Some generalized numerical radius inequalities involving Kwong functions

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Abstract
We prove several numerical radius inequalities involving positive semidefinite matrices via the Hadamard product and Kwong functions. Among other inequalities, it is shown that if $X$ is an arbitrary $n \times n$ matrix and $A, B$ are positive semidefinite, then
\[
\omega(H_{f,g}(A)) \leq k \omega(AX +XA),
\]
which is equivalent to
\[
\omega(H_{f,g}(A, B) \pm H_{f,g}(B, A)) \leq k' \{\omega((A + B)X + X(A + B)) + \omega((A - B)X - X(A - B))\},
\]
where $f$ and $g$ are two continuous functions on $(0, \infty)$ such that $h(t) = \frac{f(t)}{g(t)}$ is Kwong, $k = \max \left\{\frac{f(\lambda)}{g(\lambda)} : \lambda \in \sigma(A)\right\} \text{ and } k' = \max \left\{\frac{f(\lambda)g(\lambda)}{\lambda} : \lambda \in \sigma(A) \cup \sigma(B)\right\}.$

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1. Introduction
Let $\mathcal{M}_n$ be the $C^*$-algebra of all $n \times n$ complex matrices and $\langle \cdot, \cdot \rangle$ be the standard scalar product in $\mathbb{C}^n$. A capital letter means an $n \times n$ matrix in $\mathcal{M}_n$. For Hermitian matrices $A$ and $B$, we write $A \geq 0$ if $A$ is positive semidefinite, $A > 0$ if $A$ is positive definite, and $A \geq B$ if $A - B \geq 0$. The numerical radius of $A \in \mathcal{M}_n$ is defined by
\[
\omega(A) := \sup \{|\langle Ax, x \rangle| : x \in \mathbb{C}^n, \|x\| = 1\}.
\]
It is well known that $\omega(\cdot)$ defines a norm on $\mathcal{M}_n$, which is equivalent to the usual operator norm $\|\cdot\|$. In fact, for any $A \in \mathcal{M}_n$, $\frac{1}{2}\|A\| \leq \omega(A) \leq \|A\|$: see [11]. For further information about numerical radius inequalities we refer the reader to [4, 11, 15, 16] and references therein. We use the notation $J$ for the matrix whose entries are equal to one.

The Hadamard product (Schur product) of two matrices $A, B \in \mathcal{M}_n$ is the matrix $A \circ B$ whose $(i,j)$ entry is $a_{ij}b_{ij} \ (1 \leq i, j \leq n)$. The Schur multiplier operator $S_A$ on $\mathcal{M}_n$ is

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defined by $S_A(X) = A \circ X$ ($X \in \mathcal{M}_n$). The induced norm of $S_A$ with respect to the numerical radius norm will be denoted by

$$\|S_A\|_\omega = \sup_{X \neq 0} \frac{\omega(S_A(X))}{\omega(X)} = \sup_{X \neq 0} \frac{\omega(A \circ X)}{\omega(X)}.$$  

A continuous real valued function $f$ on an interval $(a, b) \subseteq \mathbb{R}$ is called operator monotone if $A \leq B$ implies $f(A) \leq f(B)$ for all Hermitian matrices $A, B \in \mathcal{M}_n$ with spectra in $(a, b)$. Following [3], a continuous real-valued function $f$ defined on an interval $(a, b)$ with $a > 0$ is called a Kwong function if the matrix $K_f = \left( \frac{f(\lambda_i) + f(\lambda_j)}{\lambda_i + \lambda_j} \right)_{i,j=1,2,\ldots,n}$ is positive semidefinite for any (distinct) $\lambda_1, \ldots, \lambda_n$ in $(a, b)$. It is easy to see that if $f$ is a nonzero Kwong function, then $f$ is positive and $\frac{1}{t}$ is Kwong. Kwong [13] showed that the set of all Kwong functions on $(0, \infty)$ is a closed cone and includes all non-negative operator monotone functions on $(0, \infty)$. Also, Audenaert [3] gave a characterization of Kwong functions by showing that, for given $0 \leq a < b$, a function $f$ on an interval $(a, b)$ is Kwong if and only if the function $g(x) = \sqrt{x} f(\sqrt{x})$ is operator monotone on $(a^2, b^2)$.

