MEAN ERGODIC TYPE THEOREMS

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Abstract. Let $T$ be a bounded linear operator on a Banach space $X$. Replacing the Cesàro matrix by a regular matrix $A = (a_{nj})$ Cohen studied a mean ergodic theorem. In the present paper we extend his result by taking a sequence of infinite matrices $A = (A^{(i)})$ that contains both convergence and almost convergence. This result also yields an $A$-ergodic decomposition. When $T$ is power bounded we give a characterization for $T$ to be $A$-ergodic.

1. Introduction

Let $X$ be a Banach space and $T$ be a bounded linear operator on $X$ into itself. By $M_n(T)$ we denote the Cesàro averages of $T$ given by $M_n(T) := \frac{1}{n+1} \sum_{j=0}^{n} T^j$.

An operator $T \in B(X)$ is called mean ergodic, respectively uniformly ergodic, if $\{M_n(T)\}$ is strongly, respectively uniformly, convergent in $B(X)$. Cohen [3] considered the problem of determining a class of regular matrices $A = (a_{nj})$ for which

$$L_n := \sum_{j=1}^{\infty} a_{nj} T^j$$

converges strongly to an element invariant under $T$. It is the case when $\{L_n x : n \in \mathbb{N}\}$ is weakly compact and $\lim_{n \to \infty} \sum_{j=k}^{\infty} \sum_{i=k}^{\infty} |a_{n,j+1} - a_{nj}| = 0$ uniformly in $n$ (see also [11]).

Observe that Cohen’s result is an extension of the mean ergodic theorems due to von Neumann [10], F. Riesz [8] and K. Yosida [12].

In the present paper, replacing the matrix $A = (a_{nj})$ by a sequence of infinite matrices $(A^{(i)}) = (a_{nj}^{(i)})$ we study results in an analogy of Cohen.

Now, we give some basic notations concerning the sequence of infinite matrices. Let $A$ be a sequence of infinite matrices $(A^{(i)}) = (a_{nj}^{(i)})$. Given a sequence $x = (x_j)$
we write

\[ A_n^{(i)} x = \sum_{j=1}^{\infty} a_{nj}^{(i)} x_j \]

if it exists for each \( n \) and \( i \geq 0 \). The sequence \( (x_j) \) is said to be summable to the value \( s \) by the method \( A \) if

\[ A_n^{(i)} x \to s \quad (n \to \infty, \text{ uniformly in } i). \]  

(1)

If (1) holds, we write \( x \to s(A) \).

The method \( A \) is called conservative if \( x \to s \) implies \( x \to s'(A) \). If \( A \) is conservative and \( s = s' \), we say that \( A \) is regular. We now recall a theorem which characterizes the regularity of the sequences of infinite matrices.

**Theorem 1** ([2, 9]). Let \( A \) be the sequence of infinite matrices \( (A^{(i)}) = (a_{nj}^{(i)}) \). Then, \( A \) is regular if and only if the following conditions hold:

(1) \( \sum_{j} |a_{nj}^{(i)}| < \infty, \) (for all \( n \), for all \( i \)),

(2) There exists an integer \( m \) such that \( \sup_{i \geq 0, n \geq m} \sum_{j} |a_{nj}^{(i)}| < \infty \),

(3) for all \( j \), \( \lim_{n} a_{nj}^{(i)} = 0 \), (uniformly in \( i \)),

(4) \( \lim_{n} \sum_{j} a_{nj}^{(i)} = 1 \), (uniformly in \( i \)).

In addition, we write

\[ \|A\| := \sup_{n,i} \sum_{j} |a_{nj}^{(i)}| \]  

(2)

and \( \|A\| < \infty \) to mean that, there exists a constant \( M \) such that \( \sum_{j} |a_{nj}^{(i)}| \leq M \), (for all \( n \), for all \( i \)) and the series \( \sum_{j} a_{nj}^{(i)} \) converges uniformly in \( i \) for each \( n \).

Throughout the paper we assume that the sequence of matrices \( (A^{(i)}) = (a_{nj}^{(i)}) \) satisfies the following conditions:

(i) \( A \) is regular,

(ii) \( \|A\| < \infty \),

(iii) \( \lim \sup_{k} i,n \sum_{j=k}^{\infty} |a_{nj+1}^{(i)} - a_{nj}^{(i)}| = 0 \).

2. Main results

In this section, using a sequence of infinite matrices we give a theorem analogous to one of Cohen [3].

