \delta T f \), \( f \in C^1_{\alpha, \beta}(L) \). Thus we have
\[
\theta_1(\alpha \beta^{-1}(u), v) - \theta_2(u, v) = \delta T f(\alpha \beta^{-1}(u), v) = -f([\alpha^{-1} \beta \circ \alpha \beta^{-1}(u), v]) = -f([u, v]).
\]
Define \( \varphi_g : g \to g \) by
\[
\varphi_g(u + s) = u + s + f(u).
\]
Obviously, \( \varphi_g \) is an isomorphism of vector spaces. The fact that \( \varphi_g \) is a morphism of the \( \delta \)-Bihom-Jordan-Lie algebra follows from the fact \( \theta \circ \alpha = \theta, \theta \circ \beta = \theta \). More precisely, we have
\[
\varphi_g \circ \alpha_g(u + s) = \varphi_g(\alpha(u) + s) = \alpha(u) + s + f(\alpha(u)) = \alpha(u) + s + f(u).
\]
On the other hand, we have
\[
\alpha_g \circ \varphi_g(u + s) = \alpha_g(u + s + f(u)) = \alpha(u) + s + f(u).
\]
Thus, we obtain that \( \varphi_g \circ \alpha_g = \alpha_g \circ \varphi_g \). Similarly
\[
\varphi_g \circ \beta_g = \beta_g \circ \varphi_g.
\]
We also have
\[
\varphi_g[u + s, v + t]_{\theta_1} = \varphi_g([u, v]_L + \theta_1(\alpha \beta^{-1}(u), v)) = [u, v]_L + \theta_1(\alpha \beta^{-1}(u), v) + f([u, v]_L) = ([u, v]_L, \theta_2(\alpha \beta^{-1}(u), v) = \varphi_g(u + s), \varphi_g(v + t)]_{\theta_2}.
\]
Therefore, \( \varphi_g \) is also an isomorphism of multiplicative \( \delta \)-Bihom-Jordan-Lie algebras. \( \square \)

6. The adjoint representation of \( \delta \)-Bihom-Jordan-Lie algebras

Let \((L, [\cdot, \cdot]_L, \alpha, \beta)\) be a regular \( \delta \)-Bihom-Jordan-Lie algebra. We consider that \( L \) represents on itself via the bracket with respect to the morphisms \( \alpha, \beta \). A very interesting phenomenon is that the adjoint representation of a \( \delta \)-Bihom-Jordan-Lie algebra is not unique as one will see in sequel.

**Definition 6.1.** For any integer \( s, t \), the \( \alpha^s \beta^t \)-adjoint representation of the regular \( \delta \)-Bihom-Jordan-Lie algebra \((L, [\cdot, \cdot]_L, \alpha, \beta)\), which we denote by \( \text{ad}_{s,t} \), is defined by
\[
\text{ad}_{s,t}(u)(v) = \delta[\alpha^s \beta^t(u), v]_L, \forall u, v \in L.
\]

**Lemma 6.2.** With the above notations, we have
\[
\text{ad}_{s,t}(\alpha(u)) \circ \alpha = \alpha \circ \text{ad}_{s,t}(u),
\]
\[
\text{ad}_{s,t}(\beta(u)) \circ \beta = \beta \circ \text{ad}_{s,t}(u),
\]
\[
\text{ad}_{s,t}([\beta(u), v]_L) \circ \beta = \text{ad}_{s,t}(\alpha \beta(u)) \circ \text{ad}_{s,t}(v) - \delta \text{ad}_{s,t}(\alpha(v)) \circ \text{ad}_{s,t}(\beta(u)).
\]
Thus the definition of \( \alpha^s \beta^t \)-adjoint representation is well defined.

**Proof.** For any \( u, v, w \in L \), first we show that \( \text{ad}_{s,t}(\alpha(u)) \circ \alpha = \alpha \circ \text{ad}_{s,t}(u) \)
\[
\text{ad}_{s,t}(\alpha(u))(\alpha(v)) = \delta[\alpha^{s+1} \beta^t(u), \alpha(v)]_L = \alpha(\delta[\alpha^s \beta^t(u), v]_L) = \alpha \circ \text{ad}_{s,t}(u)(v).
\]
Similarly, we have
\[
\text{ad}_{s,t}(\beta(u)) \circ \beta = \beta \circ \text{ad}_{s,t}(u).
\]
Note that the skew-symmetry condition implies
\[ \text{ad}_s(u)(v) = \delta[a^s b^t(u), v]_L \]
\[ = \delta[\beta(a^s b^t-1(u)), \alpha(\alpha^{-1}(v))]_L \]
\[ = -\delta^2[\alpha^{-1}(\beta(v)), \alpha^{s+1} b^{t-1}(u)]_L \]
\[ = -[\alpha^{-1}(\beta(v)), \alpha^{s+1} b^{t-1}(u)]_L, \forall u, v \in L. \]

On one hand, we have
\[ \text{ad}_{s,t}([\beta(u), v]_L) \circ \beta(w) = \text{ad}_{s,t}([\beta(u), v]_L)(\beta(w)) \]
\[ = -[\alpha^{-1}(\beta(w)), \alpha^{s+1} b^{t-1}(v)]_L \]
\[ = -[\alpha^{-1}(\beta^2(w), \alpha^{s+1} b^{t-1}(v)]_L. \]

On the other hand, we have
\[ \text{ad}_{s,t}(\alpha^s b^t(u) \circ \text{ad}_{s,t}(\alpha(v)) \circ \text{ad}_{s,t}(\beta(u))(w) \]
\[ = \text{ad}_{s,t}(\alpha^s b^t(u))(-[\alpha^{-1}(\beta(w), \alpha^{s+1} b^{t-1}(v)]_L \]
\[ -\delta \text{ad}_{s,t}(\alpha(v))(-[\alpha^{-1}(\beta(w), \alpha^{s+1} b^{t-1}(v)]_L \]
\[ = [\alpha^{-1}\beta([\alpha^{-1}(\beta(w), \alpha^{s+1} b^{t-1}(v)]_L), \alpha^{s+1} b^{t-1}(\alpha(\beta(u))]_L \]
\[ = [\alpha^{-1}\beta([\alpha^{-1}(\beta(w), \alpha^{s+1} b^{t-1}(v)]_L), \alpha^{s+1} b^{t-1}(\beta(v))]_L \]
\[ = \beta(\alpha^{-1}(\beta(w), \alpha^{s+1} b^{t-1}(v)]_L), \alpha^{s+2} b^{t}(w)]_L \]
\[ = -\delta[\beta(\alpha^{-1}(\beta(w), \alpha^{s+1} b^{t-1}(v)]_L \]
\[ = -\delta[\beta(\alpha^{s+1} b^{t}(u)), [\alpha^{-1}(\beta(w), \alpha^{s+1} b^{t-1}(v)]_L \]
\[ +[\alpha^{s+1} b^{t-1}(v), [\alpha^{-1}(\beta(w), \alpha^{s+2} b^{t-1}(u)]_L \]
\[ = [\alpha^{s+1} b^{t+1}(u), [\alpha^{-1}(\beta(w), \alpha^{s+1} b^{t-1}(v)]_L \]
\[ = [\alpha^1 b^{t+1}(u), \alpha^{-1}(\beta(w), \alpha^{s+1} b^{t-1}(u)]_L \]
\[ = -[\alpha^{s+1} b^t(u), [\beta(\alpha^{s+1} b^{-1}(u), \alpha(\beta^t-1)(v)]_L \]
\[ = -[\alpha^{s+1} b^t(u), [\beta(\alpha^{s+1} b^{-1}(u), \alpha(\beta^t-1)(v)]_L \]
\[ = -[\alpha^{-1}\beta^2(w), \alpha^{s+1} b^{t-1}(v)]_L. \]