The Heinz means are defined as $H_{\nu}(a, b) = \frac{a^{\frac{1}{2}} - \nu b^{\frac{1}{2}} + \nu a^{\frac{1}{2}}}{2}$ for $a > 0$ and $0 \leq \nu \leq 1$. These interesting means interpolate between the geometric and arithmetic means. In fact, the Heinz inequalities assert that $\sqrt{ab} \leq H_{\nu}(a, b) \leq \frac{a + b}{2}$, where $a, b > 0$ and $0 \leq \nu \leq 1$. There have been obtained several Heinz type inequalities for Hilbert space operators and matrices; see [5] and references therein.

For two continuous functions $f$ and $g$ on $(0, \infty)$ we denote

$$H_{f,g}(A, B) = f(A)Xg(B) + g(A)Xf(B)$$

and

$$H_{f,g}(A) = f(A)Xg(A) + g(A)Xf(A),$$

where $A, B, X \in \mathcal{M}_n$ such that $A, B$ are positive semidefinite. In particular, for $f(t) = t^\alpha$ and $g(t) = t^1 - \alpha$ ($\alpha \in [0, 1]$), we get $H_{\alpha}(A, B) = A^\alpha XB^{1-\alpha} + A^{1-\alpha}XB^\alpha$ and $H_{\alpha}(A) = A^\alpha XA^{1-\alpha} + A^{1-\alpha}XA^\alpha$. A norm $\|\cdot\|$ on $\mathcal{M}_n$ is called unitarily invariant if $\|AVU\| = \|A\|$ for all $A \in \mathcal{M}_n$ and all unitary matrices $U, V \in \mathcal{M}_n$. Let $A, B, X \in \mathcal{M}_n$ such that $A$ and $B$ are positive semidefinite. In [14] it was conjectured a general norm inequality of the Heinz inequality $\|H_{f,g}(A, B)\| \leq \|AX + XB\||$, where $f$ and $g$ are two continuous functions on $(0, \infty)$ such that $f(t)g(t) \leq t$ and the function $h(t) = \frac{f(t)}{g(t)}$ is Kwong.

In particular, if $f(t) = t^\alpha$ and $g(t) = t^{1-\alpha}$ ($\alpha \in [0, 1]$), then we state a Heinz type inequality $\|H_{\alpha}(A, B)\| \leq \|AX + XB\||$, where $A, B, X \in \mathcal{M}_n$ such that $A, B$ are positive semidefinite. For further information, we refer the reader to [5, 6] and references therein.

The numerical radius $\omega(\cdot)$ is a weakly unitarily invariant norm on $\mathcal{M}_n$, that is $\omega(U^*AU) = \omega(A)$ for every $A \in \mathcal{M}_n$ and every unitary $U \in \mathcal{M}_n$. In [1], the authors proved a Heinz type inequality for the numerical radius as follows

$$\omega(H_{\alpha}(A)) \leq \omega(AX + XA), \quad (1.1)$$

in which $A, X \in \mathcal{M}_n$ such that $A$ is positive semidefinite. They also showed that the inequality $\omega(H_{\alpha}(A, B)) \leq \omega(AX + XB)$ is not true in general.

Our research aim is to show some numerical radius inequalities via the Hadamard product and Kwong functions. By using some ideas of [8, 10] and [14], we obtain some extensions and generalizations of inequality (1.1), which are generalizations of a Hienz type inequality for the numerical radius. For instance, we prove if $A, X \in \mathcal{M}_n$ such that $A$ is positive semidefinite, then

$$\omega(H_{f,g}(A)) \leq k\omega(AX + XA),$$

where $k$ is a constant.
where \( f \) and \( g \) are two continuous functions on \((0, \infty)\) such that \( \frac{f(t)}{g(t)} \) is Kwong and \( k = \max \left\{ \frac{f(\lambda)g(\lambda)}{\lambda} : \lambda \in \sigma(A) \right\} \).

2. Main results

For our purpose we need the following lemmas.

