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Research Article

Some ordered function space topologies and ordered semi-uniformizability

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Abstract

In this work, we define some Čech based ordered function space topologies and we introduce ordered semi-uniformizability. Then we investigate ordered semi-uniformizability of the ordered function space topologies such as compact-open (interior) and point-open (interior) ordered topologies.

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1. Introduction and basic concepts

Closure operators are frequently used in mathematics and computer sciences. One of the well known closure operator is Čech closure operator. Čech closure operator was introduced by Čech in [4]. Čech closure spaces has a numerous applications. For instance, Slapal has used Čech closure operators in [19] to solve the digital image proccessing problems. The relations between Čech closure space and structural configuration of proteins were studied in [18]. The reader may find more details about Čech closure spaces in [1,2,5,6,11–13]. Ordered topological spaces were defined by Nachbin in [14]. According to [14], the triple $(X,\tau,\preceq)$ is called ordered topological space where τ is a topology and $\preceq$ is a partial order on X. Several authors have studied ordered topological spaces (see, [3,7–9]). Nailana, in [15], defined compact-open ordered and point-open ordered topology on the set of continuous and order preserving functions between the ordered topological spaces. Also, in [15], quasi-uniformizable ordered spaces were studied. According to [15] , let X be a set and $\mathcal{U}$ be a quasi-uniformity on X such that $(X,\tau(\mathcal{U}))$ is T_0, then $(X,\tau(\mathcal{U}),\cap\mathcal{U})$ is an ordered space. An ordered space $(X,\tau,\preceq)$ is said to be quasi-uniformizable if there exists a quasi-uniformity $\mathcal{U}$ on X such that $\tau=\tau(\mathcal{U}\vee\mathcal{U}^{-1})$ and $\preceq=\cap\mathcal{U}$ holds. Nachbin proved in [14] that an ordered topological space is quasi-uniformizable if and only if it is completely regular ordered space. This paper is organized as follows. In section 2, we will define compact-open (interior), point-open (interior) ordered topologies and we will study some properties of these function space topologies. Then in the last section motivated by [15], we will introduce semi-uniformizable ordered spaces and we will investigate the ordered semi-uniformizability of the ordered function space topologies which will be defined in the second section.

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Now, we will give some basic notions about closure and ordered spaces.

Following [17], a partially ordered set (poset) is a set X with a binary relation $\preceq$ which is reflexive, antisymmetric and transitive. If the relation is only reflexive and transitive then it is called preorder. In a preordered set $(X, \preceq)$, a subset A of X is called decreasing if $a \in A$, $b \in X$ and $b \preceq a$ implies $b \in A$ and called increasing if $a \in A$, $b \in X$ and $a \preceq b$ implies $b \in A$. The smallest decreasing set containing A is denoted by $\downarrow A$ and the smallest increasing set containing A is denoted by $\uparrow A$. If A is a decreasing (increasing) set, then the complement of A which will be denoted by A^c is increasing (decreasing) set. Let X and Y be ordered sets. A map $f : X \to Y$ is said to be order preserving (or, alternatively, increasing) function if $x \preceq y$ in X implies $f(x) \preceq f(y)$ in Y. Suppose X is any set and Y an ordered set. Then the set Y^X of all maps from X to Y may be ordered by pointwise order $\preceq_s$ as follows. Let $f, g \in Y^X$ and $f \preceq_s g$ if and only if $f(x) \preceq g(x)$ in Y for all $x \in X$. According to [4], let $\{\preceq_i\}_{i\in I}$ be a family of relations, then the relation consisting of all pairs, $((x_i)_{i\in I}, (y_i)_{i\in I})$, where $x_i \preceq_i y_i$ for all $i \in I$, is called the relational cartesian product of $\{\preceq_i\}_{i\in I}$ and it will be denoted by $\prod\limits_{i\in I} \preceq_i$.

Following [4], an operator c: $\mathcal{P}(X) \to \mathcal{P}(X)$ defined on the power set $\mathcal{P}(X)$ of a set X satisfying the axioms

(i) $c(\emptyset) = \emptyset$
(ii) $A \subseteq c(A)$ for all $A \subseteq X$
(iii) $c(A \cup B) = c(A) \cup c(B)$ for all $A, B \subseteq X$

is called a Čech closure operator and the pair (X, c) is called Čech closure space and we briefly call it closure space. If $cc(A) = c(A)$ holds, then it is called topological closure operator. Let consider the unit function c:$\mathcal{P}(X) \to \mathcal{P}(X)$. Then c is a topological closure operator and it is called discrete closure. A subset A of a closure space (X, c) is called closed if $c(A) = A$, open if its complement is closed. The interior operator $int_c : \mathcal{P}(X) \to \mathcal{P}(X)$ is defined by $int_c(A) = (c(A^c))^c$ for all $A \subseteq X$. A subset U of X is a neighbourhood of a point x in X if $x \in int_c(U)$ holds. The collection of all neighbourhoods of x is called neighbourhood sytem of x and will be denoted by $\mathcal{V}_c(x)$. A collection $\mathcal{W}$ of X is a local base of the neighbourhood system of a point x if and only if each $U \in \mathcal{W}$ is a neighbourhood of x and contains a $V \in \mathcal{W}$. The topological modification $\widehat{c}$ of c is the finest Kuratowski closure operator coarser than c. The corresponding topology $\tau(\widehat{c})$ consists of all open sets in (X, c). Also, $\tau(\widehat{c})$ is called the associated topology. According to [2], the collection $\mathcal{B} = \{int_c(B) \mid B \subseteq X\}$ is a base for a topology $\tau(\tilde{c})$ and its Kuratowski closure operator is denoted by $\tilde{c}$.

An ordered topological space is a nonempty set X endowed with a topology τ and a partial order $\preceq$ which will be denoted by $(X, \tau, \preceq)$. The Kuratowski closure operator of a topology τ will be denoted by cl_τ. In this work we mean by an ordered topological space, a triple $(X, \tau, \preceq)$ where τ is a topology and $\preceq$ is a preorder on X. If we endow X with a Čech closure operator c, and a preorder $\preceq$, then $(X, c, \preceq)$ is called an ordered Čech closure space. If the preorder on X is the discrete order defined as

$$a \preceq b \Leftrightarrow a = b,$$

then every ordered Čech closure space is an ordinary Čech closure space. According to [5] an ordered Čech closure space $(X, c, \preceq)$ is called

(i) upper T_1-ordered if for each pair of elements $a \npreceq b$ in X, there exists a decreasing neighbourhood U of b such that $a \notin U$.
(ii) lower T_1-ordered if for each pair of elements $a \npreceq b$ in X, there exists an increasing neighbourhood U of a such that $b \notin U$.
(iii) T_1-ordered if both (i) and (ii) are satisfied.
(iv) T_2-ordered if for each $a, b \in X$ such that $a \npreceq b$, there exists an increasing neighbourhood U of a and a decreasing neighbourhood V of b such that $U \cap V = \emptyset$.

(v) lower regular ordered if for each decreasing set $A \subseteq X$ and each element $x \notin c(A)$ there exist disjoint neighbourhoods U of x and V of A such that U is increasing and V is decreasing.

(vi) upper regular ordered if for each increasing set $A \subseteq X$ and each $x \notin c(A)$ there exist disjoint neighbourhoods U of x and V of A such that U is decreasing and V is increasing.

(vii) regular ordered if both (v) and (vi) are satisfied.

2. Some Čech based ordered function space topologies

In this section, we will construct some function space topologies based on Čech closure ordered spaces.

Following lemma is straightforward from the Lemma 15 in [15].