Thus, the definition of \( \alpha^s b^t \)-adjoint representation is well defined. The proof is completed. \( \square \)

The set of \( k \)-Bihom-cochains on \( L \) with coefficients in \( L \), which we denote by \( C^k_{\alpha,\beta}(L; L) \), is given by
\[ C^k_{\alpha,\beta}(L; L) = \{ f \in C^k(L; L)| \alpha \circ f = f \circ \alpha, \beta \circ f = f \circ \beta \}. \]

In particular, the set of 0-Bihom-cochains is given by:
\[ C^0_{\alpha,\beta}(L; L) = \{ u \in L| \alpha(u) = u, \beta(u) = u \}. \]

Associated to the \( \alpha^s b^t \)-adjoint representation, the corresponding operator
\[ d_{s,t} : C^k_{\alpha,\beta}(L; L) \to C^{k+1}_{\alpha,\beta}(L; L)(k = 1, 2) \]
is given by
\[ d_{s,t} f(u_1, u_2) = \delta[a^{1+s} b^t(u_1), f(u_2)] - [a^{1+s} b^t(u_2), f(u_1)] - \delta f([\alpha^{-1}(\beta(u_1), u_2)]; \quad (6.1) \]
For the $\alpha^{s}\beta^{t}$-adjoint representation $ad_{s,t}$, we obtain the $\alpha^{s}\beta^{t}$-adjoint complex $(C_{\alpha,\beta}^{s,t}(L;L), d_{s,t})$.

We have known that a 1-cocycle associated to the adjoint representation is a derivation for Lie algebras and Hom-Lie algebras. Similarly, we have

**Proposition 6.3.** Associated to the $\alpha^{s}\beta^{t}$-adjoint representations $ad_{s,t}$ of the regular $\delta$-Bihom-Jordan-Lie algebra $(L,[\cdot,\cdot]_{L},\alpha,\beta)$, it satisfies $\delta^{s+1} = 1$, $D \in C_{\alpha,\beta}^{s,t}(L;L)$ is a 1-cocycle if and only if $D$ is an $\alpha^{s+2}\beta^{t-1}$-derivation, i.e. $D \in \text{Der}_{\alpha^{s+2}\beta^{t-1}}(L)$.

**Proof.** The conclusion follows directly from the definition of the operator $d_{s,t}$. $D$ is closed if and only if

$$d_{s,t}(D)(u,v) = \delta[\alpha^{s+1}\beta^{t}(u),D(v)]_{L} - [\alpha^{s+1}\beta^{t}(v),D(u)]_{L} - \delta D[\alpha^{-1}\beta(u),v]_{L} = 0.$$

$D$ is an $\alpha^{s+2}\beta^{t-1}$-derivation if and only if

$$D[\alpha^{-1}\beta(u),v]_{L} = -\delta[\alpha^{s+2}\beta^{t-1}\alpha^{-1}\beta(v),D(u)]_{L} + [\alpha^{s+2}\beta^{t-1}\alpha^{-1}\beta(u),D(v)]_{L} = \delta^{s+1}([D(u),\alpha^{s+1}\beta^{t}(v)]_{L} + [\alpha^{s+1}\beta^{t}(u),D(v)]_{L}).$$

Then, $D \in C_{\alpha,\beta}^{1}(L;L)$ is a 1-cocycle if and only if $D$ is an $\alpha^{s+2}\beta^{t-1}$-derivation, i.e. $D \in \text{Der}_{\alpha^{s+2}\beta^{t-1}}(L)$.

Let $\psi \in C_{\alpha,\beta}^{2}(L;L)$ be a bilinear operator commuting with $\alpha$ and $\beta$, also $\psi(u,v) = -\delta \psi(v,u)$. Consider a $t$-parameterized family of bilinear operations

$$[u,v]_{t} = [u,v]_{L} + t\psi(u,v). \quad (6.2)$$

Since $\psi$ commutes with $\alpha$, $\beta$, then $\alpha$, $\beta$ are morphisms with respect to the bracket $[\cdot,\cdot]_{t}$ for every $t$. If all the brackets $[\cdot,\cdot]_{t}$ endow $(L,[\cdot,\cdot]_{L},\alpha,\beta)$ with regular $\delta$-Bihom-Jordan-Lie algebra structures, we say that $\psi$ generates a deformation of the regular $\delta$-Bihom-Jordan-Lie algebra $(L,[\cdot,\cdot]_{L},\alpha,\beta)$. The anti-symmetry of $[\cdot,\cdot]_{t}$ means that

$$[\beta(v),\alpha(u)]_{t} = [\beta(v),\alpha(u)]_{L} + t\psi(\beta(v),\alpha(u))$$

and

$$[\beta(u),\alpha(v)]_{t} = [\beta(u),\alpha(v)]_{L} + t\psi(\beta(u),\alpha(v)).$$