**Lemma 2.1** ([18, Theorem 3.4]). (Spectral Decomposition) Let \( A \in \mathbb{M}_n \) with eigenvalues \( \lambda_1, \ldots, \lambda_n \). Then \( A \) is normal if and only if there exists a unitary matrix \( U \) such that

\[
U^*AU = \text{diag}(\lambda_1, \ldots, \lambda_n).
\]

In particular, \( A \) is positive definite if and only if the \( \lambda_j \) (\( 1 \leq j \leq n \)) are positive.

**Lemma 2.2** ([2, Corollary 4]). Let \( A = [a_{ij}] \in \mathbb{M}_n \) be positive semidefinite. Then

\[
\|S_A\|_\omega = \max_i a_{ii}.
\]

**Lemma 2.3.** ([12]). Let \( X, Y \in \mathbb{M}_n \). Then

(i) \( \omega\left(\begin{bmatrix} X & 0 \\ 0 & Y \end{bmatrix}\right) = \max\{\omega(X), \omega(Y)\} \);

(ii) \( \max(\omega(X+Y)\omega(X-Y)\omega(X+Y)\omega(X-Y)) \leq \omega\left(\begin{bmatrix} 0 & X \\ Y & 0 \end{bmatrix}\right) \leq \frac{\omega(X+Y)+\omega(X-Y)}{2} \).

Now, we are in position to demonstrate the first result of this section by using some ideas of [8, 10, 14].

**Theorem 2.4.** Let \( A, B \in \mathbb{M}_n \) be positive semidefinite, \( X \in \mathbb{M}_n \), and let \( f, g \) be two continuous functions on \((0, \infty)\) such that \( h(t) = \frac{f(t)}{g(t)} \) is Kwong. Then

\[
\omega(H_{f,g}(A)) \leq k \omega(AX+XA),
\]

where \( k = \max_{\lambda \in \sigma(A)} \left\{ \frac{f(\lambda)g(\lambda)}{\lambda} \right\} \).

Moreover, inequality (2.1) is equivalent to the inequality

\[
\omega(H_{f,g}(A, B) \pm H_{f,g}(B, A)) \leq k' \left\{ \omega((A+B)X+X(A+B)) + \omega((A-B)X-X(A-B)) \right\},
\]

where \( k' = \max_{\lambda \in \sigma(A) \cup \sigma(B)} \left\{ \frac{f(\lambda)g(\lambda)}{\lambda} \right\} \).

**Proof.** Assume that \( A \) is positive definite. Since the numerical radius is weakly unitarily invariant, we may assume that \( A \) is diagonal matrix with positive eigenvalues \( \lambda_1, \cdots, \lambda_n \).

It follows from \( \frac{f}{g} \) is a Kwong function that

\[
Z = [z_{ij}] = \Lambda \left( \frac{f(\lambda_i)g^{-1}(\lambda_j) + f(\lambda_j)g^{-1}(\lambda_i)}{\lambda_i + \lambda_j} \right)_{i,j=1,\ldots,n}
\]

is positive semidefinite, where \( \Lambda = \text{diag}(g(\lambda_1), \cdots, g(\lambda_n)) \). It follows from Lemma 2.2 that

\[
\|S_Z\|_\omega = \max_i z_{ii} = \max_i \frac{f(\lambda_i)g(\lambda_i)}{\lambda_i} \leq k
\]

or equivalently, \( \omega(Z_{SA}) \leq k \) (\( 0 \neq X \in \mathbb{M}_n \)). If we put \( E = [\frac{1}{\lambda_i+\lambda_j}] \) and \( F = [f(\lambda_i)g(\lambda_j) + f(\lambda_j)g(\lambda_i)] \in \mathbb{M}_n \), then

\[
\omega(E \circ F \circ X) = \omega(Z \circ X) \leq k \omega(X) \quad (X \in \mathbb{M}_n).
\]

Let the matrix \( C \) be the entrywise inverse of \( E \), i.e., \( C \circ E = J \). Thus

\[
\omega(F \circ X) \leq k \omega(C \circ X) \quad (X \in \mathbb{M}_n)
\]
or equivalently
\[
\omega(H_{f,g}(A)) = \omega(f(A)Xg(A) + g(A)Xf(A)) \leq k \omega(AX +XA). \tag{2.3}
\]

