EXAMPLES OF DEFORMATIONS OF NEARLY PARALLEL $G_2$ STRUCTURES ON 7-DIMENSIONAL 3-SASAKIAN MANIFOLDS
BY CHARACTERISTIC VECTOR FIELDS

NÜLİFER ÖZDEMİR AND ŞİRİN AKTAY

(Communicated by Yusuf YAYLI)

Abstract. It is known that 7-dimensional 3-Sasakian manifolds admit nearly parallel $G_2$ structures [6, 1]. In this paper, we consider three of these nearly parallel $G_2$ structures on a 7-dimensional 3-Sasakian manifold given in [1]. We deform the fundamental 3-forms of the manifold by three characteristic vector fields of the Sasakian structure separately. Then we determine how classes of $G_2$ structures change.

1. Introduction

A $G_2$ structure on a 7-dimensional Riemannian manifold $(M, g)$ is a reduction of the structure group of the frame bundle of $M$ from $SO(7)$ to $G_2$. The Riemannian manifolds with $G_2$ structures are classified by Fernández and Gray. There are 16 classes with different defining relations [5]. An equivalent characterization of each class was done by Cabrera by using $d\varphi$ and $d^*\varphi$ in [4].

There are several ways of obtaining new $G_2$ structures from a fixed $G_2$ structure. Consider the space of 3-forms on $M$, denoted by $\Lambda^3 M$. If $M$ has $G_2$ structure $\varphi$, then this space may be written as $\Lambda^3 M = \Lambda^3_1 \oplus \Lambda^3_7 \oplus \Lambda^3_7$, where

$$\Lambda^3_1 = \{ t\varphi | t \in \mathbb{R} \},$$
$$\Lambda^3_7 = \{ *(\beta \wedge \varphi) | \beta \in \Lambda^1 M \} = \{ \omega \wedge \varphi | \omega \in \Gamma(TM) \},$$
$$\Lambda^3_27 = \{ \gamma \in \Lambda^3 M | \gamma \wedge \varphi = 0, \gamma \wedge *\varphi = 0 \}$$

and $\Lambda^k_1$ denotes a $k$-dimensional $G_2$-irreducible subspace of $\Lambda^1 M$ and $\Gamma(TM)$ is the set of smooth vector fields on $M$. One way of constructing a new $G_2$ structure is to add an element of a subspace $\Lambda^k_1$ to the fundamental 3-form $\varphi$ of the manifold. Adding an element of $\Lambda^k_1$ to $\varphi$ means conformally deforming the 3-form. Conformal deformations were studied by Fernández and Gray. It was observed how conformally changing the fundamental 3-form changes the class the manifold belongs to. In
addition conformally invariant classes were determined [5]. Adding an element of $\Lambda^3_\varphi \simeq \{ \omega \ast \varphi \mid \omega \in \Gamma(TM) \}$ gives a new $G_2$ structure on $M$ with the fundamental 3-form $\tilde{\varphi} = \varphi + \omega \ast \varphi$ for each vector field $\omega$ [7]. These deformations were studied in [7]. The new metric $\tilde{g}$ and the new Hodge-star operator in terms of old ones were obtained as:

$$\tilde{g}(u, v) = (1 + g(\omega, \omega))^{-2/3} (g(u, v) + g(u \times \omega, v \times \omega)),$$

where $\times$ is the cross product associated to the first $G_2$ structure and $u, v$ are any vector fields,

$$\tilde{*}\alpha = (1 + g(\omega, \omega))^{-1/3} (\ast \alpha + (\omega, \omega) \ast \alpha))$$

where $\alpha$ is a k-form. Using this formula $\tilde{*}\varphi$ was written as:

$$\tilde{*}\varphi = (1 + g(\omega, \omega))^{-1/3} (\ast \varphi + (\omega, \varphi) + \omega \ast \omega \ast \varphi).$$

It was not studied, however, how deforming $\varphi$ by an element of $\Lambda^3_\varphi$ changes the class the manifold belongs to. In this paper our aim is to investigate how the class the manifold belongs to changes after deforming the fundamental 3-form by a vector field on specific examples.