Lemma 2.1. *Let $(X, c_X, \preceq_X)$ and $(Y, c_Y, \preceq_Y)$ be ordered closure spaces and $A \subseteq X$, $B \subseteq Y$. Let $C^{\uparrow}(X,Y)$ denote the set of all continuous and order preserving functions from X to Y. Define $[A,B] = \{f \mid f \in C^{\uparrow}(X,Y) : f(A) \subseteq B\}$. Then $\uparrow [A,B]$=$[A, \uparrow B]$ and $\downarrow [A,B]$=$[A, \downarrow B]$ holds.*

In the light of [2], we give the following definition and our aim is to extend the concepts in [2] to the ordered case.

Definition 2.2. Let $(X, c_X, \preceq_X)$ and $(Y, c_Y, \preceq_Y)$ be two ordered closure spaces and $C^{\uparrow}(X,Y)$ denote the set of all continuous and order preserving functions from $(X, c_X, \preceq_X)$ to $(Y, c_Y, \preceq_Y)$. Define $[C,O] = \{f \in C^{\uparrow}(X,Y) | f(C) \subseteq O\}$ where C is a compact subset of X and O is an open set in Y. The family of the sets $[C,O]$ forms a subbase for a topology τ_{CO} on $C^{\uparrow}(X,Y)$ which will be called compact-open topology and endow $C^{\uparrow}(X,Y)$ with the pointwise order $\preceq_s$. Then the triple $(C^{\uparrow}(X,Y), \tau_{CO}, \preceq_s)$ is called compact-open ordered space. The sets $[C, int_{c_Y} G]$ where C is a compact subset of X and $G \subseteq Y$ form a subbase for a topology τ_{CI} on $C^{\uparrow}(X,Y)$ which will be called compact-interior topology and the triple $(C^{\uparrow}(X,Y), \tau_{CI}, \preceq_s)$ will be called compact-interior ordered space.

Theorem 2.3. *Let $(X, c_X, \preceq_X)$ and $(Y, c_Y, \preceq_Y)$ be two ordered closure spaces. If $(Y, c_Y, \preceq_Y)$ is T_2 -ordered space, then $(C^{\uparrow}(X,Y), \tau_{CI}, \preceq_s)$ is T_2-ordered.*

Proof. Let $(Y, c_Y, \preceq_Y)$ be a T_2-ordered space and $f, g \in C^{\uparrow}(X,Y)$ such that $f \npreceq_s g$. Then there exists a point x of X such that $f(x) \npreceq_Y g(x)$. Since $(Y, c_Y, \preceq_Y)$ is T_2-ordered space, there exist an increasing neighbourhood U of $f(x)$ and a decreasing neighbourhood V of $g(x)$ such that $U \cap V$=$\emptyset$. Clearly, $[x, int_{c_Y} U] \in \mathcal{V}_{\tau_{CI}}(f)$ and $[x, int_{c_Y} V] \in \mathcal{V}_{\tau_{CI}}(g)$. Since U is increasing and V is decreasing, by using Lemma 2.1, $[x,U]$ and $[x,V]$ are increasing and decreasing neighbourhoods of f and g, respectively. Since $U \cap V$=$\emptyset$, we get that $[x,U] \cap [x,V]$=$\emptyset$. Consequently, $(C^{\uparrow}(X,Y), \tau_{CI}, \preceq_s)$ is T_2-ordered space. □

We show in the next example that the converse of the above theorem is not necessarily true.

Example 2.4. Let $X = \{a,b\}$ and $Y = \{1,2,3\}$. Define the closure operators c_X and c_Y by the following:

$$c_X(\{a\}) = \{a,b\},\ c_X(\{b\}) = \{b\},\ c_X(\{a,b\}) = \{a,b\},\ c_X(\emptyset) = \emptyset$$

and

$$c_Y(\{1\}) = \{1,3\},\ c_Y(\{2\}) = \{2\},\ c_Y(\{3\}) = c_Y(\{2,3\}) = \{2,3\},$$
$$c_Y(\{1,2\}) = c_Y(\{1,3\}) = Y = c_Y(Y),\ c_Y(\emptyset) = \emptyset.$$

Morever, endow X and Y by the following preorders $\preceq_X = \{(a,a),(b,b),(a,b)\}$ and $\preceq_Y = \{(1,1),(2,2),(3,3),(1,3),(3,1)\}$. Then, $(X, c_X, \preceq_X)$ and $(Y, c_Y, \preceq_Y)$ are ordered closure spaces. Now, we construct the compact-interior topology on $C^{\uparrow}(X,Y)$. In Table 1, we give all continuous and order preserving functions from X to Y.

Table 1

x	$f_1(x)$	$f_2(x)$	$f_3(x)$	$f_4(x)$
a	1	2	3	1
b	1	2	3	3

Thus, $C^{\uparrow}(X,Y) = \{f_1, f_2, f_3, f_4\}$ and subbase elements of τ_{CI} are given by

$$[\{a\},\{1\}] = \{f_1, f_4\},\ [\{a\},\{1,3\}] = [\{b\},\{1,3\}] = [X,\{1,3\}] = \{f_1, f_3, f_4\},$$
$$[\{a\},\{2\}] = [\{b\},\{2\}] = [X,\{2\}] = \{f_2\},\ [\{b\},\{1\}] = [X,\{1\}] = \{f_1\},\ [\emptyset, Y] = C^{\uparrow}(X,Y)$$
$$\text{and } [X,\emptyset] = \emptyset$$

Therefore, we get that

$$\tau_{CI} = \{\{f_1\},\{f_2\},\{f_1,f_2\},\{f_1,f_4\},\{f_1,f_3,f_4\},\{f_1,f_2,f_4\},\emptyset, C^{\uparrow}(X,Y)\}$$

and $(C^{\uparrow}(X,Y), \tau_{CI}, \preceq_s)$ is T_2-ordered space. Indeed, for every two functions $f_i,\ f_j \in C^{\uparrow}(X,Y)$ such that $f_i \not\preceq_s f_j$ $(1 \le i,j \le 4)$, there exist an increasing neighbourhood U of f_i and a decreasing neighbourhood V of f_j such that $U \cap V = \emptyset$. Nevertheless, $(Y, c_Y, \preceq_Y)$ is not T_2-ordered, since $1 \not\preceq_Y 2$ and the intersection of every increasing neighbourhood of 1 and every decreasing neighbourhood of 2 is nonempty.

Theorem 2.5. *Let $(X, c_X, \preceq_X)$ and $(Y, c_Y, \preceq_Y)$ be ordered closure spaces. Then the following assertions are true:*

(i) *$(C^{\uparrow}(X,Y), \tau_{CO}, \preceq_s)$ is T_1-ordered $\Leftrightarrow$ $(Y, \widehat{c}_Y, \preceq_Y)$ is T_1-ordered.*

(ii) *$(C^{\uparrow}(X,Y), \tau_{CO}, \preceq_s)$ is T_2-ordered $\Leftrightarrow$ $(Y, \widehat{c}_Y, \preceq_Y)$ is T_2-ordered.*

Proof. (i) Let $(C^{\uparrow}(X,Y), \tau_{CO}, \preceq_s)$ be a T_1-ordered space and $y_1, y_2 \in Y$ such that $y_1 \not\preceq_Y y_2$. Let f and g be constant functions, $f(X) = \{y_1\}$ and $g(X) = \{y_2\}$. So, it is clear that $f \not\preceq_s g$. Since $(C^{\uparrow}(X,Y), \tau_{CO}, \preceq_s)$ is T_1-ordered , there exist an increasing neighbourhood U of f such that $g \notin U$ and a decreasing neighbourhood V of g such that $f \notin V$. Taking into account the definiton of compact-open topology we have $f \in \bigcap\limits_{i=1}^{n} [C_i, O_i] \subseteq U$ and $g \in \bigcap\limits_{i=1}^{n} [C'_i, O'_i] \subseteq V$ where C_i and C'_i are compact subsets of X, O_i and O'_i are open subsets of Y for all $i \in \{1,2,...,n\}$. Since U is increasing we have