Now, if \( A \) is positive semidefinite, we may assume that \( A = \begin{bmatrix} A_1 & 0 \\ 0 & 0 \end{bmatrix} \), where \( A_1 \in M_k \) \((k < n)\) is a positive definite matrix. Let \( X = \begin{bmatrix} X_1 & X_2 \\ X_3 & X_4 \end{bmatrix} \), where \( X_1 \in M_k \) and \( X_4 \in M_{n-k} \). Then we have
\[
\omega(H_{f,g}(A)) = \omega\left( \begin{bmatrix} f(A_1)X_1g(A_1) + g(A_1)X_1f(A_1) & 0 \\ 0 & 0 \end{bmatrix} \right)
\leq k \omega\left( \begin{bmatrix} A_1X_1 + X_1A_1 & 0 \\ 0 & 0 \end{bmatrix} \right) \tag{by Lemma 2.3(i)}
= k \omega(A_1X_1 + X_1A_1) \tag{by (2.3)}
\leq k \omega(AX +XA) \tag{by Lemma 2.3(i)}.
\]

Hence, we reach inequality (2.1). Moreover, if we replace \( A \) and \( X \) by \( \begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix} \) and \( \begin{bmatrix} 0 & X \\ X & 0 \end{bmatrix} \) in inequality (2.1), respectively, then
\[
\omega\left( \begin{bmatrix} 0 & H_{f,g}(A, B) \\ H_{f,g}(B, A) & 0 \end{bmatrix} \right) \leq k' \omega\left( \begin{bmatrix} 0 & AX + XB \\ XA + BX & 0 \end{bmatrix} \right),
\]
whence
\[
\max \left\{ \omega(H_{f,g}(A, B) \pm H_{f,g}(B, A)) \right\}
\leq 2 \omega\left( \begin{bmatrix} 0 & f(A)Xg(B) + g(A)Xf(B) \\ g(B)Xf(A) + f(B)Xg(A) & 0 \end{bmatrix} \right) \tag{by Lemma 2.3(ii)}
\leq 2k' \omega\left( \begin{bmatrix} 0 & AX + XB \\ XA + BX & 0 \end{bmatrix} \right) \tag{by (2.4)}
\leq k' \left( \omega(AX +XB +XA +BX) + \omega(AX +XB -XA -BX) \right) \tag{by Lemma 2.3(iii)}.
\]

Thus, we have inequality (2.2). Also, if we put \( B = A \) in inequality (2.2), then we reach inequality (2.1).

If we take \( f(t) = t^\alpha \) and \( g(t) = t^{1-\alpha} \) in Theorem 2.4 for each \( 0 \leq \alpha \leq 1 \), then we get the next result.

**Corollary 2.5** ([1, Theorem 2.4]). Let \( A, B \in M_n \) be positive semidefinite, \( X \in M_n \), and let \( 0 \leq \alpha \leq 1 \). Then
\[
\omega(H_\alpha(A)) \leq \omega(AX +XA). \tag{2.5}
\]

Moreover, inequality (2.5) is equivalent to the inequality
\[
\omega(H_\alpha(A, B) \pm H_\alpha(B, A)) \leq \omega((A + B)X + X(A + B)) + \omega((A - B)X - X(A - B)).
\]
Corollary 2.6. Let $A, B \in \mathcal{M}_n$ be positive semidefinite, $X \in \mathcal{M}_n$, and let $f$ be a non-negative operator monotone function on $[0, \infty)$ such that $f'(0) = \lim_{x \to 0^+} f'(x) < \infty$ and $f(0) = 0$. Then
\[
\omega(f(A)X + XF(A)) \leq f'(0) \omega(AX + XA).
\] (2.6)
Moreover, inequality (2.6) is equivalent to the inequality
\[
\omega(X(f(A) + f(B)) + (f(A) + f(B))X)
\leq f'(0) \left(\omega((A + B)X + X(A + B)) + \omega((A - B)X - X(A - B))\right).
\]

Proof. A function $g$ is non-negative operator increasing on $[0, \infty)$ if and only if $\frac{f(t)}{g(t)}$ is non-negative operator increasing on $[0, \infty)$; see [9]. Hence $\frac{f(t)}{g(t)}$ is operator increasing. Then $\frac{f(t)}{t}$ is decreasing. If $0 \leq x \leq t$, then $\frac{f(t)}{t} \leq \frac{f(x)}{x}$. Now, by taking $x \to 0^+$ we have $\frac{f(t)}{t} \leq f'(0)$. If we put $g(t) = 1 (t \in [0, \infty))$ in Theorem 2.4, it follows from $k = k' \leq f'(0)$ that we get the required result.