Then $[\beta(v),\alpha(u)]_{t} = -\delta[\beta(u),\alpha(v)]_{t}$ if and only if

$$\psi(\beta(v),\alpha(u)) = -\delta \psi(\beta(u),\alpha(v)). \quad (6.3)$$
By computing the Bihom-Jacobi identity of $[\cdot, \cdot]_L$
\[
[\beta^2(u), [\beta(v), \alpha(u)]_L]_L + [\beta^2(v), [\beta(u), \alpha(v)]_L]_L + [\beta^2(w), [\beta(u), \alpha(v)]_L]_L \\
= [\beta^2(u), [\beta(v), \alpha(w)]_L + t\psi(\beta(v), \alpha(w))]_L \\
+ [\beta^2(v), [\beta(u), \alpha(u)]_L + t\psi(\beta(w), \alpha(u))]_L \\
+ [\beta^2(w), [\beta(u), \alpha(v)]_L + t\psi(\beta(u), \alpha(v))]_L
\]
\[
= [\beta^2(u), [\beta(v), \alpha(w)]_L + t\psi(\beta(u), [\beta(v), \alpha(w)])_L \\
+ [\beta^2(v), [\beta(w), \alpha(u)]_L + t\psi(\beta(v), \alpha(u))]_L \\
+ [\beta^2(w), [\beta(u), \alpha(v)]_L + t\psi(\beta(u), \alpha(v))]_L
\]
This is equivalent to the conditions
\[
\psi([\beta^2(u), [\beta(v), \alpha(w)])_L + [\beta^2(v), [\beta(u), \alpha(u)]_L + [\beta^2(w), [\beta(u), \alpha(v)]_L]) = 0, \quad (6.4)
\]
\[
\psi([\beta^2(u), [\beta(v), \alpha(w)])_L + [\beta^2(v), [\beta(u), \alpha(u)]_L + [\beta^2(w), [\beta(u), \alpha(v)]_L]) = 0. \quad (6.5)
\]
Obviously, (6.4) and (6.3) means that $\psi$ must itself define a $\delta$-Bihom-Jordan-Lie algebra structure on $L$. Furthermore, (6.5) means that $\psi$ is closed with respect to the $\alpha^{-1}\beta$-adjoint representation $d_{-1,1}$. i.e. $d_{-1,1}\psi = 0$.
\[
d_{-1,1}\psi(u, v, w) \\
= \delta[\beta^2(u), [\beta(v), \alpha(w)]_L - [\beta^2(v), [\beta(u), \alpha(w)]_L + \delta[\beta^2(w), [\beta(u), \alpha(w)]_L \\
- \psi([\alpha^{-1}\beta(u), [\beta(v), \alpha(w)]_L + \delta[\beta^2(w), [\beta(u), \alpha(w)]_L] - \psi([\alpha^{-1}\beta(u), w]_L, \beta(v)) \\
= \delta[\beta^2(u), [\beta(v), \alpha(w)]_L + \delta[\beta^2(v), [\beta(u), \alpha(w)]_L + \delta[\beta^2(w), [\beta(u), \alpha(w)]_L \\
+ \delta\psi([\beta(u), [\alpha^{-1}\beta(u), w]_L] + \delta\psi([\beta(v), [\alpha^{-1}\beta(u), w]_L] + \delta\psi([\beta(u), [\alpha^{-1}\beta(u), w]_L) \\
= 0.
\]
A deformation is said to be trivial if there is a linear operator $N \in C^1_{\alpha, \beta}(L; L)$ such that for $T_i = id + tN$, there holds
\[
T_i[u, v]_L = [T_i(u), T_i(v)]_L. \quad (6.6)
\]
**Definition 6.4.** A linear operator $N \in C^1_{\alpha, \beta}(L, L)$ is called a Bihom-Nijenhuis operator if we have
\[
[Nu, Nv]_L = Nu, v]_N, \quad (6.7)
\]
where the bracket $[\cdot, \cdot]_N$ is defined by
\[
[u, v]_N \equiv [Nu, v]_L + [u, Nv]_L - Nu, v]_L. \quad (6.8)
\]
**Theorem 6.5.** Let $N \in C^1_{\alpha, \beta}(L, L)$ be a Bihom-Nijenhuis operator. Then a deformation of the regular $\delta$-Bihom-Jordan-Lie algebra $(L, [\cdot, \cdot]_L, \alpha, \beta)$ can be obtained by putting
\[
\psi(u, v) = \delta d_{-1,1}N(u, v) = [u, v]_N.
\]
Furthermore, this deformation is trivial.
Thus, on the other hand, we have

\[ \mathcal{O}_{u,v,w} \psi(\beta^2(u), \psi(\beta(v), \alpha(w))) = \mathcal{O}_{u,v,w} \psi(\beta^2(u), [N \beta(v), \alpha(w)] + \beta(v), N \alpha(u)) - N[\beta(v), \alpha(w)]) = \mathcal{O}_{u,v,w} \psi(\beta^2(u), [\beta(v), N \alpha(u)] - \psi(\beta^2(u), N[\beta(v), \alpha(w)]) = \mathcal{O}_{u,v,w} [N \beta^2(u), [N \beta(v), \alpha(w)] + [N \beta^2(v), [\beta(w), N \beta(u)] + \beta^2(w), N[\beta(u), \alpha(v)])_N + \mathcal{O}_{u,v,w} N[\beta^2(v), N[\beta(w), \alpha(u)] - [N \beta^2(v), N[\beta(w), \alpha(u)] = -N[\beta^2(v), [\beta(w), \alpha(u)]) + 2[N \beta^2(v), [\beta(w), \alpha(u)])].

Furthermore, also by the fact that \( N \) is a Bihom-Nijenhuis operator and we take in (6.7) and (6.8), \( u = \beta^2(v) = \beta^2(w) \) and \( v = [\beta(w), \alpha(u)] \), we have

\[ \mathcal{O}_{u,v,w} [N \beta^2(u), [N \beta(v), \alpha(w)] + [N \beta^2(v), [\beta(w), N \beta(u)] + [N \beta^2(w), N[\beta(u), \alpha(v)])_N = 0. \]

By the Bihom-Jacobi identity of \( L \), we have

\[ \mathcal{O}_{u,v,w} N[\beta^2(v), N[\beta(w), \alpha(u)] - [N \beta^2(v), N[\beta(w), \alpha(u)] = \mathcal{O}_{u,v,w} N[\beta^2(v), [\beta(w), \alpha(u)]. \]

Then,

\[ \mathcal{O}_{u,v,w} \psi(\beta^2(u), \psi(\beta(v), \alpha(w))) = -N[\beta^2(v), [\beta(w), \alpha(u)]) - N[\beta^2(u), [N \beta(v), \alpha(w)] + [\beta^2(u), [\beta(u), N \beta(v)] = -N[\beta^2(Nv), [\beta(w), \alpha(u)] + [\beta^2(u), [N \beta(v), \alpha(w)] + [\beta^2(u), [\beta(u), N \beta(v)] = 0. \]

Thus \( \psi \) generates a deformation of the \( \delta \)-Bihom-Jordan-Lie algebra \((L, [\cdot, \cdot]_L, \alpha, \beta)\).

Let \( T_t = \text{id} + tN \), then we have

\[ T_t[u,v]_L = (\text{id} + tN)[uv]_L + t\psi(u,v) = (\text{id} + tN)[uv]_L + t[u,v]_N = [u,v]_L + t[t[u,v]_L + N[u,v]_L] + t^2N[u,v]_N. \]

On the other hand, we have

\[ [T_t(u), T_t(v)]_L = [u + tNu, v + tNv]_L = [u, v]_L + t([N[u,v]_L + [u,Nv]_L] + t^2[Nu,Nv]_L. \]

By the equations (6.7) and (6.8), we have

\[ T_t[u,v]_L = [T_t(u), T_t(v)]_L, \]

which implies that the deformation is trivial. \( \square \)

7. \( T^* \)-extensions of \( \delta \)-Bihom-Jordan-Lie algebras

The last part deals with \( T^* \)-extension. We provide in this section, for \( \delta \)-Bihom-Jordan-Lie algebras, characterizations of \( T^* \)-extensions and observations about \( T^* \)-extensions of nilpotent and solvable \( \delta \)-Bihom-Jordan-Lie algebras. This method was introduced by Martin Bordemann in [2].
Definition 7.1. Let \((L, \cdot, \cdot)_L, \alpha, \beta)\) be a \(\delta\)-Bihom-Jordan-Lie algebra. A bilinear form \(f\) on \(L\) is said to be nondegenerate if
\[
L^\perp = \{ x \in L | f(x, y) = 0, \forall y \in L \} = 0;
\]
\(\alpha\beta\)-invariant if
\[
f([\beta(x), \alpha(y)], \alpha(z)) = f(\alpha(x), [\beta(y), \alpha(z)]), \forall x, y, z \in L;
\]
symmetric if
\[
f(x, y) = f(y, x).
\]
A subspace \(I\) of \(L\) is called isotropic if \(I \subseteq I^\perp\).

Definition 7.2. Let \((L, \cdot, \cdot)_L, \alpha, \beta)\) be a \(\delta\)-Bihom-Jordan-Lie algebra over a field \(\mathbb{K}\). If \(L\) admits a nondegenerate invariant symmetric bilinear form \(f\), then we call \((L, f, \alpha, \beta)\) a quadratic \(\delta\)-Bihom-Jordan-Lie algebra. In particular, a quadratic vector space \(V\) is a vector space admitting a nondegenerate symmetric bilinear form.