$f \in \uparrow \bigcap\limits_{i=1}^{n} [C_i, O_i] \subseteq U$. Then $f \in \bigcap\limits_{i=1}^{n} [C_i, \uparrow O_i] \subseteq U$ and this implies that $\bigcap\limits_{i=1}^{n} \uparrow O_i$ is an increasing neighbourhood of y_1 and $y_2 \notin \bigcap\limits_{i=1}^{n} \uparrow O_i$. To show this let assume $y_2 \in \uparrow O_i$ for all $1 \le i \le n$. Thus, $g \in \bigcap\limits_{i=1}^{n} [C_i, \uparrow O_i]$ and this implies $g \in U$, which contradicts with $g \notin U$. Similarly, $\bigcap\limits_{i=1}^{n} \downarrow O'_i$ is a decreasing neighbourhood of y_2 and $y_1 \notin \bigcap\limits_{i=1}^{n} O'_i$. Conversely, let $f, g \in C^{\uparrow}(X,Y)$ and $f \not\preceq_s g$. Then there exists a point x of X such that $f(x) \not\preceq_Y g(x)$. Since $(Y, \widehat{c}_Y, \preceq_Y)$ is T_1-ordered, there exist an open increasing neighbourhood U of $f(x)$ such that $g(x) \notin U$ and open decreasing neighbourhood V of $g(x)$ such that $f(x) \notin V$. Therefore, $[x, U]$ is an increasing neighbourhood of f such that $g \notin [x, U]$ and $[x, V]$ is a decreasing neighbourhood of g such that $f \notin [x, V]$. Consequently, $(C^{\uparrow}(X,Y), \tau_{CO}, \preceq_s)$ is T_1-ordered space.

(ii) The statement is similar to (i) and we omitted it. □

Theorem 2.6. *Let $(X, c_X, \preceq_X)$ and $(Y, c_Y, \preceq_Y)$ be ordered closure spaces. Then the following assertions are true:*

(i) $(C^{\uparrow}(X,Y),\tau_{CI},\preceq_s)$ *is* T_1*-ordered* $\Leftrightarrow$ $(Y,\tilde{c}_Y,\preceq_Y)$ *is* T_1*-ordered.*
(ii) $(C^{\uparrow}(X,Y),\tau_{CI},\preceq_s)$ *is* T_2*-ordered* $\Leftrightarrow$ $(Y,\tilde{c}_Y,\preceq_Y)$ *is* T_2*-ordered.*

Proof. We will only prove (ii). The statement (i) can be shown by using the same arguments as for Theorem 2.5. Let $(C^{\uparrow}(X,Y),\tau_{CI},\preceq_s)$ be a T_2-ordered and $y_1,y_2\in Y$ such that $y_1 \not\preceq_Y y_2$. Consider the constant functions f and g such that $f(X)=\{y_1\}$ and $g(X)=\{y_2\}$. Then, $f\not\preceq_s g$. Since $(C^{\uparrow}(X,Y),\tau_{CI},\preceq_s)$ is T_2-ordered there exist an increasing subbasic element $[C,int_{c_Y}V]$ and a decreasing subbasic element $[C',int_{c_Y}V']$ such that $[C,int_{c_Y}V]\cap[C',int_{c_Y}V']=\emptyset$. Thus, $int_{c_Y}V$ and $int_{c_Y}V'$ are increasing and decreasing neighbourhoods of y_1 and y_2, respectively. Morever, $int_{c_Y}V\cap int_{c_Y}V'=\emptyset$. Consequently, $(Y,\tilde{c}_Y,\preceq_Y)$ is T_2-ordered. On the other hand, let $f,g\in C^{\uparrow}(X,Y)$ and $f\not\preceq_s g$. Then, there exists a point x of X such that $f(x)\not\preceq_Y g(x)$. Since $(Y,\tilde{c}_Y,\preceq_Y)$ is T_2-ordered there exist an increasing base element $int_{c_Y}U$ containing $f(x)$ and a decreasing base element $int_{c_Y}U'$ containing $g(x)$ such that $int_{c_Y}U\cap int_{c_Y}U'=\emptyset$. Clearly, $[x,int_{c_Y}U]$ and $[x,int_{c_Y}U']$ are increasing and decreasing neighbourhoods of f and g, respectively. Consequently, $(C^{\uparrow}(X,Y),\tau_{CI},\preceq_s)$ is T_2-ordered. $\square$

Proposition 2.7. *Let* $(X,c_X,\preceq_X)$ *and* $(Y,c_Y,\preceq_Y)$ *be ordered closure spaces.If* $(Y,\widehat{c}_Y,\preceq_Y)$ $((Y,\tilde{c}_Y,\preceq_Y))$ *is regular ordered space, then* $(C^{\uparrow}(X,Y),\tau_{CO},\preceq_s)$ $((C^{\uparrow}(X,Y),\tau_{CI},\preceq_s))$ *is regular ordered.*

Proof. Let $f\in C^{\uparrow}(X,Y)$ and $[C,V]$ be an increasing subbase element containing f. Then for each $x\in C$, $f(x)\in V$ such that V is increasing and there exists an increasing open set U_x such that $f(x)\in U_x\subseteq\widehat{c}_Y(U_x)\subseteq V$, since $(Y,\widehat{c}_Y,\preceq_Y)$ is lower regular. The family $\{U_x \mid x\in C\}$ is an interior cover of $f(C)$. Since $f(C)$ is compact, there is a finite subcover $\{U_{x_i}\mid i\in\{1,2,...,n\}\}$. The set $U=\bigcup_{i=1}^{n}U_{x_i}$ is open increasing and $f(C)\subseteq U\subseteq\widehat{c}_Y(U)\subseteq V$ holds. Furthermore, the open increasing set $[C,U]$ satisfies the inclusions $[C,U]\subseteq cl_{\tau_{CO}}[C,U]\subseteq[C,\widehat{c}_Y(U)]\subseteq[C,V]$. Thus, $(C^{\uparrow}(X,Y),\tau_{CO},\preceq_s)$ is lower regular ordered space and similarly, it can be shown that it is upper regular ordered space. Consequently, $(C^{\uparrow}(X,Y),\tau_{CO},\preceq_s)$ is regular ordered space. Second part can be shown similarly. $\square$

Definition 2.8. Let $(X,c_X,\preceq_X)$ and $(Y,c_Y,\preceq_Y)$ be ordered closure spaces. The sets $[\{x\},V]=\{f\in C^{\uparrow}(X,Y)\mid f(x)\in V\}$, where V is an open set in (Y,c_Y), is a subbase for a topology τ_{PO} on $C^{\uparrow}(X,Y)$ which will be called point-open topology and endow $C^{\uparrow}(X,Y)$ with the pointwise order $\preceq_s$. Then the triple $(C^{\uparrow}(X,Y),\tau_{PO},\preceq_s)$ is called point-open ordered space and the sets

$$[\{x\},int_{c_Y}V]=\{f\in C^{\uparrow}(X,Y)\mid f(x)\in int_{c_Y}V\}$$

form a subbase for a topology τ_{PI} which will be called point-interior topology and the triple $(C^{\uparrow}(X,Y),\tau_{PI},\preceq_s)$ is called point-interior ordered space.