We first cite the following lemma due to Fujii et al. [10], which will be needed in the next theorem.

Lemma 2.7 ([10, Lemma 3.1]). Let $\lambda_1, \cdots, \lambda_n$ be any positive real numbers and $-2 < t \leq 2$. If $f$ and $g$ are two continuous functions on $(0, \infty)$ such that $\frac{f(t)}{g(t)}$ is Kwong, then the $n \times n$ matrix
\[
Y = \left(\frac{f(\lambda_i)g^{-1}(\lambda_j) + f(\lambda_j)g^{-1}(\lambda_i)}{\lambda_i^2 + t\lambda_i\lambda_j + \lambda_j^2}\right)_{i,j=1,\ldots,n}
\]
is positive semidefinite.

Theorem 2.8. Let $A, B \in \mathcal{M}_n$ be positive semidefinite, $X \in \mathcal{M}_n$, $f$, $g$ be two continuous functions on $(0, \infty)$ such that $\frac{f(t)}{g(t)}$ is Kwong, and let $-2 < t \leq 2$. Then
\[
\omega(A^{\frac{1}{2}}(H_{f,g}(A))A^{\frac{1}{2}}) \leq \frac{2k}{t + 2} \omega(A^2X + tAXA + XA^2),
\] (2.7)
where $k = \max_{\lambda \in \sigma(A)} \left\{ \frac{f(\lambda)g(\lambda)}{\lambda} \right\}$.
Moreover, inequality (2.7) is equivalent to the inequality
\[
\omega(A^{\frac{1}{2}}(H_{f,g}(A,B))B^{\frac{1}{2}}) \leq \frac{4k'}{t + 2} \omega(A^2X + tAXB + XB^2),
\] (2.8)
where $k' = \max_{\lambda \in \sigma(A) \cap \sigma(B)} \left\{ \frac{f(\lambda)g(\lambda)}{\lambda} \right\}$.

Proof. First, we show inequality (2.7). It is enough to show the inequality in the case $A$ is positive definite. Since the numerical radius is weakly unitarily invariant, we may assume that $A$ is diagonal matrix with positive eigenvalues $\lambda_1, \cdots, \lambda_n$. Let $\Sigma = \text{diag} \left(\lambda_1^2 g(\lambda_1), \cdots, \lambda_n^2 g(\lambda_n)\right)$. It follows from Lemma 2.7 that
\[
Z = [z_{ij}] = \Sigma \left(\frac{(t + 2) (f(\lambda_i)g^{-1}(\lambda_j) + f(\lambda_j)g^{-1}(\lambda_i))}{2(\lambda_i^2 + t\lambda_i\lambda_j + \lambda_j^2)}\right)_{i,j=1,\ldots,n}
\]
is positive semidefinite for $-2 < t \leq 2$. In addition, all diagonal entries of $Z$ are no more than $k$. Therefore,
\[
\|SZ\|_\omega = \max_i z_{ii} = \max_i \frac{f(\lambda_i)g(\lambda_i)}{\lambda_i} \leq k,
\]
whence \( \frac{\omega(Z \circ X)}{\omega(X)} \leq k \) \((0 \neq X \in \mathcal{M}_n)\). Now, let \( M = \left[ \frac{1}{\lambda_i + \lambda_j + \lambda_j^2} \right]_{i,j=1,\ldots,n} \) and
\[
P = \left[ \frac{1+2\lambda_j^2 f(\lambda_i)g(\lambda_j) + f(\lambda_j)g(\lambda_j)\lambda_i^2}{2} \right]_{i,j=1,\ldots,n}.
\]
Then
\[
\omega(M \circ P \circ X) = \omega(Z \circ X) \leq k \omega(X) \quad (0 \neq X \in \mathcal{M}_n).
\]
Let the matrix \( N \) be the entrywise inverse of \( M \), i.e., \( M \circ N = J \). Hence
\[
\omega(P \circ X) \leq k \omega(N \circ X) \quad (0 \neq X \in \mathcal{M}_n)
\]
or equivalently
\[
\omega(A^{\frac{1}{2}} (H_{f,g}(A)) A^{\frac{1}{2}}) \leq \frac{2k}{t+2} \omega(A^2 X + tAXA + XA^2),
\]
where \( X \in \mathcal{M}_n \), \(-2 < t \leq 2 \) and \( k = \max \left\{ \frac{f(\lambda)g(\lambda)}{\lambda} : \lambda \in \sigma(A) \right\} \). Hence we have inequality (2.7).