We now present a lemma which will be used in the proof of the main theorem.
Lemma 2. Let $T$ and $A_n^{(i)}$ be bounded linear operators on a Banach space $X$ into itself such that $TA_n^{(i)} = A_n^{(i)}T$ for all $n$ and $i$. If
\[ \lim_{n \to \infty} A_n^{(i)}(x - Tx) = 0, \quad \text{(uniformly in } i), \] (3)
and
\[ A_n^{(i)}x \to x_0(w), \quad (n \to \infty, \text{ uniformly in } i), \]
then $Tx_0 = x_0$, where $(w)$ indicates the weak convergence.

Proof. By $X'$ we denote the dual space of $X$. Let $f \in X'$. Then, by weak convergence (uniformly in $i$) of $(A_n^{(i)}x)$ we have
\[ \lim_{n} \sup_{i} f(A_n^{(i)}x - x_0) = 0. \] (4)
Since $T$ is a linear and continuous operator on $X$, we also have
\[ \lim_{n} \sup_{i} f(TA_n^{(i)}x - Tx_0) = 0. \] (5)
It follows from (3) and the fact that $f \in X'$,
\[ \lim_{n} \sup_{i} f(A_n^{(i)}x - A_n^{(i)}Tx) = 0. \] (6)
Using the commutativity $TA_n^{(i)} = A_n^{(i)}T$ for each $n$ and $i$, one may write
\[ f(x_0 - Tx_0) = f(x_0 - A_n^{(i)}x) + f(A_n^{(i)}x - A_n^{(i)}Tx) + f(TA_n^{(i)}x - Tx_0). \] (7)
Applying the operator $\lim sup_{n} \sup_{i}$ to both sides of (7) we get that
\[ \left| \lim_{n} \sup_{i} f(x_0 - Tx_0) \right| \leq \left| \lim_{n} \sup_{i} f(x_0 - A_n^{(i)}x) \right| + \left| \lim_{n} \sup_{i} f(A_n^{(i)}x - A_n^{(i)}Tx) \right| \\
+ \left| \lim_{n} \sup_{i} f(TA_n^{(i)}x - Tx_0) \right|. \] (8)
Then by (4), (5), (6) and (8), we conclude that $f(x_0 - Tx_0) = 0$ for all $f \in X'$. This implies that $Tx_0 = x_0$.

We now present the main result of the paper.

Theorem 3. Let $X$ be a Banach space and $T : X \to X$ be a bounded linear operator. Suppose that there exists an $H > 0$ such that $\|T\| \leq H$ for all $j \in \mathbb{N}$. Suppose that the sequence of infinite matrices $(A^{(i)}) = (a_{nj}^{(i)})$ satisfies the conditions (i)-(iii) and define $A_n^{(i)}x = \sum_{j=1}^{\infty} a_{nj}^{(i)}T^jx$. Assume that there exists a subsequence
\[ \{A_{n_p}^{(i)}x\} \subset \{A_n^{(i)}x\} \] such that
\[ \lim_{p} \sup_{i} A_{n_p}^{(i)}x = x_0(w), \] (9)
where \( x_0 \in X \). Then, \( Tx_0 = x_0 \) and \( \lim_{n \to \infty} A_n^{(i)} x = x_0 \) (uniformly in \( i \)). Denote by 
\( P \) the strong limit in \( B(X) \) of \( \{A_n^{(i)} x\} \). Then it is the projection onto the space \( N(I - T) \) of \( T \)-fixed points corresponding to the ergodic decomposition 
\( X = R(I - T) \oplus N(I - T) \) and \( P = P^2 = TP = PT \).

**Proof.** From the hypothesis there exists an \( H > 0 \) such that \( \|T^j\| \leq H \) for all \( j \in \mathbb{N} \). Since \( \|A\| < \infty \), for \( x \in X \) we have

\[
\left\| A_n^{(i)} x \right\| = \left\| \sum_{j=1}^{\infty} a_{n,j}^{(i)} T^j x \right\| \leq H \left\| x \right\| \sum_{j=1}^{\infty} |a_{n,j}^{(i)} | < H \left\| x \right\| \| A\|.
\]

(10)

Since \( X \) is complete, each \( \{A_n^{(i)} x\} \) is defined on \( X \). By taking supremum over \( \|x\| = 1 \) in both sides of (10), we get, for all \( n \) and \( i \), that

\[
\| A_n^{(i)} \| \leq H \| A\|.
\]

(11)

Also we have

\[
TA_n^{(i)} x = \sum_{j=1}^{\infty} a_{n,j}^{(i)} T^{j+1} x = A_n^{(i)} T x.
\]