2. Preliminaries

Let $(M, g)$ be a 7-dimensional Riemannian manifold with $G_2$ structure $\varphi$. It is known that for each $p \in M$, $\nabla \varphi$ belongs to the space

$$W = \{ \alpha \in T^*_p M \otimes \Lambda^2 T^*_p M \mid \alpha(x, y \wedge z \wedge (y \times z)) = 0 \text{ for all } x, y, z \in T^*_p M \}. $$

The space $W$ can be decomposed as $W = W_1 \oplus W_2 \oplus W_3 \oplus W_4$ where $W_i$ are $G_2$ irreducible subspaces [5]. A $G_2$ structure is said to be of type $\mathcal{P}$, $\mathcal{W}_1$, $\mathcal{W}_2$, $\mathcal{W}_3$, $\mathcal{W}_4$ or $\mathcal{W}$, if the covariant derivative $\nabla \varphi$ lies in $\{ 0 \}$. $W_i$, $W_i \oplus W_j$, $W_i \oplus W_j \oplus W_k$ or $W$, respectively, for $i, j, k = 1, 2, 3, 4$ [4]. Characterization of Cabrera is written in Table 1 above. Note that $\ast d\varphi \wedge \varphi = - \ast d \varphi \wedge \ast \varphi \ast \varphi$, $\alpha = -\frac{1}{4} \ast (\ast d\varphi \wedge \varphi)$.

| $\mathcal{P}$ | $d\varphi = 0$ and $d \ast \varphi = 0$ |
| $\mathcal{W}_1$ | $d\varphi = k \ast \varphi$ and $d \ast \varphi = 0$ |
| $\mathcal{W}_2$ | $d\varphi = 0$ |
| $\mathcal{W}_3$ | $d \ast \varphi = 0$ and $d \varphi \wedge \varphi = 0$ |
| $\mathcal{W}_4$ | $d\varphi = \alpha \wedge \varphi$ and $d \ast \varphi = \beta \wedge \ast \varphi$ |
| $\mathcal{W}_1 \oplus \mathcal{W}_2$ | $d\varphi = k \ast \varphi$ and $d \ast \varphi \wedge \ast \varphi = 0$ |
| $\mathcal{W}_1 \oplus \mathcal{W}_3$ | $d \ast \varphi = 0$ |
| $\mathcal{W}_2 \oplus \mathcal{W}_3$ | $d \varphi \wedge \varphi = 0$ and $d \ast \varphi \wedge \varphi = 0$ |
| $\mathcal{W}_1 \oplus \mathcal{W}_4$ | $d\varphi = \alpha \wedge \varphi + f \ast \varphi$ and $d \ast \varphi = \beta \wedge \ast \varphi$ |
| $\mathcal{W}_2 \oplus \mathcal{W}_4$ | $d\varphi = \alpha \wedge \varphi$ |
| $\mathcal{W}_3 \oplus \mathcal{W}_4$ | $d \ast \varphi = 0$ or $d \ast \varphi \wedge \ast \varphi = 0$ |
| $\mathcal{W}_1 \oplus \mathcal{W}_2 \oplus \mathcal{W}_3$ | $d \varphi = \beta \wedge \ast \varphi$ |
| $\mathcal{W}$ | $d \varphi \wedge \varphi = 0$ |

Table 1. Defining relations for classes of $G_2$ structures
\[ \beta = -\frac{1}{3} \ast (d\varphi \wedge \varphi) \text{ and } f = \frac{1}{3} \ast (\varphi \wedge d\varphi) \] [4]. Following definitions and properties can be found in [2, 3].