Now, if we replace \( A \) and \( X \) by \( \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix} \) and \( \begin{pmatrix} 0 & X \\ 0 & 0 \end{pmatrix} \) inequality (2.7), respectively, then
\[
\omega \left( \begin{bmatrix} 0 & A^{\frac{1}{2}} (H_{f,g}(A,B)) B^{\frac{1}{2}} \\ 0 & 0 \end{bmatrix} \right) \leq \frac{2k'}{t+2} \omega \left( \begin{bmatrix} 0 & A^2 X + tAXB + XB^2 \\ 0 & 0 \end{bmatrix} \right).
\]

Hence
\[
\frac{1}{2} \omega(A^{\frac{1}{2}} (H_{f,g}(A,B)) B^{\frac{1}{2}}) \leq \frac{2k'}{t+2} \omega \left( \begin{bmatrix} 0 & A^2 X + tAXB + XB^2 \\ 0 & 0 \end{bmatrix} \right)
\]
(by Lemma 2.3)
\[
\leq \frac{2k'}{t+2} \omega \left( \begin{bmatrix} 0 & A^2 X + tAXB + XB^2 \\ 0 & 0 \end{bmatrix} \right)
\]
(by Lemma 2.3).

Thus, we reach inequality (2.8). Also, if we put \( B = A \) in inequality (2.7), then we get inequality (2.8).

**Corollary 2.9.** Let \( A \in \mathcal{M}_n \) be positive semidefinite. If \( f \) is a positive operator monotone function on \((0, \infty)\), then
\[
\omega(A^{\frac{1}{2}} f(A)Xf(A)^{-1}A^{\frac{1}{2}} + A^{\frac{1}{2}} f(A)^{-1}Xf(A)A^{\frac{1}{2}}) \leq \frac{4}{t+2} \omega(A^2 X + tAXA + XA^2),
\]
where \( X \in \mathcal{M}_n \) and \(-2 < t \leq 2 \).

**Proof.** Since \( f \) positive operator monotone on \((0, \infty)\), then \( g(t) = \frac{f(t)}{t} \) is operator monotone on \((0, \infty)\) and also \( \frac{f(t)}{g(t)} = tf^2(t) \) is Kwong function [14]. So \( f \) and \( g \) satisfy the conditions of Theorem 2.8. Hence we have the desired inequality.

**Example 2.10.** The function \( f(t) = \log(1 + t) \) is operator monotone on \((0, \infty)\); see [9]. If we put \( g(t) = 1 \), then \( \frac{f(t)}{g(t)} = \log(1 + t) \) is Kwong [13]. Using Theorem 2.4 we have
\[
\omega(A^{\frac{1}{2}} (\log(I + A)X + X \log(I + A))A^{\frac{1}{2}}) \leq \frac{2}{t+2} \omega(A^2 X + tAXA + XA^2),
\]
where \( A, X \in \mathcal{M}_n \) such that \( A \) is positive semidefinite and \(-2 < t \leq 2 \).
Now, we infer the following lemma due to Zhan [17], which will be needed in the next theorem.