(12)

By the hypothesis, we have for any \( \varepsilon > 0 \) that there exists a \( k_0 = k_0(\varepsilon) \in \mathbb{N} \) such that for all \( k \geq k_0 \)

\[
\sup_{i,n,j} |a_{n,j}^{(i)} - a_{n,j+1}^{(i)}| < \varepsilon.
\]

Hence, we get, for each \( x \in X \), that

\[
\left\| A_n^{(i)} (x - Tx) \right\| = \left\| a_{n,1}^{(i)} T x + \sum_{j=1}^{k_0} (a_{n,j+1}^{(i)} - a_{n,j}^{(i)}) T^j+1 x \right\|
\]

\[
\leq H \left\| x \right\| \left( \sup_{i} |a_{n,1}^{(i)}| + \sum_{j=1}^{k_0} |a_{n,j+1}^{(i)} - a_{n,j}^{(i)}| + \sup_{i,n,j \geq k_0} |a_{n,j}^{(i)} - a_{n,j+1}^{(i)}| \right)
\]

\[
\leq H \left\| x \right\| \left( 2 \sup_{i} \sum_{j=1}^{k_0} |a_{n,j}^{(i)}| + \varepsilon \right).
\]

Then, for \( n > n_\varepsilon \) we also have \( \sup_{i,j} |a_{n,j}^{(i)}| < \varepsilon \) which yields

\[
\left\| A_n^{(i)} (x - Tx) \right\| \leq H \left\| x \right\| 3\varepsilon.
\]
This implies
\[ \lim_{n \to \infty} A_n(i)(x - Tx) = 0, \quad \text{(uniformly in } i). \quad (13) \]
Furthermore, from (9), (12) and (13), the conditions of Lemma 2 are satisfied. Thus, one can get \( Tx_0 = x_0 \).

Now, we consider the linear subspace \( X_0 \) spanned by \( x - Tx \) for \( x \in X \). We will show that \( x_0 - x \in X_0 \). To achieve this, we follow the idea given by Cohen [3]. Assume that \( x_0 - x \notin X_0 \). Then, one can easily see that there exists an \( f \in X' \) such that
\[ f(u) = 0, \quad u \in X_0; \quad f(x - x_0) = 1. \]
Since \( T^k x - T^{k+1} x \in X_0 \) for \( k = 0, 1, 2, \ldots \), we have \( f(T^k x - T^{k+1} x) = 0 \). Then, it is easy to show that \( f(x - T^j x) = 0 \). So we obtain
\[ f(x) = f(T^j x), \quad j = 1, 2, \ldots \quad (14) \]
Moreover, from (11) and (13), it follows that
\[ \lim_{n \to \infty} A_n(i) u = 0, \quad u \in X_0. \quad (15) \]
Since \( f \in X' \), one can get by (14) that
\[ f(A_n(i)x) = \sum_{j=1}^{\infty} a_{nj}^{(i)} f(T^j x) = \left( \sum_{j=1}^{\infty} a_{nj}^{(i)} \right) f(x) \]
which yields
\[ \lim_{n \to \infty} f(A_n(i)x) = f(x). \quad (16) \]
By (9) and (16) we obtain
\[ 0 = \lim_{p \to \infty} f(A_{np}(i)x - x_0) = \lim_{p \to \infty} (f(A_{np}(i)x) - f(x_0)) = f(x) - f(x_0) = f(x - x_0). \]
This is a contradiction. Then we necessarily have \( x_0 - x \in X_0 \). Since \( Tx_0 = x_0 \) we have \( T^j x_0 = x_0 \) for \( j = 1, 2, \ldots \). Hence we have
\[ A_n(i)x_0 = \sum_{j=1}^{\infty} a_{nj}^{(i)} T^j x_0 = \left( \sum_{j=1}^{\infty} a_{nj}^{(i)} \right) x_0 \quad (17) \]
from which we immediately get
\[ \lim_{n \to \infty} A_n(i)x_0 = x_0. \quad (18) \]
Since \( x = x_0 + (x - x_0) \), we get from (15) and (18) that
\[ \lim_{n \to \infty} A_n(i)x = x_0, \]
which proves the first claim.