Theorem 2.9. *Let* $(X,c_X,\preceq_X)$ *and* $(Y,c_Y,\preceq_Y)$ *be ordered closure spaces. Then the following assertions are true:*

(i) $(C^{\uparrow}(X,Y),\tau_{PO},\preceq_s)$ *is* T_1*-ordered* $\Leftrightarrow$ $(Y,\widehat{c}_Y,\preceq_Y)$ *is* T_1*-ordered.*
(ii) $(C^{\uparrow}(X,Y),\tau_{PO},\preceq_s)$ *is* T_2*-ordered* $\Leftrightarrow$ $(Y,\widehat{c}_Y,\preceq_Y)$ *is* T_2*-ordered.*

Proof. We will only prove the second assertion, (i) follows using the same arguments as for Theorem 2.5. Let $(C^{\uparrow}(X,Y),\tau_{PO},\preceq_s)$ be a T_2-ordered space and $y_1,y_2\in Y$ such that $y_1\not\preceq_Y y_2$. Define the constant functions f and g by $f(x)=y_1$ and $g(x)=y_2$ for all $x\in X$. It is clear that $f\not\preceq_s g$. Since $(C^{\uparrow}(X,Y),\tau_{PO},\preceq_s)$ is T_2-ordered, there exist an increasing subbasic set $[x,U]$ containing f and a decreasing subbasic set $[x,V]$ containing g such that $[x,U]\cap[x,V]=\emptyset$. Therefore, U is an increasing neighbourhood of y_1 and V is a decreasing neighbourhood of y_2 and $U\cap V=\emptyset$. Hence $(Y,\widehat{c}_Y,\preceq_Y)$ is T_2-ordered.

Conversely, let $(Y, \widehat{c}_Y, \preceq_Y)$ be a T_2-ordered space and $f, g \in C^{\uparrow}(X,Y)$ such that $f \npreceq_s g$. Then there exists a point x of X such that $f(x) \npreceq_Y g(x)$. Since $(Y, \widehat{c}_Y, \preceq_Y)$ is T_2-ordered, there exist an open increasing set U containing $f(x)$ and open decreasing set V containing $g(x)$ such that $U \cap V = \emptyset$. Therefore, $[x, U]$ is an increasing neighbourhood of f and $[x, V]$ is a decreasing neighbourhood of g such that $[x,U] \cap [x,V] = \emptyset$. Thus, $(C^{\uparrow}(X,Y), \tau_{PO}, \preceq_s)$ is T_2-ordered. □

Theorem 2.10. *Let $(X, c_X, \preceq_X)$ and $(Y, c_Y, \preceq_Y)$ be ordered closure spaces. Then the following assertions are true:*

(i) $(C^{\uparrow}(X,Y), \tau_{PI}, \preceq_s)$ *is T_1-ordered* $\Leftrightarrow$ $(Y, \tilde{c}_Y, \preceq_Y)$ *is T_1-ordered.*
(ii) $(C^{\uparrow}(X,Y), \tau_{PI}, \preceq_s)$ *is T_2-ordered* $\Leftrightarrow$ $(Y, \tilde{c}_Y, \preceq_Y)$ *is T_2-ordered.*

Proof. It is similar to previous statements. □

Proposition 2.11. *Let $(X, c_X, \preceq_X)$ and $(Y, c_Y, \preceq_Y)$ be ordered closure spaces. Then $(Y, \widehat{c}_Y, \preceq_Y)$ $((Y, \tilde{c}_Y, \preceq_Y))$ is regular ordered if and only if $(C^{\uparrow}(X,Y), \tau_{PO}, \preceq_s)$ $((C^{\uparrow}(X,Y), \tau_{PI}, \preceq_s))$ is regular ordered.*

Proof. We will only prove the second part. First part can be obtained similarly. Let $(Y, \tilde{c}_Y, \preceq_Y)$ be a regular ordered space. Let $f \in C^{\uparrow}(X,Y)$ and $[x, int_{c_Y}U]$ be an increasing subbasic open set containing f. Then $f(x) \in int_{c_Y}U$. Also, $int_{c_Y}U$ is an increasing open set in $(Y, \tilde{c}_Y, \preceq_Y)$. Since Y is regular ordered, there exists an increasing subbasic open set $int_{c_Y}V$ containing $f(x)$ and $\tilde{c}_Y(int_{c_Y}V) \subseteq int_{c_Y}U$ holds. Therefore, $[x, int_{c_Y}V]$ is an increasing neighbourhood of f and we get that $f \in cl_{\tau_{PI}}([x, int_{c_Y}V]) \subseteq [x, \tilde{c}_Y(int_{c_Y}V)] \subseteq [x, int_{c_Y}U]$. Hence $C^{\uparrow}(X,Y)$ is lower regular ordered and similarly it is upper regular ordered. Consequently, $C^{\uparrow}(X,Y)$ is a regular ordered space. Conversely, let $(C^{\uparrow}(X,Y), \tau_{PI}, \preceq_s)$ be a regular ordered space. We show that $(Y, \tilde{c}_Y, \preceq_Y)$ is a regular ordered space. To this end, consider a function

$$\Phi : (Y, \tilde{c}_Y, \preceq_Y) \to (C^{\uparrow}(X,Y), \tau_{PI}, \preceq_s)$$

given by $\Phi(y) = f_y$ where $f_y(x) = y$ for all $x \in X$. Then Φ is an order embedding, that is,

$$\Phi : (Y, \tilde{c}_Y, \preceq_Y) \to (\Phi(Y), (\tau_{PI})_{\Phi(Y)}, (\preceq_s)_{\Phi(Y)})$$

is an order homeomorphism (i.e, Φ is a homeomorphism and the functions Φ and Φ^{-1} are order preserving), where $(\tau_{PI})_{\Phi(Y)}$ is the induced topology on $\Phi(Y)$ by τ_{PI} and $(\preceq_s)_{\Phi(Y)} = \preceq_s \cap (\Phi(Y) \times \Phi(Y))$. By using Theorem 5.2 (ii) given in [2], we get that Φ is an embedding. Also, it is clear that Φ and Φ^{-1} are order preserving. Therefore, Φ is an order embedding. Now we show that the subspace $(\Phi(Y), (\tau_{PI})_{\Phi(Y)}, (\preceq_s)_{\Phi(Y)})$ is regular ordered. Let $[a, int_{c_Y}U] \cap \Phi(Y)$ be an increasing subbase element where $a \in X$, $U \subseteq Y$ and $f_y \in [a, int_{c_Y}U] \cap \Phi(Y)$. Then, $[a, int_{c_Y}U]$ is an increasing neighbourhood of f_y in $(C^{\uparrow}(X,Y), \tau_{PI}, \preceq_s)$. Indeed, let $f \in [a, int_{c_Y}U]$ and $g \in C^{\uparrow}(X,Y)$ such that $f \preceq_s g$. We need to show that $g \in [a, int_{c_Y}U]$. Define the constant functions $c_{f(a)}$ and $c_{g(a)}$ where $c_{f(a)}(x) = f(a)$ and $c_{g(a)}(x) = g(a)$ for all $x \in X$. Then, $c_{f(a)} \preceq_s c_{g(a)}$. Since $[a, int_{c_Y}U] \cap \Phi(Y)$ is increasing, we get that $c_{g(a)} \in [a, int_{c_Y}U] \cap \Phi(Y)$. Thus, $g \in [a, int_{c_Y}U]$. Therefore, $[a, int_{c_Y}U]$ is an increasing set in $(C^{\uparrow}(X,Y), \preceq_s)$. Since $(C^{\uparrow}(X,Y), \tau_{PI}, \preceq_s)$ is regular ordered, there exists an increasing neighbourhood W of f_y such that

$$f_y \in cl_{\tau_{PI}}(W) \subseteq [a, int_{c_Y}U]$$

holds. Morever, it is clear that $W \cap \Phi(Y)$ is an increasing neighbourhood of f_y in $(\Phi(Y), (\tau_{PI})_{\Phi(Y)}, (\preceq_s)_{\Phi(Y)})$. Thus, we have that

$$f_y \in cl_{(\tau_{PI})_{\Phi(Y)}}(W \cap \Phi(Y)) \subseteq [a, int_{c_Y}U] \cap \Phi(Y)$$

Therefore, $(\Phi(Y), (\tau_{PI})_{\Phi(Y)}, (\preceq_s)_{\Phi(Y)})$ is lower regular ordered and similarly it is upper regular ordered. Thus, $(\Phi(Y), (\tau_{PI})_{\Phi(Y)}, (\preceq_s)_{\Phi(Y)})$ is regular ordered. Since Φ is an order homeomorphism between Y and $\Phi(Y)$, we get that $(Y, \tilde{c}_Y, \preceq_Y)$ is regular ordered. □

3. Semi uniform spaces and ordered semi uniformizability

In this section, we first recall some basic notions about semi-uniform spaces and then motivated from [15] we will introduce semi-uniformizable ordered spaces.