Let \((L', \cdot, \cdot)_{L'}, \alpha_1, \beta_1)\) be another \(\delta\)-Bihom-Jordan-Lie algebra. Two quadratic \(\delta\)-Bihom-Jordan-Lie algebras \((L, f, \alpha, \beta)\) and \((L', f', \alpha_1, \beta_1)\) are said to be isometric if there exists a \(\delta\)-Bihom-Jordan-Lie algebra isomorphism \(\phi : L \to L'\) such that
\[
f(x, y) = f'(\phi(x), \phi(y)), \forall x, y \in L.
\]

Lemma 7.3. Let \(\text{ad}\) be the adjoint representation of a \(\delta\)-Bihom-Jordan-Lie algebra \((L, \cdot, \cdot)_L, \alpha, \beta)\). Let us consider \(L^*\) the dual space of \(L\), \(\tilde{\alpha}, \tilde{\beta} : L^* \to L^*\) two homomorphisms defined by
\[
\tilde{\alpha}(f) = f \circ \alpha, \quad \tilde{\beta}(f) = f \circ \beta, \quad \forall f \in L^*.
\]
Then the linear map \(\pi : L \to \text{End}(L^*)\) defined by, \(\pi(x)(f)(y) = -\delta f \circ \text{ad}(x)(y), \forall x, y \in L\), is a representation of \(L\) on \((L^*, \tilde{\alpha}, \tilde{\beta})\) if and only if
\[
\alpha \circ \text{ad}(x) = \text{ad}x \circ \alpha; \quad (7.1)
\]
\[
\beta \circ \text{ad}(\beta)(x) = \text{ad}x \circ \beta; \quad (7.2)
\]
\[
\text{ad}(\alpha(x)) \circ \text{ad}(\beta(y)) - \delta \text{ady} \circ \text{ad}(\alpha\beta(x)) = \beta \circ \text{ad}[\beta(x), y]_L. \quad (7.3)
\]
We call the representation \(\pi\) the coadjoint representation of \(L\).

Proof. Firstly, we have
\[
(\pi(\alpha(x)) \circ \tilde{\alpha})(f) = -\delta \tilde{\alpha}(f) \circ \alpha(x) = -\delta f \circ \alpha \circ \text{ad}(x),
\]
and
\[
\tilde{\alpha}(\pi(x))(f) = -\delta \tilde{\alpha}(f \circ \text{ad}x) = -\delta f \circ \text{ad}x \circ \alpha.
\]
Similarly,
\[
(\pi(\beta(x)) \circ \tilde{\beta})(f) = -\delta \tilde{\beta}(f) \circ \beta(x) = -\delta f \circ \beta \circ \text{ad}(x),
\]
and
\[
\tilde{\beta}(\pi(x))(f) = -\delta \tilde{\beta}(f \circ \text{ad}x) = -\delta f \circ \text{ad}x \circ \beta.
\]
Therefore,
\[
(\pi([\beta(x), y]) \circ \tilde{\beta})(f) = -\delta f \circ \beta \circ \text{ad}[\beta(x), y];
\]
\[
(\pi(\alpha\beta(x)) \circ \pi(y) - \delta \pi(\beta(y)) \circ \pi(\alpha(x)))(f)
\]
\[
= -\delta \pi(\alpha\beta(x))(f \circ \text{ady}) + \pi(\beta(y))(f \circ \text{ad}(x));
\]
\[
= f \circ \text{ady} \circ \text{ad}\beta x - \delta f \circ \text{ad}(x) \circ \text{ad}\beta(y)
\]
\[
= -\delta f \circ (\text{ad}(x) \circ \text{ad}\beta(y) - \text{ady} \circ \text{ad}\beta(x)).
\]
Then we have
\[
\begin{align*}
\pi(\alpha(x)) \circ \tilde{\alpha} &= \tilde{\alpha}(\pi(x)); \\
\pi(\beta(x)) \circ \tilde{\beta} &= \tilde{\beta}(\pi(x)); \\
\pi([\beta(x), y]) \circ \tilde{\beta} &= \pi(\alpha \beta(x)) \circ \pi(y) - \delta \pi(\beta(y)) \circ \pi(\alpha(x)).
\end{align*}
\]
Then \( \pi \) is a representation of \( L \) on \( (L^*, \tilde{\alpha}, \tilde{\beta}) \).

**Lemma 7.4.** Under the above notations, let \((L, [\cdot, \cdot]_L, \alpha, \beta)\) be a \( \delta \)-Bihom-Jordan-Lie algebra, and \( \omega : L \times L \to L^* \) be a bilinear map. Assume that the coadjoint representation exists. The space \( L \oplus L^* \), provided with the following bracket and a linear map defined respectively by
\[
[x + f, y + g]_{L \oplus L^*} = [x, y]_L + \omega(x, y) + \delta \pi(x)g - \pi(\alpha^{-1} \beta(y) \tilde{\alpha} \tilde{\beta}^{-1}(f)),
\]
\[
\alpha'(x + f) = \alpha(x) + f \circ \alpha,
\]
\[
\beta'(x + f) = \beta(x) + f \circ \beta.
\]

Then \((L \oplus L^*, [\cdot, \cdot]_{L \oplus L^*}, \alpha', \beta')\) is a \( \delta \)-Bihom-Jordan-Lie algebra if and only if \( \omega \) is a 2-cocycle: \( L \times L \to L^* \), i.e. \( \omega \in Z^2(L, L^*) \).

**Proof.** For any elements \( x + f, y + g, z + h \in L \oplus L^* \). We have
\[
\begin{align*}
[\beta'(x + f), \alpha'(y + g)] &= [\beta(x) + f \circ \beta, \alpha(y) + g \circ \alpha] \\
&= [\beta(x), \alpha(y)]_L + w(\beta(x), \alpha(y)) + \delta \pi(\beta(x))(g \circ \alpha) - \pi(\alpha^{-1} \beta(\alpha(y)))(\tilde{\alpha} \tilde{\beta}^{-1}(f \circ \beta)) \\
&= [\beta(x), \alpha(y)]_L + w(\beta(x), \alpha(y)) + \delta \pi(\beta(x))(g \circ \alpha) - \pi(\beta(y))(f \circ \alpha).
\end{align*}
\]
Similarly, we have
\[
[\beta'(y + g), \alpha'(x + f)] = [\beta(y) + g \circ \alpha, \alpha(x) + f \circ \alpha] + \delta \pi(\beta(y))(f \circ \alpha) - \pi(\beta(x))(g \circ \alpha).
\]
Then, we have \([\beta'(x + f), \alpha'(y + g)] = -\delta[\beta'(y + g), \alpha'(x + f)]\) if and only if
\[
w(\beta(x), \alpha(y)) = -\delta w(\beta(y), \alpha(x)).
\]
Therefore,
\[
\begin{align*}
[\beta^2(x + f), [\beta'(y + g), \alpha'(z + h)]] &= [\beta^2(x) + f \circ \beta^2, [\beta(y) + g \circ \alpha, \alpha(z) + h \circ \alpha]] \\
&= [\beta^2(x) + f \circ \beta^2, [\beta(y), \alpha(z)]_L + w(\beta(y), \alpha(z)) + \delta \pi(\beta(y))(h \circ \alpha) - \pi(\alpha^{-1} \beta(\alpha(z)))(\tilde{\alpha} \tilde{\beta}^{-1}(g \circ \beta))] \\
&= [\beta^2(x) + f \circ \beta^2, [\beta(y), \alpha(z)]_L + w(\beta(y), \alpha(z)) + \delta \pi(\beta(y))(h \circ \alpha) - \pi(\beta(z))(f \circ \alpha)].
\end{align*}
\]
And
\[
\begin{align*}
[\beta^2(y + g), [\beta'(z + h), \alpha'(x + f)]] &= [\beta^2(y) + g \circ \alpha, [\beta(z) + h \circ \alpha, \alpha(x) + f \circ \alpha]] \\
&= [\beta^2(y) + g \circ \alpha, [\beta(z), \alpha(x)]_L + w(\beta^2(y), \beta(z), \alpha(x)) + \delta \pi(\beta^2(y))(g \circ \alpha) - \pi(\alpha^{-1} \beta(\beta(z), \alpha(x)))(f \circ \alpha)].
\end{align*}
\]
\[ \left[ \beta^2(z + h), [\beta'(x + f), \alpha'(y + g)] \right] \]
\[ = \left[ \beta^2(z), [\beta(x), \alpha(y)] \right]_L + \omega(\beta^2(z), [\beta(x), \alpha(y)]_L) \]
\[ + \delta \pi(\beta^2(z))w(\beta(x), \alpha(y)) + \pi(\beta^2(z))\pi(\beta(x))(g \circ \alpha) \]
\[ - \delta \pi(\beta^2(z))\pi(\beta(y))(f \circ \alpha) - \pi(\alpha^{-1}\beta[\beta(x), \alpha(y)])(h \circ \beta \circ \alpha). \]