**Definition 2.1.** Let \( (M, g) \) be a Riemannian manifold with Levi-Civita covariant derivative \( \nabla \) of \( g \). \( (M, g) \) is called a Sasakian manifold if there exists a Killing vector field of unit length with the property that the endomorphism defined by \( \Phi := \nabla \xi \) satisfies

\[ (\nabla_x \Phi)(y) = g(\xi, y)x - g(x, y)\xi \]

for all vector fields \( x, y \).

The triple \( (\xi, \eta, \Phi) \) where \( \eta \) is the metric dual of \( \xi \) is called a Sasakian structure on \( (M, g) \). The Killing vector field \( \xi \) and the one-form \( \eta \) are respectively called the characteristic vector field and the characteristic one-form of the Sasakian structure.

Any Sasakian manifold \( (M, g) \) with the Sasakian structure \( (\xi, \eta, \Phi) \) satisfies the following relations:

\[
\begin{align*}
\Phi \circ \Phi(y) &= -y + \eta(y)\xi, \\
\Phi(\xi) &= 0, \quad \eta(\Phi(y)) = 0, \\
g(x, \Phi(y)) + g(\Phi(x), y) &= 0, \\
g(\Phi(y), \Phi(x)) &= g(y, x) - \eta(y)\eta(x), \\
d\eta(x, y) &= 2g(\Phi(x), y)
\end{align*}
\]

for all vector fields \( x, y \).

**Definition 2.2.** Let \( (M, g) \) be a Riemannian manifold. \( (M, g) \) is called a 3-Sasakian manifold if there are three Sasakian structures \( (\xi_i, \eta_i, \Phi_i) \) for \( i = 1, 2, 3 \) such that \( g(\xi_i, \xi_j) = \delta_{ij}, [\xi_1, \xi_2] = 2\xi_3, [\xi_2, \xi_3] = 2\xi_1 \) and \( [\xi_3, \xi_1] = 2\xi_2 \).

Each 3-Sasakian manifold has the properties below:

\[
\begin{align*}
\eta_i(\xi_j) &= \delta_{ij}, \\
\Phi_i(\xi_j) &= -\varepsilon_{ijk}\xi_k, \\
\Phi_i \circ \Phi_j - \xi_i \otimes \eta_j &= -\varepsilon_{ijk}\Phi_k - \delta_{ij}Id.
\end{align*}
\]

Let \( (M, g) \) be a 7-dimensional 3-Sasakian manifold with Sasakian structures \( (\xi_i, \eta_i, \Phi_i) \) for \( i = 1, 2, 3 \). The tangent bundle \( TM \) can be written as \( TM = T^v + T^h \), where \( T^v \) is the vertical subbundle spanned by \( \{\xi_1, \xi_2, \xi_3\} \) and the horizontal subbundle \( T^h \) is the orthogonal complement of \( T^v \) [1]. For topological reasons, we assume that \( M \) is compact and simply-connected. The structure group of \( (M, g) \) is \( SU(2) \subset G_2 \subset SO(7) \). In [1], a locally orthonormal frame \( \{e_1, \cdots, e_7\} \) such that \( e_1 = \xi_1, e_2 = \xi_2 \) and \( e_3 = \xi_3 \) was given together with endomorphisms \( \Phi_i \) acting on \( T^h \) by the following matrices:

\[
\Phi_1 := \begin{bmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{bmatrix}, \quad \Phi_2 := \begin{bmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}, \quad \Phi_3 := \begin{bmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}.
\]