Lemma 2.11 ([17, Lemma 5]). Let $\lambda_1, \cdots, \lambda_n$ be any positive real numbers, $r \in [-1, 1]$ and $-2 < t \leq 2$. Then the $n \times n$ matrix

$$L = \left( \frac{\lambda_i + \lambda_j}{\lambda_i^2 + t \lambda_i \lambda_j + \lambda_j^2} \right)_{i,j=1,\cdots,n}$$

is positive semidefinite.

Now, we shall show the following result related to Zhan [17].

Proposition 2.12. Let $A, X \in M_n$ such that $A$ is positive semidefinite, $\beta > 0$ and $1 \leq 2r \leq 3$. Then

$$\omega(A^r X A^{2-r} + A^{2-r} X A^r) \leq \omega \left( 2(1 - 2\beta + 2\beta r_0) A X A + \frac{4\beta(1 - r_0)}{t + 2}(A^2 X + t A X A + X A^2) \right),$$

where $-2 < t \leq 2\beta - 2$ and $r_0 = \min\{\frac{1}{2} + |1 - r|, 1 - |1 - r|\}$.

Proof. Since the numerical radius is weakly unitarily invariant, we may assume that $A$ is diagonal matrix with positive eigenvalues $\lambda_1, \cdots, \lambda_n$. Since $1 \leq 2r \leq 3$, then $\frac{1}{2} \leq r \leq \frac{3}{2}$.

Let $t_0 = \frac{1 - 2\beta + 2\beta r_0}{1 - r_0}(t + 2) + t$. It follows from $-2 < t \leq 2\beta - 2$ and $\frac{1}{4} \leq 1 - r_0 \leq \frac{1}{4}$, that $t_0 > 0$ and $-2 < t_0 \leq 2$, where $t_0 = \frac{t}{2(1-r_0)} + \frac{1}{\beta(1-r_0)} - 2$. Hence, by using Lemma 2.11, the $n \times n$ matrix

$$W = [w_{ij}] = \frac{t + 2}{4\beta(1-r_0)} \Lambda^r \left( \frac{\lambda_i^{2-2r} + \lambda_j^{2-2r}}{\lambda_i^2 + t_0 \lambda_i \lambda_j + \lambda_j^2} \right)_{i,j=1,\cdots,n}$$

is positive semidefinite for $\frac{1}{2} \leq r \leq \frac{3}{2}$, where $\Lambda = \text{diag} (\lambda_1, \cdots, \lambda_n)$. Therefore,

$$\|S_W\|_\omega = \max_i w_{ii} = \max_i \frac{(t + 2) \lambda_i (2\lambda_i^{2-2r}) \lambda_i^r}{4\beta(1-r_0)(t_0 + 2) \lambda_i^2} = 1,$$

whence $\frac{\omega(W \circ X)}{\omega(X)} \leq 1$ ($0 \neq X \in M_n$). Now, let $O = \left[ \frac{\lambda_i^2 + t_0 \lambda_i \lambda_j + \lambda_j^2}{\lambda_i^2 + t \lambda_i \lambda_j + \lambda_j^2} \right]_{i,j=1,\cdots,n}$ and

$$M = \left[ \frac{1}{2(1 - 2\beta + 2\beta r_0) \lambda_i \lambda_j + \frac{4\beta(1 - r_0)}{t + 2}(\lambda_i^2 X + t \lambda_i \lambda_j + \lambda_j^2)} \right]_{i,j=1,\cdots,n}.$$

Then

$$\omega(O \circ M \circ X) = \omega(W \circ X) \leq \omega(X) \quad (0 \neq X \in M_n).$$

Let the matrix $N$ be the entrywise inverse of $M$, i.e., $M \circ N = J$. Hence

$$\omega(O \circ X) \leq \omega(N \circ X) \quad (0 \neq X \in M_n)$$

or equivalently

$$\omega(A^r X A^{2-r} + A^{2-r} X A^r) \leq \omega \left( 2(1 - 2\beta + 2\beta r_0) A X A + \frac{4\beta(1 - r_0)}{t + 2}(A^2 X + t A X A + X A^2) \right),$$

where $-2 < t \leq 2\beta - 2$ and $r_0 = \min\{\frac{1}{2} + |1 - r|, 1 - |1 - r|\}$. \hfill \Box

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References