Since \( \pi \) is the coadjoint representation of \( L \), we have
\[ \pi(\alpha^{-1}\beta[\beta(x), \alpha(y)])_L h \circ \beta \circ \alpha \]
\[ = \pi([\beta(\alpha^{-1}\beta(x)), \beta(y)]_L) \circ \beta(h \circ \alpha) \]
\[ = \pi(\alpha\beta(\alpha^{-1}\beta(x)))\pi(\beta(y))(h \circ \alpha) - \delta \pi(\beta(\beta(y)))\pi(\alpha(\alpha^{-1}\beta(x)))(h \circ \alpha) \]
\[ = \pi(\beta^2(x))\pi(\beta(y))(h \circ \alpha) - \delta \pi(\beta^2(\beta))(\pi(\beta(x))(h \circ \alpha)). \]

Similarly,
\[ \pi(\alpha^{-1}\beta[\beta(y), \alpha(z)]_L) f \circ \beta \circ \alpha = \pi(\beta^2(y))\pi(\beta(z))(f \circ \alpha) - \delta \pi(\beta^2(z))\pi(\beta(y))(f \circ \alpha), \]
and
\[ \pi(\alpha^{-1}\beta[\beta(z), \alpha(x)]_L) g \circ \beta \circ \alpha = \pi(\beta^2(\beta))(\pi(\beta(x))(g \circ \alpha) - \delta \pi(\beta^2(x))\pi(\beta(z))(g \circ \alpha). \]

Consequently, \( [\beta^2(x + f), [\beta'(y + g), \alpha'(z + h)] + [\beta^2(y + g), [\beta'(z + h), \alpha'(x + f)] + [\beta^2(z + h), [\beta'(x + f), \alpha'(y + g)] = 0 \) if and only if
\[ 0 = \omega(\beta^2(x), \left[ \beta(y), \alpha(z) \right]\_L) + \delta \pi(\beta^2(x))w(\beta(y), \alpha(z)) \]
\[ + w(\beta^2(y), \left[ \beta(z), \alpha(x) \right]\_L) - \delta \pi(\beta^2(y))w(\beta(z), \alpha(x)) \]
\[ + \delta w(\beta^2(z), \left[ \beta(x), \alpha(y) \right]\_L) + \delta \pi(\beta^2(z))w(\beta(x), \alpha(y)) \]
\[ - \delta w(\left[ \beta(y), \alpha(z) \right]\_L, \beta^2(x)) + w(\beta^2(y), \left[ \beta(x), \alpha(y) \right]\_L) - \delta w(\left[ \beta(x), \alpha(y) \right]\_L, \beta^2(z)) \]
\[ = \delta d_{-1,1} \omega(x, y, z). \]

That is \( \omega \in Z^2_{\alpha, \beta}(L, L^*) \). Then confirmation holds if and only if \( \omega \in Z^2(L, L^* \). Consequently, we prove the lemma. \( \square \)

Clearly, \( L^* \) is an abelian Bihom-ideal of \( (L \oplus L^*, \cdot, \cdot, \alpha', \beta') \) and \( L \) is isomorphic to the factor \( \delta \)-Bihom-Jordan-Lie algebra \( (L \oplus L^*)/L^* \). Moreover, consider the following symmetric bilinear form \( q_L \) on \( L \oplus L^* \) for all \( x + f, y + g \in L \oplus L^* \),
\[ q_L(x + f, y + g) = f(y) + g(x). \]

Then we have the following lemma.

**Lemma 7.5.** Let \( L, L^*, \omega \) and \( q_L \) be as above. Then the 4-tuple \( (L \oplus L^*, q_L, \alpha', \beta') \) is a quadratic \( \delta \)-Bihom-Jordan-Lie algebra if and only if \( \omega \) is Jordan-cyclic in the following sense:
\[ \omega(\beta(x), \alpha(y))(\alpha(z)) = \omega(\beta(y), \alpha(z))(\alpha(x)) \text{ for all } x, y, z \in L. \]

**Proof.** If \( x + f \) is orthogonal to all elements of \( L \oplus L^* \), then \( f(y) = 0 \) and \( g(x) = 0 \), which implies that \( x = 0 \) and \( f = 0 \). So the symmetric bilinear form \( q_L \) is nondegenerate.
Now suppose that \( x + f, y + g, z + h \in L \oplus L^* \), then
\[
q_L(\beta'(x + f), \alpha'(y + g), \alpha'(z + h)) = q_L(\beta(x + f \circ \beta, \alpha(y + g \circ \alpha)) + \pi(\beta(x))g \circ \alpha - \pi(\alpha^{-1}\beta\alpha(y))\beta^{-1}(f \circ \beta), \alpha(z) + h \circ \alpha)
\]
\[
= \omega(\beta(x), \alpha(y)) + \delta(\pi(\beta(x))g \circ \alpha)(\alpha(z)) - \pi(\beta(y))(f \circ \alpha)(\alpha(z)) + h \circ \alpha([\beta(x), \alpha(y)]_L)
\]
\[
= \omega(\beta(x), \alpha(y))/(\alpha(z)) - \delta \circ \alpha([\beta(x), \alpha(z)]_L) + f \circ \alpha([\beta(y), \alpha(z)]_L) + h \circ \alpha([\beta(x), \alpha(z)]_L).
\]