The corresponding orthonormal coframe is denoted by \( \{\eta_1, \cdots, \eta_7\} \). According to this frame, the exterior derivatives of characteristic 1-forms are computed as:

\[
d\eta_1 = -2(\eta_{24} + \eta_{45} + \eta_{57}), \quad d\eta_2 = 2(\eta_{13} - \eta_{46} + \eta_{57}), \quad d\eta_3 = -2(\eta_{12} + \eta_{47} + \eta_{56}).
\]
The following three nearly parallel $G_2$ structures (i.e. $d\varphi_i = -4 \ast \varphi_i$) on 7-dimensional 3-Sasakian manifolds are expressed in [1]:

$$\varphi_1 = \frac{1}{2} \eta_1 \wedge d\eta_1 - \frac{1}{2} \eta_2 \wedge d\eta_2 - \frac{1}{2} \eta_3 \wedge d\eta_3,$$

$$\varphi_2 = -\frac{1}{2} \eta_1 \wedge d\eta_1 + \frac{1}{2} \eta_2 \wedge d\eta_2 - \frac{1}{2} \eta_3 \wedge d\eta_3,$$

$$\varphi_3 = -\frac{1}{2} \eta_1 \wedge d\eta_1 - \frac{1}{2} \eta_2 \wedge d\eta_2 + \frac{1}{2} \eta_3 \wedge d\eta_3.$$

3. **Deformations of one of the nearly parallel $G_2$ structures by characteristic vector fields**

Let $(M, g)$ be a 7-dimensional 3-Sasakian manifold equipped with Sasakian structures $(\xi_i, \eta_i, \phi_i)_{i=1,2,3}$, having the nearly parallel $G_2$ structure $\varphi_1$ given in the previous section. On an open subset of $M$, we choose the local orthonormal frame given in [1] with the properties mentioned in the previous section. We denote the corresponding coframe via the Riemannian metric by $\{\eta_1, \cdots, \eta_7\}$. Now we deform $\varphi_1$ by three characteristic vector fields. We begin by deforming $\varphi_1$ by $\xi_1$. We write the new deformed 3-form $\tilde{\varphi}_1 = \varphi_1 + \xi_1 \ast \varphi_1$ by

$$\tilde{\varphi}_1 = \frac{1}{2} \eta_1 \wedge d\eta_1 - \frac{1}{2} \eta_2 \wedge d\eta_2 - \frac{1}{2} \eta_3 \wedge d\eta_3 + \frac{1}{2} \eta_3 \wedge d\eta_2 - \frac{1}{2} \eta_2 \wedge d\eta_3,$$

locally

$$\tilde{\varphi}_1 = \eta_{123} - \eta_{145} - \eta_{167} + \eta_{246} - \eta_{257} + \eta_{347} + \eta_{356} + \eta_{357} - \eta_{346} + \eta_{256} + \eta_{247}.$$

Now we investigate the class the Riemannian manifold $\tilde{M} := (M, \tilde{g})$ belongs to. We compute

$$d\tilde{\varphi}_1 = d\varphi_1 - d(\ast(\eta_1 \wedge \varphi_1)).$$

Since $\ast(\eta_1 \wedge \varphi_1) = -\frac{1}{2} \eta_3 \wedge d\eta_2 + \frac{1}{2} \eta_2 \wedge d\eta_3$, $d(\ast(\eta_1 \wedge \varphi_1)) = 0$ and thus $d\tilde{\varphi}_1 = d\varphi_1$.

This may locally be written as

$$d\tilde{\varphi}_1 = 4\{ -\eta_{1247} - \eta_{1256} - \eta_{1346} - \eta_{1357} + \eta_{2345} + \eta_{2367} - \eta_{4567} \}.$$

Since $d\tilde{\varphi}_1 \neq 0$, the defining relations of $G_2$ structures imply $\tilde{M} \notin \mathcal{P}$ and $\tilde{M} \notin \mathcal{W}_2$.