Let X be a set and let $\triangle = \{(x,x) \mid x \in X\}$. $A \subseteq X \times X$ is called symmetric if $A = A^{-1}$, where $A^{-1} = \{(x,y) \mid (y,x) \in A\}$. For $A, B \subseteq X \times X$, $A \circ B = \{(x,y) : (x,z) \in A$ and $(z,y) \in B$ for some $z \in X\}$. A filter $\mathfrak{U}$ of subsets on $X \times X$ is called a semi-uniformity if $\triangle \subseteq U$ and $U^{-1} \in \mathfrak{U}$ for all $U \in \mathfrak{U}$. The pair $(X, \mathfrak{U})$ is called semi-uniform space. Semi-uniform structure induces a Čech closure operator in such a way that let consider the collection $[\mathfrak{U}][x]$ of all subsets of X of the form $U[x] = y \in X|(x,y) \in U$, where $U \in \mathfrak{U}$, then there exists a unique closure operator c for X such that $[\mathfrak{U}][x]$ is a local base at x in (X,c). This closure operator is called the closure induced by $\mathfrak{U}$ and denoted by $c_{\mathfrak{U}}$. Also, it is possible to define a topology by using a semi-uniformity. Let $\mathfrak{U}$ be a semi-uniformity on X. Then there is a unique topology $\tau_{\mathfrak{U}}$ on X such that $[\mathfrak{U}][x_0]$ is a base for the $\tau_{\mathfrak{U}}$-neighbourhoods of $x_0 \in X$, where base means a family of subsets of X containing x. Morever, $G \in \tau_{\mathfrak{U}}$ iff $\forall x \in G$ there is a $V \in \mathfrak{U}$ such that $V[x] \subseteq G$. Although every $\tau_{\mathfrak{U}}$-neighbourhood of $x_0 \in X$ is a member of $[\mathfrak{U}][x_0]$, the converse is not neccesarily true and this causes considerable complications. To get around to this problem topological semi-uniformity definition was given in [16] by the following:

Definition 3.1. Let $(X, \mathfrak{U})$ be a semi-uniform space. If $c_{\mathfrak{U}} = \tau_{\mathfrak{U}}$-closure holds, then $\mathfrak{U}$ is called topological semi-uniformity.

The following definition was given by Čech in [4].

Definition 3.2. The product of a family $\{(X_i, \mathfrak{U}_i) \mid i \in I\}$ of semi-uniform spaces, denoted by $\prod\{(X_i, \mathfrak{U}_i) \mid i \in I\}$ is defined to be the semi-uniform space $(X, \mathfrak{U})$ where X is the product of the family $\{X_i\}$ and $\mathfrak{U}$ called the product semi-uniformity is the collection of all subsets of $X \times X$ containing a set of the form $\{(x,y) \mid (x,y) \in X \times X, i \in J \Rightarrow (\pi_i(x), \pi_i(y)) \in U_i\}$ where J is a finite subset of I and $U_i \in \mathfrak{U}_i$ for each i.

Now we will give the following lemma and then we will introduce the semi-uniformizable ordered spaces.

Lemma 3.3. *Let $(X, \mathfrak{U})$ be a semi-uniform space. For every $x, y \in X$, the relation $\preceq_{\mathfrak{U}}$ is defined by the following:*

$$x \preceq_{\mathfrak{U}} y \Leftrightarrow U[y] \subseteq U[x] \text{ for all } U \in \mathfrak{U} .$$

Then $\preceq_{\mathfrak{U}}$ is a preorder.

Proof. i) Clearly, $x \preceq_{\mathfrak{U}} x$ for all $x \in X$.
ii) Let $x, y, z \in X$ such that $x \preceq_{\mathfrak{U}} y$ and $y \preceq_{\mathfrak{U}} z$. Let consider any $U \in \mathfrak{U}$. Then, $U[y] \subseteq U[x]$ and $U[z] \subseteq U[y]$. By transivity of the inclusion we have $U[z] \subseteq U[x]$. Hence, $x \preceq_{\mathfrak{U}} z$. Therefore, $\preceq_{\mathfrak{U}}$ is a preorder. □

Definition 3.4. Let $(X, c, \preceq)$ be an ordered closure space and $\mathfrak{U}$ be a semi-uniformity on X. If $c = c_{\mathfrak{U}}$ and $\preceq = \preceq_{\mathfrak{U}}$, then X is called semi-uniformizable ordered space. We can give this definition in the concept of topological ordered spaces. Let $(X, \tau, \preceq)$ be a topological ordered space. If $\tau = \tau_{\mathfrak{U}}$ and $\preceq = \preceq_{\mathfrak{U}}$, then X is called semi-uniformizable ordered topological space. If $\mathfrak{U}$ is a topological semi-uniform structure, then $(X, \tau, \preceq)$ is called topological semi-uniformizable ordered space.

We give the following example to illustrate the Definition 3.4.

Example 3.5. Let $X = \{a,b,c\}$ and $\mathcal{U} = \{U_1, U_2, U_3, U_4\}$ be a semi-uniformity on X, where

$$U_1 = \triangle \cup \{(a,b),(b,a),((b,c),(c,b)\},\ U_2 = \triangle \cup \{(a,b),(b,a),((b,c),(c,b),(a,c)\},$$
$$U_3 = \triangle \cup \{(a,b),(b,a),((b,c),(c,b),(a,c),(c,a)\},$$
$$U_4 = \triangle \cup \{(a,b),(b,a),((b,c),(c,b),(c,a)\}$$

Then the closure operator $c_{\mathcal{U}}$ induced by $\mathcal{U}$ will be the following one:

$$c_{\mathcal{U}}(\{a\}) = \{a,b\},\ c_{\mathcal{U}}(\{b\}) = \{a,b,c\},\ c_{\mathcal{U}}(\{c\}) = \{b,c\},$$
$$c_{\mathcal{U}}(\{a,b\}) = c_{\mathcal{U}}(\{b,c\}) = c_{\mathcal{U}}(\{a,c\}) = X = c_{\mathcal{U}}(X),\ c_{\mathcal{U}}(\emptyset) = \emptyset$$

Also, the preorder induced by $\mathcal{U}$ is $\preceq_{\mathcal{U}} = \{(a,a),(b,b),(c,c),(b,a),(b,c)\}$. Therefore, $(X, c_{\mathcal{U}}, \preceq_{\mathcal{U}})$ is a semi-uniformizable ordered space induced by $\mathcal{U}$ and the topology induced by $\mathcal{U}$ is $\tau_{\mathcal{U}} = \{\emptyset, X\}$. Thus, $(X, \tau_{\mathcal{U}}, \preceq_{\mathcal{U}})$ is a semi-uniformizable ordered topological space.