On the other hand,
\[
q_L(\alpha'(x + f), [\beta'(y + g), \alpha'(z + h)]_L) = q_L(\alpha(x + f \circ \alpha, [\beta(y) + g \circ \beta, \alpha(z) + h \circ \alpha])_L)
\]
\[
= q_L(\alpha(x + f \circ \alpha, [\beta(y), \alpha(z)]_L + \omega(\beta(y), \alpha(z)) + \delta(\pi(\beta(y))g \circ \alpha - \pi(\alpha^{-1}\beta\alpha(y))\beta^{-1}(g \circ \beta))
\]
\[
= q_L(\alpha(x + f \circ \alpha, [\beta(y), \alpha(z)]_L + \omega(\beta(y), \alpha(z)) + \delta(\pi(\beta(y))g \circ \alpha - \pi(\beta(y))(g \circ \alpha))
\]
\[
= f \circ \alpha([\beta(y), \alpha(z)]_L + \omega(\beta(y), \alpha(z))(\alpha(x)) + \delta(\pi(\beta(y))g \circ \alpha(\alpha(x))
\]
\[
= \omega(\beta(y), \alpha(z))(\alpha(x)) + g \circ \alpha([\beta(z), \alpha(x)]_L) + f \circ \alpha([\beta(y), \alpha(z)]_L)
\]
\[
- \delta h \circ \alpha([\beta(y), \alpha(x)]_L).
\]

Hence the lemma follows. \(\square\)

Now, for a Jordan cyclic 2-cocycle \( \omega \) we shall call the quadratic \( \delta \)-Bihom-Jordan-Lie algebra \( (L \oplus L^*, q_L, \alpha, \beta) \) the \( T^* \)-extension of \( L \) (by \( \omega \)) and denote the \( \delta \)-Bihom-Jordan-Lie algebra \( (L \oplus L^*, [,], \alpha', \beta') \) by \( T^*_\omega L \).

**Definition 7.6.** Let \( L \) be a \( \delta \)-Bihom-Jordan-Lie algebra over a field \( \mathbb{K} \). We inductively define a derived series
\[
(L^{(n)})_{n \geq 0} : L^{(0)} = L, L^{(n+1)} = [L^{(n)}, L^{(n)}],
\]
and a central descending series
\[
(L^n)_{n \geq 0} : L^0 = L, L^{n+1} = [L^n, L].
\]

\( L \) is called solvable and nilpotent (of length \( k \)) if and only if there is a (smallest) integer \( k \) such that \( L^{(k)} = 0 \) and \( L^k = 0 \), respectively.

In the following theorem we discuss some properties of \( T^*_\omega L \).

**Theorem 7.7.** Let \( (L, [,], \alpha, \beta) \) be a \( \delta \)-Bihom-Jordan-Lie algebra over a field \( \mathbb{K} \).

1. If \( L \) is solvable (nilpotent) of length \( k \), then the \( T^* \)-extension \( T^*_\omega L \) is solvable (nilpotent) of length \( r \), where \( k \leq r \leq k + 1 \) \( (k \leq r \leq 2k - 1) \).
2. If \( L \) is decomposed into a direct sum of two Bihom-ideals of \( L \), so is the trivial \( T^* \)-extension \( T^*_0 L \).
**Proof.** (1) Firstly we suppose that $L$ is solvable of length $k$. Since $(T^*_wL)^{(n)}/L^* \cong L^{(n)}$ and $L^{(k)} = 0$, we have $(T^*_wL)^{(k)} \subseteq L^*$, which implies $(T^*_wL)^{(k+1)} = 0$ because $L^*$ is abelian, and it follows that $T^*_wL$ is solvable of length $k$ or $k + 1$.

Suppose now that $L$ is nilpotent of length $k$. Since $(T^*_wL)^{(n)}/L^* \cong L^n$ and $L^k = 0$, we have $(T^*_wL)^k \subseteq L^*$. Let $g \in (T^*_wL)^k \subseteq L^*, b \in L$, $x_1 + f_1, \ldots, x_{k-1} + f_{k-1} \in T^*_wL$, $1 \leq i \leq k - 1$, we have

$$[[\cdots,[x_1,f_1]_{L^*},\cdots,x_{k-1},f_{k-1}]_{L^*},b] = \delta^k \gamma(\delta x_1) \gamma(\delta x_2) \cdots \gamma(\delta x_{k-1}) \gamma(\delta b)(b)$$

$$= g([x_1],[\beta^{-1}(\delta x_2),\ldots,[\beta^{-1}(\delta x_{k-1}),\beta^{-1}(\delta b)]_{L^*},L]_L)$$

This proves that $(T^*_wL)^{2k-1} = 0$. Hence $T^*_wL$ is nilpotent of length at least $k$ and at most $2k - 1$.

(2) Suppose that $0 \neq L = I \oplus J$, where $I$ and $J$ are two nonzero Bihom-ideals of $(L,[,]_L,\alpha,\beta)$. Let $I^*$ (resp. $J^*$) denote the subspace of all linear forms in $L^*$ vanishing on $J$ (resp. $I$). Clearly, $I^*$ (resp. $J^*$) can canonically be identified with the dual space of $I$ (resp. $J$) and $L^* \cong I^* \oplus J^*$.

Since $[I^*,L]_{L^*L^*}(J) = I^*[([L,J]_L) \subseteq I^*[([J,L]_L) \subseteq I^*(J) = 0$ and $[I,J]_{L^*L^*}(J) = L^*[([I,J]_L) \subseteq L^*[I,J]_L \subseteq L^*[I,J]_L \subseteq L^*$, then $[I^*,L]_{L^*L^*} = [I^*,L]_{L^*L^*} + [I^*,L]_{L^*L^*} + [I^*,L]_{L^*L^*} \subseteq I^* \oplus I^* = T^*_wL$.

$T^*_wL$ is a Bihom-ideal of $L$ and so is $T^*_wL$ in the same way. Hence $T^*_wL$ can be decomposed into the direct sum $T^*_wI \oplus T^*_wJ$ of two nonzero Bihom-ideals of $T^*_wL$. □

In the proof of a criterion for recognizing $T^*$-extensions of a $\delta$-Bihom-Jordan-Lie algebra, we will need the following result.

**Lemma 7.8.** Let $(L,q_L,\alpha,\beta)$ be a quadratic $\delta$-Bihom-Jordan-Lie algebra of even dimension $n$ over a field $\mathbb{K}$ and $I$ be an isotropic $n/2$-dimensional subspace of $L$. If $I$ is a Bihom-ideal of $(L,[,]_L,\alpha,\beta)$, then $[\beta(I),\alpha(I)] = 0$.

**Proof.** Since $\dim L + \dim I^\perp = n/2 + \dim I^\perp = n$ and $I \subseteq I^\perp$, we have $I = I^\perp$. If $I$ is a ideal of $(L,[,]_L,\alpha,\beta)$, then $q_L(\alpha(L),[\beta(I),\alpha(I)]) = q_L([\beta(L),\alpha(I)],\alpha(I^\perp)) \subseteq q_L([\beta(L),I],\alpha(I^\perp)) \subseteq q_L(I,\alpha(I)) = 0$, which implies $[\beta(I),\alpha(I)] = [\beta(I),\alpha(I^\perp)] \subseteq \alpha(I^\perp) = 0$. □

**Theorem 7.9.** Let $(L,q_L,\alpha,\beta)$ be a quadratic $\delta$-Bihom-Jordan-Lie algebra of even dimension $n$ over a field $\mathbb{K}$ of characteristic not equal to two. Then $(L,q_L,\alpha,\beta)$ is isometric to a $T^*$-extension $(T^*_wB,q_B,\alpha',\beta')$ if and only if $n$ is even and $(L,[,]_L,\alpha,\beta)$ contains an isotropic $\delta$-Bihom-ideal $I$ of dimension $n/2$. In particular, $B \cong L/I$, with $B^*$ satisfying $\alpha(B^*) \subseteq B^*$ and $\beta(B^*) \subseteq B^*$.