Assume $\alpha$ is a 1-form on $M$ such that $d\tilde{\varphi}_1 = \alpha \wedge \tilde{\varphi}_1$. This 1-form may be locally written as $\alpha = \sum \alpha_i \eta_i$ for smooth functions $\alpha_i$. Then

$$\alpha \wedge \tilde{\varphi}_1 = -\alpha_1 \eta_{1234} - \alpha_5 \eta_{1235} - \alpha_6 \eta_{1236} - \alpha_7 \eta_{1237} + \alpha_2 \eta_{1245} + \alpha_1 \eta_{1246} + \alpha_1 \eta_{1247} + \alpha_3 \eta_{1256} - \alpha_1 \eta_{1257} + \alpha_2 \eta_{1267} + \alpha_3 \eta_{1345} - \alpha_1 \eta_{1346} + \alpha_1 \eta_{1347} + \alpha_1 \eta_{1356} + \alpha_1 \eta_{1357} + \alpha_3 \eta_{1367} + \alpha_6 \eta_{1456} + \alpha_7 \eta_{1457} + \alpha_4 \eta_{1467} + \alpha_5 \eta_{1567} - (\alpha_2 + \alpha_3) \eta_{2346} + (\alpha_2 - \alpha_3) \eta_{2347} + (\alpha_2 - \alpha_3) \eta_{2356} + (\alpha_2 + \alpha_3) \eta_{2357} + (\alpha_5 - \alpha_4) \eta_{2456} + (\alpha_4 + \alpha_5) \eta_{2457} + (\alpha_6 - \alpha_7) \eta_{2467} - (\alpha_6 + \alpha_7) \eta_{2567} - (\alpha_4 + \alpha_5) \eta_{3456} + (\alpha_4 - \alpha_5) \eta_{3457} + (\alpha_6 + \alpha_7) \eta_{3467} + (\alpha_6 - \alpha_7) \eta_{3567}.$$

The coefficient of $\eta_{2345}$ in $d\tilde{\varphi}_1$ is 4, while in $\alpha \wedge \tilde{\varphi}_1$ there does not exist an $\eta_{2345}$ term. Thus there is no such 1-form $\alpha$ with the property $d\tilde{\varphi}_1 = \alpha \wedge \tilde{\varphi}_1$ even locally. Hence $\tilde{M} \notin \mathcal{W}_4$ and $\tilde{M} \notin \mathcal{W}_2 \oplus \mathcal{W}_4$. 
We apply the Hodge-star formula for a k-form given in the introduction to the 3-form $\tilde{\varphi}_1$ and also the identities given in the appendix of [7] to obtain $\tilde{\tilde{\varphi}}_1$:

\[
\tilde{\tilde{\varphi}}_1 = 2^{-1/3}\left\{\ast\varphi_1 + \ast(\xi_1 \ast \varphi_1) + \xi_1 \ast (\tilde{\xi}_1 \ast \varphi_1)\right\}
\]

Thus there is no such non-zero constant. This eliminates the classes nonzero.

Comparing the coefficients of $\eta$ and $\tilde{\varphi}$ and $\tilde{\varphi}$ gives $-4 = 2^{-1/3}k$ and $-4 = 2^{3/4}k$, a contradiction. Thus there is no such non-zero constant. This eliminates the classes $W_1$ and $W_1 \oplus W_2$.

The exterior derivative of $\tilde{\tilde{\varphi}}_1$ is

\[
d^{\tilde{\tilde{\varphi}}}_1 = 2^{-1/3}\left\{\frac{1}{2} \eta_2 \wedge d\eta_1 \wedge d\eta_2 - \frac{1}{2} \eta_1 \wedge d\eta_2 \wedge d\eta_2
\]

\[
\frac{1}{2} \eta_1 \wedge d\eta_1 \wedge d\eta_3 - \frac{1}{2} \eta_1 \wedge d\eta_3 \wedge d\eta_3\right\}.
\]

This is locally equivalent to $d^{\tilde{\tilde{\varphi}}}_1 = 2^{5/3}\eta_1\eta_346 + 2^{5/3}\eta_1\eta_346 - 2^{8/3}\eta_1\eta_34567$ which is nonzero.