Remark 3.6. Let $(X, c, \preceq)$ be an ordered closure space induced by a topological semi-uniform structure $\mathcal{U}$, then it is clear that the closure operator c is a topological closure operator.

Lemma 3.7. *Let (X, c) be a closure space and τ_c be the topology associated with c. If c is induced by a semi-uniformity $\mathcal{U}$ on X, then $\tau_c = \tau_{\mathcal{U}}$ holds.*

Proof. Let $\mathcal{U}$ be a semi-uniformity on X and $c = c_{\mathcal{U}}$. We will show that $\tau_c = \tau_{\mathcal{U}}$. Let $G \in \tau_c$ and $x \in G$. Notice that

$$\begin{aligned} x \notin G^c &\Rightarrow x \notin c(G^c) \\ &\Rightarrow \exists U \in \mathcal{U} \ni U[x] \cap G^c = \emptyset \\ &\Rightarrow \exists U \in \mathcal{U} \ni U[x] \subseteq G. \end{aligned}$$

Hence, $G \in \tau_{\mathcal{U}}$. In the other direction, let $G \in \tau_{\mathcal{U}}$ and let $x \in c(G^c)$. Hence $U[x] \cap G^c \neq \emptyset$ holds. We claim that $x \notin G$. Suppose that $x \in G$, then there exists $U' \in \mathcal{U}$ such that $U'[x] \subseteq G$ and whence $U'[x] \cap G^c = \emptyset$. But this contradicts with $x \in c(G^c)$. Hence $x \notin G$. Therefore $G \in \tau_c$ and consequently, $\tau_c = \tau_{\mathcal{U}}$ holds. □

According to [10], let $(X, \tau, \preceq)$ be an ordered topological space where $\preceq$ is a preorder on X. A subset A of X with the induced topology τ_A and induced preorder $\preceq_A$ is called subspace.

Lemma 3.8. *Every subspace of a topological semi-uniformizable ordered space is a topological semi-uniformizable ordered space.*

Proof. Let $(X, \tau, \preceq)$ be a topological semi-uniformizable ordered space and induced by a topological semi-uniformity $\mathcal{U}$. Let $A \subseteq X$. We claim that the semi-uniformity $\mathcal{U}_A = (A \times A) \cap \mathcal{U}$ induces the subspace $(A, \tau_A, \preceq_A)$. Firstly, we would like to show that $\preceq_A = \preceq_{\mathcal{U}_A}$. Let $x, y \in A$. Then

$$\begin{aligned} x \preceq_A y &\Leftrightarrow x \preceq y \\ &\Leftrightarrow x \preceq_{\mathcal{U}} y \\ &\Leftrightarrow x \preceq_{\mathcal{U}_A} y. \end{aligned}$$

Therefore, $\preceq_A = \preceq_{\mathcal{U}_A}$. Now we will show that $\tau_A = \tau_{\mathcal{U}_A}$. Let $G \in \tau_A$ and $x \in G$. By the definition of τ_A, there exists $H \in \tau$ such that $G = H \cap A$. Also,

$$\begin{aligned} x \in G &\Rightarrow x \in H \text{ and } x \in A \\ &\Rightarrow \exists U \in \mathcal{U} \ni U[x] \subseteq H \text{ and } x \in A \\ &\Rightarrow ((A \times A) \cap U)[x] \subseteq G. \end{aligned}$$

Whence $G \in \tau_{\mathcal{U}_A}$. Now, let $G \in \tau_{\mathcal{U}_A}$ and $x \in G$. By the definition of $\tau_{\mathcal{U}_A}$, there exists $U \in \mathcal{U}$ such that $((A \times A) \cap U)[x] \subseteq G$. Since $U[x]$ is a neighbourhood of the point x, there exists $H \in \tau$ such that $x \in H \subseteq U[x]$. Therefore we get that $x \in A \cap H \subseteq G$. Consequently, $G \in \tau_A$ and whence $\tau_A = \tau_{\mathcal{U}_A}$. □

Theorem 3.9. *Let $(X, c_X, \preceq_X)$ and $(Y, c_Y, \preceq_Y)$ be ordered closure spaces. The following assertions are true:*

(i) *$(Y, \widehat{c}_Y, \preceq_Y)$ is semi-uniformizable if and only if $(C^{\uparrow}(X,Y), \tau_{PO}, \preceq_s)$ is topological semi-uniformizable.*

(ii) *$(Y, \tilde{c}_Y, \preceq_Y)$ is semi-uniformizable if and only if $(C^{\uparrow}(X,Y), \tau_{PI}, \preceq_s)$ is topological semi-uniformizable.*

Proof. (i) Let $(Y, \widehat{c}_Y, \preceq_Y)$ be a semi-uniformizable ordered space and induced by a semi-uniformity $\mathcal{U}$. In [4], it was shown that the product of semi-uniformizable spaces is semi-uniformizable and the product closure $\prod_{x\in X}(\widehat{c}_Y)_x$ is induced by the product uniform $\mathcal{U}_{\mathrm{p}} = \prod_{x\in X}\mathcal{U}_x$ where $\mathcal{U}_x = \mathcal{U}$ and $(\widehat{c}_Y)_x = \widehat{c}_Y$ for all $x \in X$. Also, it is clear that $\mathcal{U}_{\mathrm{p}}$ is topological semi-uniformity since $\widehat{c}_Y$ is a topological closure operator. By Lemma 3.7, $\tau(\prod_{x\in X}(\widehat{c}_Y)_x)$ is induced by $\mathcal{U}_{\mathrm{p}}$. Hence $(\prod_{x\in X}Y_x, \tau(\prod_{x\in X}(\widehat{c}_Y)_x))$ is induced by $\mathcal{U}_{\mathrm{p}}$, where $Y_x = Y$ for all $x \in X$. Now we show that $\prod_{x\in X}(\preceq_Y)_x$ is induced by $\preceq_{\mathcal{U}_p}$, where $(\preceq_Y)_x = \preceq_Y$ for all $x \in X$ (Denote $\prod_{x\in X}(\preceq_Y)_x$ by $\preceq_p$). Let $a, b \in \prod_{x\in X}Y_x$ such that $a \preceq_p b$. Using the definition of product order, we have $a_x \preceq_Y b_x$ for all $x \in X$, where $\pi_x(a) = a_x, \pi_x(b) = b_x$. Let us take any $U \in \mathcal{U}_{\mathrm{p}}$. We have to show that $U[b] \subseteq U[a]$ holds. Let $z \in U[b]$. Then $(b, z) \in U$. From the definition of product semi uniformity, there exist $U_x \in \mathcal{U}$ such that $(b, z) \in \prod_{x\in X}U_x \subseteq U$, where except for a finite number of x's, $U_x = Y \times Y$. Therefore, $(b_x, z_x) \in U_x$ for all $x \in X$. Since $a_x \preceq_Y b_x$, we have that $U_x[b_x] \subseteq U_x[a_x]$. Thus, $z_x \in U_x[a_x]$. Therefore $(a, z) \in U$ and $z \in U[a]$. Whence $U[b] \subseteq U[a]$. We get that $\preceq_p \subseteq \preceq_{\mathcal{U}_p}$. Now, we show the other side. Let $a, b \in \prod_{x\in X}Y_x$ such that $a \preceq_{\mathcal{U}_p} b$. We want to show that $a \preceq_p b$. To this end we have to show $a_x \preceq_{\mathcal{U}} b_x$ for all $x \in X$. Let $U \in \mathcal{U}$ and $z \in X$. Define $U' = \prod_{x\in X}U_x$ where $U_z = U$ for $z \in X$ and except for the point z, $U_x = Y \times Y$. Since $a \preceq_{\mathcal{U}_p} b$, $U'[b] \subseteq U'[a]$. We claim that $U[b_z] \subseteq U[a_z]$. Let $t \in U[b_z]$ and $t' = (t, t, ..., t, ..)$. Then it is clear that $(b, t') \in U'$. Therefore, $t' \in U'[a]$. Clearly, $t \in U[a_z]$. By now we obtained that $(\prod_{x\in X}Y_x, \tau(\prod_{x\in X}(\widehat{c}_Y)_x), \prod_{x\in X}(\preceq_Y)_x)$ is induced by $\mathcal{U}_p$. In [2], it was shown that $\tau(\prod_{x\in X}(\widehat{c}_Y)_x) = \tau_{PO}$ and, by Lemma 3.8, we have that $(C^{\uparrow}(X,Y), \tau_{PO}, \preceq_s)$ is semi-uniformizable ordered space. Conversely, let $(C^{\uparrow}(X,Y), \tau_{PO}, \preceq_s)$ be a topological semi-uniformizable ordered space and consider a function