We can write \( x = x_0 + (x - x_0) \) such that \( x_0 \in N(I - T) \) and \( (x - x_0) \in R(I - T) \).
$R(I - T)$. Now let $\varepsilon > 0$ and let $z \in \overline{R(I - T)} \cap N(I - T)$. Following [4] we then have $\|z - (u - Tu)\| < \varepsilon/(3H\|A\|)$ for $u \in X$. Hence

$$\left\|A_n^{(i)}(z - (u - Tu))\right\| < \left\|\sum_{j=1}^{\infty} a_{nj}^{(i)}T^j\right\| \left\|z - (u - Tu)\right\| < \frac{\varepsilon}{3}. \quad (19)$$

Since $z \in \overline{R(I - T)} \cap N(I - T)$, we observe that

$$A_n^{(i)}z = \sum_{j=1}^{\infty} a_{nj}^{(i)}T^j z = \sum_{j=1}^{\infty} a_{nj}^{(i)}z \quad (20)$$

from which we get

$$\lim_{n} \sup_{i} A_n^{(i)}z = z. \quad (21)$$

By (15), (19) and (21), we conclude that

$$\lim_{n} \sup_{i} A_n^{(i)}x = x_0.$$ 

Remark 4. If we define the sequence of matrices $(A^{(i)}) = (a_{nj}^{(i)})$ by

$$a_{nj}^{(i)} = \begin{cases} \frac{1}{n + 1}, & i \leq j \leq i + n, \\ 0, & \text{otherwise} \end{cases}$$

then $A$ reduces to almost convergence method of Lorentz [6]. Observe that $(a_{nj}^{(i)})$ defined as above satisfies the conditions (i)-(iii) imposed in Section 1. Some results concerning the almost convergence of the sequence of operators may be found in [1] and [7].
Given a sequence \( A \) of matrices \( \{A(i)\} = \{a(i)_{kj}\} \), if the limit of \( \{A_n^{(i)}x\} \) exists then we call the operator \( T \) an \( \mathcal{A} \)-ergodic operator. Motivated by that of Proposition 2.2 in [3] we have the following

**Theorem 5.** Let \( X \) be a Banach space, \( T \) be a bounded linear operator on \( X \) into itself. Assume that there exists an \( H > 0 \) such that \( \|T^j\| \leq H \) for all \( j \in \mathbb{N} \). Let \( \{A(i)\} = \{a(i)_{kj}\} \) be a sequence of infinite matrices satisfying the conditions (i)-(iii). Then, the operator \( T \) is \( \mathcal{A} \)-ergodic if and only if \( (I - T)(I - T)X = (I - T)X \).

**Proof.** Let the operator \( T \) be \( \mathcal{A} \)-ergodic. Then, by Theorem 3 we have
\[
X = \overline{R(I - T)} \oplus N(I - T).
\]
The necessity is proved by applying the operator \( (I - T) \).

Assume that \( (I - T)(I - T)X = (I - T)X \). We have, for \( x \in N(I - T) \), that
\[
A_n^{(i)}x = \sum_{j=1}^{\infty} a(n)_j T^j x = \sum_{j=1}^{\infty} a(n)_j x.
\]
Hence, we get
\[
\|A_n^{(i)}x - x\| \rightarrow 0, \quad (n \rightarrow \infty, \text{uniformly in } i).
\]  
(22)

Now, let \( x \in \overline{R(I - T)} \). Hence, there exists \( x_k \in R(I - T) \) so that \( x_k \rightarrow x \). One can get
\[
\left\| A_n^{(i)}x \right\| \leq \left\| A_n^{(i)}x_k \right\| + \left\| A_n^{(i)}(x_k - x) \right\|.
\]
If we choose \( k \) in order to make \( \|x_k - x\| \) sufficiently small, we find that \( \|A_n(x_k - x)\| \) is also sufficiently small (no matter what \( n \) may be) because of the fact that \( \mathcal{A} \) satisfies (ii) and \( T \) is power bounded. Combining this with (15), we observe, for \( x \in \overline{R(I - T)} \), that
\[
\|A_n^{(i)}x\| \rightarrow 0, \quad (n \rightarrow \infty, \text{uniformly in } i).
\]  
(23)

Thus, by (22) and (23) the sequence \( \{A_n^{(i)}\} \) is strongly convergent on \( \overline{R(I - T)} \oplus N(I - T) \). Since \( (I - T)(I - T)X = (I - T)X \), for \( y \in X \) there exists \( z \in \overline{R(I - T)} \) such that \( (I - T)z = (I - T)y \). We then get \( h = y - z \in N(I - T) \). Since we have \( y = h + z \) such that \( h \in N(I - T) \) and \( z \in \overline{R(I - T)} \), the proof is completed. \( \Box \)

**References**


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