Let $\beta$ be a smooth 1-form on $M$ satisfying $d^{\tilde{\tilde{\varphi}}}_1 = \beta \wedge \tilde{\tilde{\varphi}}_1$. Then $\beta$ may be locally written as $\beta = \sum \beta_i \eta_i$ for smooth functions $\beta_i$.

\[
2^{1/3} \beta \wedge \tilde{\tilde{\varphi}}_1 = -2 \beta_1 \eta_1\eta_2 - (\beta_2 - \beta_3)\eta_1\eta_2 + (\beta_2 - \beta_3)\eta_1\eta_3 + (\beta_2 - \beta_3)\eta_1\eta_3 + (\beta_3 - \beta_4)\eta_1\eta_3 - 2\beta_1 \eta_1\eta_2
\]

Comparing the coefficients of $\eta_1\eta_2$ and $\eta_1\eta_3$ in $2^{1/3}d^{\tilde{\tilde{\varphi}}}_1$ and $2^{1/3} \beta \wedge \tilde{\tilde{\varphi}}_1$ respectively gives $\beta_1 = -2$, $\beta_3 = -4$. This is a contradiction. Hence there does not exist a 1-form $\beta$ satisfying the defining relation $d^{\tilde{\tilde{\varphi}}}_1 = \beta \wedge \tilde{\tilde{\varphi}}_1$. This means that $M \notin W_1 \oplus W_4$.

If we take $\alpha = -2\eta_1$ and $f = -2^{1/3}$, then direct computation yields the equality $d\tilde{\varphi}_1 = \alpha \wedge \tilde{\varphi}_1 + f \tilde{\tilde{\varphi}}_1$. Hence $M \in W_1 \oplus W_2 \oplus W_3$.

Next we deform the 3-form $\varphi_1$ by $\xi_2$. The new deformed 3-form $\tilde{\varphi}_1 = \varphi_1 + \xi_2 \ast \varphi_1$ is

\[
\tilde{\varphi}_1 = \frac{1}{2} \eta_1 \wedge d\eta_1 - \frac{1}{2} \eta_2 \wedge d\eta_2 - \frac{1}{2} \eta_3 \wedge d\eta_3 + \frac{1}{2} \eta_3 \wedge d\eta_1 + \frac{1}{2} \eta_1 \wedge d\eta_3
\]

which is locally

\[
\tilde{\varphi}_1 = \eta_1\eta_4 - \eta_1\eta_5 + \eta_2\eta_4 - \eta_2\eta_5 + \eta_3\eta_6 + \eta_4\eta_6 - \eta_5\eta_6
\]

We compute $\tilde{\tilde{\varphi}}_1$:

\[
\tilde{\tilde{\varphi}}_1 = 2^{-1/3}\left\{\ast\varphi_1 + \ast(\xi_2 \ast \varphi_1) + \xi_2 \ast (\tilde{\xi}_2 \ast \varphi_1)\right\}
\]

\[
= 2^{-1/3}\left\{\ast\varphi - \eta_2 \wedge \varphi_1 + * (\eta_2 \wedge \varphi_1)\right\}.
\]
Note that
\[ *\varphi_1 = -\frac{1}{8} d\eta_1 \wedge d\eta_1 + \frac{1}{8} d\eta_2 \wedge d\eta_2 + \frac{1}{8} d\eta_3 \wedge d\eta_3, \]
and
\[ *\{(\eta_1 \wedge *\varphi_1) = \frac{1}{8} d\eta_2 \wedge d\eta_2. \]
This yields \( d^*\tilde{\varphi}_1 = 0 \). The new \( G_2 \) structure can be an element of \( \mathcal{P}, W_1, W_3 \) or \( W_1 \oplus W_3 \). The exterior derivative of \( \tilde{\varphi}_1 \) is
\[ d\tilde{\varphi}_1 = \frac{1}{2} d\eta_1 \wedge d\eta_1 - \frac{1}{2} d\eta_2 \wedge d\eta_2 - \frac{1}{2} d\eta_3 \wedge d\eta_3 + d\eta_1 \wedge d\eta_1. \]
This may locally be written as
\[ d\tilde{\varphi}_1 = 4\{(\eta_1-1245)-\eta_1247+\eta_1256-\eta_11346-\eta_1357+\eta_2345+\eta_2347+\eta_2356+\eta_2367-\eta_4567\}. \]
Since \( d\tilde{\varphi}_1 \neq 0 \), the class \( \mathcal{P} \) is eliminated.