$$\Phi : (Y, \widehat{c}_Y, \preceq_Y) \to (C^{\uparrow}(X,Y), \tau_{PO}, \preceq_s)$$

given by $\Phi(y) = f_y$ where $f_y(x) = y$ for all $x \in X$. Then, it is clear that Φ is an order embedding. By Lemma 3.8, we know that the subspace of $(C^{\uparrow}(X,Y), \tau_{PO}, \preceq_s)$ is a topological semi-uniformizable ordered space. Since Φ is order embedding, we get that $(Y, \widehat{c}_Y, \preceq_Y)$ is semi-uniformizable ordered space.

(ii) In [2], it was shown that $\tau(\prod_{x\in X}(\tilde{c}_Y)_x) = \tau_{PI}$. Therefore, (ii) follows using the similar arguments as for (i). □

Lemma 3.10. *Let $(X, c_X, \preceq_X)$ and $(Y, c_Y, \preceq_Y)$ be ordered closure spaces and $\mathcal{U}$ be a semi-uniformity on $C^{\uparrow}(X,Y)$. Define $U^c[g] = \{f \in C^{\uparrow}(X,Y) \mid f$ is a constant function and $(f,g) \in U\}$. Let $f, g \in C^{\uparrow}(X,Y)$. Then the relation $\preceq^c_{\mathcal{U}}$, which is defined by the following:*

$$f \preceq^c_{\mathcal{U}} g \Leftrightarrow U^c[g] \subseteq U^c[f] \text{ for all } U \in \mathcal{U}$$

is a preorder.

Proof. (i) Clearly, $f \preceq^c_{\mathcal{U}} f$
(ii) Let $f, g, h \in C^{\uparrow}(X,Y)$ such that $f \preceq^c_{\mathcal{U}} g$ and $g \preceq^c_{\mathcal{U}} h$. Let take $U \in \mathcal{U}$. Then, $U^c[g] \subseteq U^c[f]$ and $U^c[h] \subseteq U^c[g]$. Thus $U^c[h] \subseteq U^c[f]$ and hence $f \preceq^c_{\mathcal{U}} h$. Consequently, "$\preceq^c_{\mathcal{U}}$" is a preorder. □

Proposition 3.11. *Let $(X, c_X, \preceq_X)$ and $(Y, c_Y, \preceq_Y)$ be ordered closure spaces. Consider an ordered topological space $(C^{\uparrow}(X,Y), \tau_{PO}, \preceq_s)((C^{\uparrow}(X,Y), \tau_{PI}, \preceq_s))$. If a topological semi-uniformity $\mathcal{U}$ on $C^{\uparrow}(X,Y)$ induces $\tau_{PO}(\tau_{PI})$ and $\preceq_s = \preceq^c_{\mathcal{U}}$, then $(Y, \widehat{c}_Y, \preceq_Y)$ $((Y, \tilde{c}_Y, \preceq_Y))$ is a semi-uniformizable ordered closure space.*

Proof. Let $\tau_{PO} = \tau_{\mathcal{U}}$ and $\preceq_s = \preceq^c_{\mathcal{U}}$. Define the constant functions f_a and f_b by $f_a(x) = a$ and $f_b(x) = b$ for all $x \in X$. Let $\widehat{U} = \{(a,b) \mid (f_a, f_b) \in U\}$ for all $U \in \mathcal{U}$. Clearly, the collection $\mathcal{B} = \{\widehat{U} \mid U \in \mathcal{U}\}$ is a base for a semi-uniformity on Y. Let $\widehat{\mathcal{U}}$ be the semi-uniformity generated by $\mathcal{B}$. We claim that $\widehat{c}_Y = c_{\widehat{\mathcal{U}}}$. To prove this equality, we will show that for any point y of Y, the family $\{\widehat{U}[y] \mid U \in \mathcal{U}\}$ is a neighbourhood base at y in $(Y, \widehat{c}_Y, \preceq_Y)$. To this end we show the following two conditions:

i) $\widehat{U}[y] \in \mathcal{V}_{\widehat{c_Y}}(y)$ for all $U \in \mathcal{U}$,
ii) $\forall V \in \mathcal{V}_{\widehat{c_Y}}(y)\ \exists U \in \mathcal{U}$ such that $\widehat{U}[y] \subseteq V$.

Let us consider a constant function f_y defined by $f_y(x) = y$ for all $x \in X$. It is clear that $U[f_y] \in \mathcal{V}_{\tau_{PO}}(f_y)$ for any $U \in \mathcal{U}$. By the definition of the compact open topology there exists a subbasic element $[z, O]$ such that $f_y \in [z, O] \subseteq U[f_y]$. Therefore, $y \in O \subseteq \widehat{U}[y]$ and whence $\widehat{U}[y] \in \mathcal{V}_{\widehat{c_Y}}(y)$. Now we show ii). Take any $V \in \mathcal{V}_{\widehat{c_Y}}(y)$. We can take V as an open set. Then for any $x \in X$, we have that $[x, V] \in \tau_{PO}$ and $f_y \in [x, V]$. Since $\tau_{PO} = \tau_{\mathcal{U}}$, there exists $U \in \mathcal{U}$ such that $U[f_y] \subseteq [x, V]$. Clearly, $\widehat{U}[y] \subseteq V$. Indeed, notice that for any point z of X

$$\begin{aligned} z \in \widehat{U}[y] &\Rightarrow (y, z) \in \widehat{U} \\ &\Rightarrow (f_y, f_z) \in U \\ &\Rightarrow f_z \in U[f_y] \\ &\Rightarrow z \in V. \end{aligned}$$

Hence $\widehat{U}[y] \subseteq V$ and we have that $\widehat{c}_Y = c_{\widehat{\mathcal{U}}}$. Now we show that $\preceq_Y = \preceq_{\widehat{\mathcal{U}}}$. Let $y_1, y_2 \in Y$ such that $y_1 \preceq_Y y_2$. Let $\widehat{U} \in \widehat{\mathcal{U}}$. Define the constant functions f_{y_1}, f_{y_2} by $f_{y_1}(x) = y_1$ and $f_{y_2}(x) = y_2$ for all $x \in X$. Therefore $f_{y_1} \preceq_s f_{y_2}$. Since $\preceq_s = \preceq^c_{\widehat{\mathcal{U}}}$, we have $f_{y_1} \preceq^c_{\widehat{\mathcal{U}}} f_{y_2}$. Hence $U^c[f_{y_1}] \subseteq U^c[f_{y_2}]$ for all $U \in \mathcal{U}$. Also, $\widehat{U}[y_2] \subseteq \widehat{U}[y_1]$. Indeed, for any point z of X

$$\begin{aligned} z \in \widehat{U}[y_2] &\Rightarrow (y_2, z) \in \widehat{U} \\ &\Rightarrow (f_{y_2}, f_z) \in U \\ &\Rightarrow f_z \in U^c[f_{y_2}] \\ &\Rightarrow f_z \in U^c[f_{y_1}] \\ &\Rightarrow (f_{y_1}, f_z) \in U \\ &\Rightarrow z \in \widehat{U}[y_1]. \end{aligned}$$