**Proof.** $(\Rightarrow)$ Since $\dim B = \dim B^*$, $\dim T^*_wB$ is even. Moreover, it is clear that $B^*$ is a Bihom-ideal of half the dimension of $T^*_wB$ and by the definition of $q_B$, we have $q_B(B^*,B^*) = 0$, i.e., $B^* \subseteq (B^*)^\perp$ and so $B^*$ is isotropic.

$(\Leftarrow)$ Suppose that $I$ is an $n/2$-dimensional isotropic $\delta$-Bihom-ideal of $L$. By Lemma 7.8, $[\beta(I),\alpha(I)] = 0$. Let $B = L/I$ and $p : L \to B$ be the canonical projection. Since $\text{ch}\mathbb{K} \neq 2$, we can choose an isotropic complement subspace $B_0$ to $I$ in $L$, i.e., $L = B_0 + I$ and $B_0 \subseteq B_0^\perp$. Then $B_0^\perp = B_0$ since $\dim B_0 = n/2$.

Denote by $p_0$ (resp. $p_1$) the projection $L \to B_0$ (resp. $L \to I$) and let $q^*_L$ denote the homogeneous linear map $I \to B^* : i \mapsto q^*_L(i)$, where $q^*_L(i)(p(x)) := q_L(i,x), \forall x \in L$. We claim that $q^*_L$ is a linear isomorphism. In fact, if $p(x) = p(y)$, then $x - y \in I$, hence
$q_L(i, x - y) \in q_L(I, I) = 0$ and so $q_L(i, x) = q_L(i, y)$, which implies $q^*_L$ is well-defined and it is easily seen that $q^*_L$ is linear. If $q^*_L(i) = q^*_L(j)$, then $q^*_L(i)(p(x)) = q^*_L(j)(p(x)), \forall x \in L$, i.e., $q_L(i, x) = q_L(j, x)$, which implies $i - j \in L^\perp$, hence $q^*_L$ is injective. Note that \( \dim I = \dim B^* \), then $q^*_L$ is surjective.

In addition, $q^*_L$ has the following property:

\[
q^*_L(\beta(x), \alpha(i)) = \delta q^*_L(\beta(x), \alpha(i))p(\alpha(y)) + \delta p(\beta(x))q^*_L(\alpha(i))p(\alpha(y)),
\]

where $x, y \in L$, $i \in I$. A similar computation shows that

\[
q^*_L(\beta(x), \alpha(i)) = [\beta(x), q^*_L(\alpha(i))]_{L \perp B^*}, \quad q^*_L(\beta(i), \alpha(x)) = [q^*_L(\beta(i)), p(\beta(x))]_{L \perp B^*}.
\]

Define a homogeneous bilinear map

\[
\omega : B \times B \longrightarrow B^*
\]

\[
(p(b_0), p(b'_0)) \longmapsto q^*_L(p_1([b_0, b'_0])),
\]

where $b_0, b'_0 \in B_0$. Then $w$ is well-defined since the restriction of the projection $p$ to $B_0$ is a linear isomorphism.

Let $\varphi$ be the linear map $L \rightarrow B \oplus B^*$ defined by $\varphi(b_0 + i) = p(b_0) + q^*_L(i), \forall b_0 + i \in B_0 + I = L$. Since the restriction of $p$ to $B_0$ and $q^*_L$ are linear isomorphisms, $\varphi$ is also a linear isomorphism. Note that

\[
\varphi(\beta(b_0 + i), \alpha(b'_0 + i'))_{L} = \varphi(\beta(b_0), \alpha(b'_0))_{L} + \beta(b_0), \alpha(i')_{L} + \beta(i), \alpha(b'_0)_{L}
\]

\[
= \varphi(p_0(\beta(b_0), \alpha(b'_0))_{L} + \pi(\beta(b_0), \alpha(b'_0))_{L} + \beta(b_0), \alpha(i')_{L} + \beta(i), \alpha(b'_0)_{L})
\]

\[
= p_1(\beta(b_0), \alpha(b'_0))_{L} + q^*_L(\pi(\beta(b_0), \alpha(b'_0))_{L} + \beta(b_0), \alpha(i')_{L} + \beta(i), \alpha(b'_0)_{L})
\]

\[
= [p_1(\beta(b_0), \alpha(b'_0))_{L} + \omega(p(\beta(b_0)), p(\alpha(b'_0)))]_{L} + \delta(p(\beta(b_0))(q^*_L(\alpha(i'))
\]

\[
- \pi(p(\beta(b_0))(q^*_L(\alpha(i))))
\]

\[
= [\varphi(\beta(b_0)), \alpha(b'_0) + q^*_L(\alpha(i'))_{B \oplus B^*}
\]

\[
= \varphi(\beta(b_0 + i), \alpha(b'_0 + i'))_{B \oplus B^*}
\]

Then $\varphi$ is an isomorphism of algebras, and so $(B \oplus B^*, [\cdot, \cdot]_{B \oplus B^*}, \alpha, \beta)$ is a $\delta$-Bihom-Jordan-Lie algebra. Furthermore, we have

\[
q_B(\varphi(b_0 + i), \varphi(b'_0 + i')) = q_B(p(b_0) + q^*_L(i), p(b'_0) + q^*_L(i'))
\]

\[
= q^*_L(i)(p(b'_0)) + q^*_L(i')(p(b_0))
\]

\[
= q_L(i, b'_0) + q_L(i', b_0)
\]

\[
= q_L(b_0 + i, b'_0 + i'),
\]

then $\varphi$ is isometric. The relation

\[
q_B(\beta(\varphi(x)), \alpha'(\varphi(\alpha(y)))) = q_B(\alpha'(\varphi(\alpha(z))))
\]

\[
= q_B([\varphi(\beta(x)), \varphi(\alpha(y)), \varphi(\alpha(z))] = q_B(\varphi(\beta(x), \alpha(y)), \varphi(\alpha(z)))) = q_L([\beta(x), \alpha(y)], \alpha(z))
\]

\[
= q_L(\alpha(x), [\beta(y), \alpha(z)]) = q_B(\varphi(\beta(y)), \varphi(\alpha(z)))
\]

\[
= q_B(\alpha'(\varphi(z)), [\beta'(\varphi(y)), \alpha'(\varphi(z))])
\]
which implies that $q_B$ is a nondegenerate invariant symmetric bilinear form, and so $(B \oplus B^*, q_B, \alpha', \beta')$ is a quadratic $\delta$-Bihom-Jordan-Lie algebra. In this way, we get a $T^*$-extension $T_w^* B$ of $B$ and consequently, $(L, q_L, \alpha, \beta)$ and $(T_w^* B, q_B, \alpha', \beta')$ are isomorphic as required.

Let $(L, [\cdot, \cdot]_L, \alpha, \beta)$ be a $\delta$-Bihom-Jordan-Lie algebra over a field $K$, and let $\omega_1 : L \times L \to L^*$ and $\omega_2 : L \times L \to L^*$ be two different Jordan cyclic 2-cocycles. The $T^*$-extensions $T_w^* L$ and $T_w^* L$ of $L$ are said to be equivalent if there exists an isomorphism of $\delta$-Bihom-Jordan-Lie algebras $\phi : T_w^* L \to T_w^* L$ which is the identity on the Bihom-ideal $L^*$ and which induces the identity on the factor $\delta$-Bihom-Jordan-Lie algebra $T_w^* L/L^* \cong L \cong T_w^* L/L^*$. The two $T^*$-extensions $T_w^* L$ and $T_w^* L$ are said to be isometrically equivalent if they are equivalent and $\phi$ is an isometry.