Assume \( d\tilde{\varphi}_1=k\tilde{\varphi}_1 \) for a non-zero constant \( k \). Since
\[ \tilde{\varphi}_1 = 2^{-1/3}\{\eta_1245+\eta_1247+\eta_1256-\eta_11267-2\eta_1346+2\eta_1357+\eta_2345-\eta_2347-\eta_2356-\eta_2367+2\eta_4567\}, \]
comparing the coefficients of \( \eta_1245 \) and \( \eta_1346 \) in \( d\tilde{\varphi}_1 \) and \( k\tilde{\varphi}_1 \) respectively gives
\[ 4 = -2^{-1/3}k \]
and \( 4 = -2^{2/3}k \), a contradiction. Thus there is no such non-zero constant. This excludes the class \( W_1 \).

Computed locally, \( d\tilde{\varphi}_1 \wedge \tilde{\varphi}_1 = -44\eta_1234567 \neq 0 \) and hence \( M \notin W_3 \). Therefore \( M \) is in the class \( W_1 \oplus W_3 \) with the new \( G_2 \) structure \( \tilde{\varphi}_1 \).

Finally we deform the 3-form \( \varphi_1 \) by \( \xi_3 \). The new deformed 3-form \( \tilde{\varphi}_1 = \varphi_1 + \xi_3 \wedge \varphi_1 \) is
\[ \tilde{\varphi}_1 = \frac{1}{2} \eta_1 \wedge d\eta_1 - \frac{1}{2} \eta_2 \wedge d\eta_2 - \frac{1}{2} \eta_3 \wedge d\eta_3 - \frac{1}{2} \eta_2 \wedge d\eta_1 - \frac{1}{2} \eta_1 \wedge d\eta_2 \]
which is locally
\[ \tilde{\varphi}_1 = \eta_123 - \eta_145 - \eta_167 + \eta_246 + \eta_1257 + \eta_347 + \eta_356 + \eta_267 + \eta_245 - \eta_157 + \eta_146. \]

We compute \( \tilde{\varphi}_1 \):
\[ \tilde{\varphi}_1 = 2^{-1/3}\{\varphi_1 + (\xi_3 \wedge \varphi_1) + \xi_3 \wedge (\xi_3 \wedge \varphi_1)\} \]
\[ = 2^{-1/3}\{\varphi_1 + \eta_3 \wedge \varphi_1 + (\eta_3 \wedge (\eta_3 \wedge \varphi_1))\}. \]
Since
\[ *\varphi_1 = -\frac{1}{8} d\eta_1 \wedge d\eta_1 + \frac{1}{8} d\eta_2 \wedge d\eta_2 + \frac{1}{8} d\eta_3 \wedge d\eta_3, \]
and
\[ *(\eta_3 \wedge *\varphi_1) = \frac{1}{8} d\eta_3 \wedge d\eta_3, \]
we have $d\tilde{\varphi}_1 = 0$. The new $G_2$ structure can be an element of $\mathcal{P}$, $\mathcal{W}_1$, $\mathcal{W}_2$ or $\mathcal{W}_1 \oplus \mathcal{W}_3$. The exterior derivative of $\tilde{\varphi}_1$ is

$$d\tilde{\varphi}_1 = \frac{1}{2} d\eta_1 \wedge d\eta_1 - \frac{1}{2} d\eta_2 \wedge d\eta_2 - \frac{1}{2} d\eta_3 \wedge d\eta_3 - d\eta_1 \wedge d\eta_2.$$ 

This may be locally written as

$$d\tilde{\varphi}_1 = 4\{-\eta_{1247} - \eta_{1256} + \eta_{1345} + \eta_{1346} - \eta_{1357} + \eta_{1367} + \eta_{2345} - \eta_{2346} + \eta_{2357} + \eta_{2367} - \eta_{4567}\}.$$

Since $d\tilde{\varphi}_1 \neq 0$, the class $\mathcal{P}$ is eliminated.