**Proposition 7.10.** Let $L$ be a $\delta$-Bihom-Jordan-Lie algebra over a field $K$ of characteristic not equal to 2, and $\omega_1, \omega_2$ be two Jordan cyclic 2-cocycles $L \times L \to L^*$. Then we have

(i) $T_w^* L$ is equivalent to $T_w^* L$ if and only if there is $z \in C^1(L, L^*)$ such that

$$\omega_1(x, y) - \omega_2(x, y) = \delta \pi(x)z(y) - \pi(\alpha^{-1} \beta(y)\tilde{\alpha}\tilde{\beta}^{-1} z(x) - z([x, y], L), \forall x, y \in L). \quad (7.7)$$

If this is the case, then the symmetric part $z_s$ of $z$, defined by $z_s(x)(y) := \frac{1}{2}(z(x)(y) + z(y)(x))$, for all $x, y \in L$, induces a symmetric invariant bilinear form on $L$.

(ii) $T_w^* L$ is isometrically equivalent to $T_w^* L$ if and only if there is $z \in C^1(L, L^*)$ such that $(29)$ holds for all $x, y \in L$ and the symmetric part $z_s$ of $z$ vanishes.

**Proof.** (i) $T_w^* L$ is equivalent to $T_w^* L$ if and only if there is an isomorphism of $\delta$-Bihom-Jordan-Lie algebras $\Phi : T_w^* L \to T_w^* L$ satisfying $\Phi|_{L^*} = 1_{L^*}$ and $x - \Phi(x) \in L^*, \forall x \in L$.

Suppose that $\Phi : T_w^* L \to T_w^* L$ is an isomorphism of $\delta$-hom-Jordan-Lie algebra and define a linear map $z : L \to L^*$ by $z(x) := \Phi(x) - x$, then $z \in C^1(L, L^*)$ and for all $x + f, y + g \in T_w^* L$, we have

$$\Phi([x, f], y + g) = \Phi([x, y], L) + \omega_1(x, y) + \delta \pi(x)g - \pi(\alpha^{-1} \beta(y)\tilde{\alpha}\tilde{\beta}^{-1} f) = [x, y]_L + z([x, y], L) + \omega_1(x, y) + \delta \pi(x)g - \pi(\alpha^{-1} \beta(y)\tilde{\alpha}\tilde{\beta}^{-1} f).$$

On the other hand,

$$\Phi(x + f, y + g) = [x + z(x) + f, y + z(y) + g] = [x, y]_L + \omega_2(x, y) + \delta \pi(x)g + \delta \pi(x)z(y) - \pi(\alpha^{-1} \beta(y)\tilde{\alpha}\tilde{\beta}^{-1} z(x) - \pi(\alpha^{-1} \beta(y)\tilde{\alpha}\tilde{\beta}^{-1} f).$$

Since $\Phi$ is an isomorphism, $(7.7)$ holds.

Conversely, if there exists $z \in C^1(L, L^*)$ satisfying $(7.7)$, then we can define $\Phi : T_w^* L \to T_w^* L$ by $\Phi(x + f) := x + z(x) + f$. It is easy to prove that $\Phi$ is an isomorphism of $\delta$-Bihom-Jordan-Lie algebras such that $\Phi|_{L^*} = id_{L^*}$ and $x - \Phi(x) \in L^*, \forall x \in L$, i.e. $T_w^* L$ is equivalent to $T_w^* L$.

Consider the symmetric bilinear form $q_L : L \times L \to K, (x, y) \mapsto z_s(x)(y)$ induced by $z_s$. Note that

$$\omega_1(\beta(x), \alpha(y))(\alpha(m)) - \omega_2(\beta(x), \alpha(y))(\alpha(m))$$

$$= \delta \pi(\beta(x))z(\alpha(y))(\alpha(m)) - \pi(\alpha^{-1} \beta(y)\tilde{\alpha}\tilde{\beta}^{-1} z(\beta(x))(\alpha(m)) - z([\beta(x), \alpha(y)], L)(\alpha(m))$$

$$= \delta \pi(\beta(x))z(\alpha(y))(\alpha(m)) - \pi(\alpha^{-1} \beta(y)\tilde{\alpha}\tilde{\beta}^{-1} z(\beta(x))(\alpha(m)) - z([\beta(x), \alpha(y)], L)(\alpha(m))$$

$$= -\delta z(\alpha(y))(\beta(x), \alpha(m)]_L + z(\alpha(x))([\beta(y), \alpha(m)]_L - z([\beta(x), \alpha(y)]_L)(\alpha(m)),$$
and
\[ \omega_1(\beta(y), \alpha(m))(\alpha(x)) - \omega_2(\beta(y), \alpha(m))(\alpha(x)) = \delta(\beta(y))(\alpha(m))(\alpha(x)) - \pi(\alpha(m))\beta(y)(\alpha(x)) - z([\beta(y), \alpha(m)]_L)(\alpha(x)) \]
\[ = -\delta(\alpha(m))(\beta(y))(\alpha(x)) + z(\alpha(y))(\beta(m))(\alpha(x)) - z([\beta(y), \alpha(m)]_L)(\alpha(x)) \]
\[ = z(\alpha(m))(\beta(y))(\alpha(x)) - \delta(\alpha(y))(\beta(x))(\alpha(m)) - z([\beta(y), \alpha(m)]_L)(\alpha(x)). \]

Since both \( \omega_1 \) and \( \omega_2 \) are Jordancyclic, the right hand sides of above two equations are equal. Hence
\[ -\delta(\alpha(y))(\beta(x))(\alpha(m)) + z(\alpha(y))(\beta(y))(\alpha(m)) = z(\alpha(m))(\beta(y))(\alpha(x)) - \delta(\alpha(y))(\beta(x))(\alpha(m)) - z([\beta(y), \alpha(m)]_L)(\alpha(x)). \]

That is
\[ z(\alpha(y))(\beta(y))(\alpha(m))L + z(\alpha(x))(\beta(y))(\alpha(m))L - z([\beta(y), \alpha(m)]_L)(\alpha(m)) \]
\[ = z(\alpha(y))(\beta(y))(\alpha(m))L - \delta(\alpha(y))(\beta(x))(\alpha(m))L - z([\beta(y), \alpha(m)]_L)(\alpha(x)). \]

Since \( \text{chK} \neq 2 \), \( q_L(\alpha(x), [\beta(y), \alpha(m)]) = q_L([\beta(x), \alpha(y)], \alpha(m)) \), which proves the invariance of the symmetric bilinear form \( q_L \) induced by \( z_s \).

(ii) Let the isomorphism \( \Phi \) be defined as in (i). Then for all \( x + f, y + g \in L \oplus L^* \), we have
\[ q_B(\Phi(x + f), \Phi(y + g)) = q_B(x + z(x) + f, y + z(y) + g) \]
\[ = z(x)(y) + f(y) + z(y)(x) + g(x) \]
\[ = z(x)(y) + z(y)(x) + f(y) + g(x) \]
\[ = 2z_s(x)(y) + q_B(x + f, y + g). \]

Thus, \( \Phi \) is an isometry if and only if \( z_s = 0 \). \( \square \)

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