Assume $d\tilde{\varphi}_1 = k\tilde{\varphi}_1$ for a non-zero constant $k$. Since

$$\tilde{\varphi}_1 = 2^{-1/3}\{-\frac{1}{2} d\eta_1 \wedge d\eta_1 + \frac{1}{2} d\eta_2 \wedge d\eta_2 + \frac{1}{8} d\eta_1 \wedge d\eta_3 + \frac{1}{3} d\eta_1 \wedge d\eta_2 + \frac{1}{8} d\eta_3 \wedge d\eta_2 \}$$

or, locally

$$\tilde{\varphi}_1 = 2^{-1/3}\{-\frac{1}{2} d\eta_1 \wedge d\eta_1 + \frac{1}{2} d\eta_2 \wedge d\eta_2 + \frac{1}{8} d\eta_1 \wedge d\eta_3 + \frac{1}{3} d\eta_1 \wedge d\eta_2 + \frac{1}{8} d\eta_3 \wedge d\eta_2 \},$$

comparing the coefficients of $\eta_{1247}$ and $\eta_{1357}$ in $d\tilde{\varphi}_1$ and $k\tilde{\varphi}_1$ respectively gives $4 = -2^{2/3}k$ and $4 = -2^{-1/3}k$, a contradiction. Thus there is no such non-zero constant. This excludes the class $\mathcal{W}_1$.

Computed locally, $d\tilde{\varphi}_1 \wedge \tilde{\varphi}_1 = -44 \eta_{1234567} \neq 0$ and hence $M \notin \mathcal{W}_3$. Therefore $M$ is in the class $\mathcal{W}_1 \oplus \mathcal{W}_3$ with the new $G_2$ structure $\tilde{\varphi}_1$.

To sum up, if we take the nearly parallel structure $\varphi_1$ in a 7-dimensional 3-Sasakian manifold and deform this structure by the characteristic vector fields $\xi_1$, $\xi_2$, $\xi_3$ of the Sasakian structure, we get new $G_2$ structures of types $\mathcal{W}_1 \oplus \mathcal{W}_2 \oplus \mathcal{W}_4$, $\mathcal{W}_1 \oplus \mathcal{W}_3$ and $\mathcal{W}_1 \oplus \mathcal{W}_3$ respectively. Similarly, if we deform $\varphi_2$ (respectively $\varphi_3$) by $\xi_1$ and $\xi_3$ (resp. by $\xi_1$ and $\xi_2$), we get $G_2$ structures of type $\mathcal{W}_1 \oplus \mathcal{W}_3$. If we deform $\varphi_2$ (resp. $\varphi_3$) by $\xi_2$ (resp. $\xi_3$), we get $G_2$ structures of type $\mathcal{W}_1 \oplus \mathcal{W}_2 \oplus \mathcal{W}_4$ such that $d\tilde{\varphi}_1 = \alpha \wedge \tilde{\varphi}_1 + f\tilde{\varphi}_1$ hold for $\alpha = -2\eta_i$ and $f = -2^{2/3}$, for $i = 2, 3$.

**References**


Department of Mathematics, Anadolu University, 26470 Eskişehir, Turkey

E-mail address: nozdemir@anadolu.edu.tr

Department of Mathematics, Anadolu University, 26470 Eskişehir, Turkey

E-mail address: sirins@anadolu.edu.tr