Hence $\widehat{U}[y_2] \subseteq \widehat{U}[y_1]$. Conversely, let $y_1, y_2 \in Y$ and $y_1 \preceq_{\widehat{\mathcal{U}}} y_2$. Consider the constant functions f_{y_1} and f_{y_2} defined by $f_{y_1}(x) = y_1$ and $f_{y_2}(x) = y_2$ for all $x \in X$. Clearly, $U^c[f_{y_2}] \subseteq U^c[f_{y_1}]$ for all $U \in \mathcal{U}$. Thus $f_{y_1} \preceq^c_{\widehat{\mathcal{U}}} f_{y_2}$ and whence $f_{y_1} \preceq_s f_{y_2}$. Therefore $y_1 \preceq_Y y_2$. Consequently, we have that $\preceq_Y = \preceq_{\widehat{\mathcal{U}}}$. □

Corollary 3.12. *Let $(X, c_X, \preceq_X)$ and $(Y, c_Y, \preceq_Y)$ be ordered closure spaces and (X, c_X) be a discrete space. Then, $(Y, \widehat{c}_Y, \preceq_Y)((Y, \tilde{c}_Y, \preceq_Y))$ is semi-uniformizable ordered space if and only if $(C^{\uparrow}(X,Y), \tau_{CO}, \preceq_s)((C^{\uparrow}(X,Y), \tau_{CI}, \preceq_s))$ is topological semi-uniformizable ordered space.*

Proof. Since X is discrete, $\tau_{PO}(\tau_{PI})$ coincides with $\tau_{CO}(\tau_{CI})$. Therefore, proof follows from Theorem 3.9. □

The following definition was given by Williams in [20] .

Definition 3.13. A filter $\mathfrak{U}$ of subsets on $X \times X$ is called locally-uniformity if it is a semi-uniformity and $\forall U \in \mathfrak{U}$ and $x \in X$, there is a $V \in \mathfrak{U}$ such that $(V \circ V)[x] \subseteq U[x]$. Then $(X, \mathfrak{U})$ is called locally-uniform space.

Remark 3.14. Locally-uniform spaces are topological semi-uniform spaces.

Definition 3.15. Let $(X, c_X, \preceq_X)$ be an ordered closure space and $\mathfrak{U}$ be a locally-uniformity on Y. Define;

$$(K, U) = \{(f, g) \in C^{\uparrow}(X,Y) \times C^{\uparrow}(X,Y) \mid (f(x), g(x)) \in U \text{ for all } x \in K\}$$

where K is a compact subset of X and U be an element of $\mathfrak{U}$. The collection of the sets (K, U) is a base for a locally-uniformity $\mathfrak{U}_k$ on $C^{\uparrow}(X,Y)$ and $\mathfrak{U}_k$ is called locally-uniformity of $\mathfrak{U}$-local uniform convergence on compact sets. Then the triple $(C^{\uparrow}(X,Y), \tau_{\mathfrak{U}_k}, \preceq_{\mathfrak{U}_k})$ is called ordered topological space of $\mathfrak{U}$-local uniform convergence on compact sets.

The following Lemma is obvious from Theorem 1.2 in [20] and Corollary 1.17 in [16].

Lemma 3.16. *Let $(X, \mathfrak{U})$ be a locally-uniform space. Then for all $U \in \mathfrak{U}$ and $x \in X$ there exists a symmetric and closed set $V \in \mathfrak{U}$ such that $V[x] \subseteq U[x]$.*

Theorem 3.17. *Let $(X, c_X, \preceq_X)$ be an ordered closure space and $(Y, \widehat{c}_Y, \preceq_Y)$ $((Y, \tilde{c}_Y, \preceq_Y))$ be an ordered closure space induced by a locally-uniformity $\mathfrak{U}$ on Y. Then compact-open (interior) ordered topological space coincides with the ordered space of $\mathfrak{U}$-locally uniform convergence on compact sets.*

Proof. Let $[K, U]$ be a subbasic open set in the compact-open topology and $g \in [K, U]$. Then there exists $E \in \mathfrak{U}$ such that $E[g(K)] \subseteq U$. Indeed,

$$\begin{aligned} x \in g(K) &\Rightarrow x \in U \\ &\Rightarrow \exists D_x \in \mathfrak{U} \ni D_x[x] \subseteq U \\ &\Rightarrow \exists E_x \in \mathfrak{U} \ni E_x^2[x] \subseteq D_x[x] \subseteq U \end{aligned}$$

and since $g(K)$ is compact we have that $g(K) \subseteq E_{x_1}[x_1] \cup E_{x_2}[x_2] \cup ... \cup E_{x_n}[x_n]$. Define $E = E_{x_1} \cap ... \cap E_{x_n}$. Then it is clear that $E[g(K)] \subseteq U$. Consider $D = \{(f, h) \mid (f(x), h(x)) \in E$ for all $x \in K\}$. It is clear that $D \in \mathfrak{U}_k$. Morever, $g \in D[g] \subseteq [K, U]$. Indeed

$$\begin{aligned} h \in D[g] &\Rightarrow (g(x), h(x)) \in E \text{ for all } x \in K \\ &\Rightarrow h(x) \in E[g(x)] \text{ for all } x \in K \\ &\Rightarrow h(x) \in E[g(K)] \subseteq U \\ &\Rightarrow h \in [K, U]. \end{aligned}$$

Hence $\tau_{CO} \subseteq \tau_{\mathfrak{U}_k}$. Now let $G \in \tau_{\mathfrak{U}_k}$ and $f \in G$. By the definition of $\tau_{\mathfrak{U}_k}$ there exists a compact subset K of X and $U \in \mathfrak{U}$ such that

$$f \in (K, U)[f] \subseteq G$$

By Theorem 1.2 in [20] and using Lemma 3.16, for all $x \in K$ there exists a closed and symmetric subset T_x of $Y \times Y$ such that $T_x^3[f(x)] \subseteq U[f(x)]$. Put $T = \bigcap\limits_{x \in K} T_x$. Clearly, T

is closed and symmetric. Since $f(K)$ is compact, $f(K) \subseteq \bigcup_{i=1}^{n} T[f(x_i)]$ holds. Define $K_i = K \cap f^{-1}(T[f(x_i)])$ and $G_i = int_{\widehat{c_Y}}(T^2[f(x_i)])$ for all $i \in \{1, 2, ..., n\}$. Put $A = \bigcap_{i=1}^{n} [K_i, G_i]$. Then $f \in A \subseteq (K, U)[f]$. Indeed let $g \in A$. Thus $g(K_i) \subseteq G_i$ for all $i \in \{1, 2, ..., n\}$. Therefore

$$\begin{aligned} g(a) \in G_i \text{ for all } a \in K_i \quad &\Rightarrow \quad g(a) \in int_{\widehat{c_Y}}(T^2[f(x_i)]) \text{ for all } a \in K_i \\ &\Rightarrow \quad g(a) \in T^2[f(x_i)] \text{ for all } a \in K_i \\ &\Rightarrow \quad (f(x_i), g(a)) \in T^2 \text{ for all } a \in K_i. \end{aligned}$$

Also $f(a) \in T[f(x_i)]$ for all $a \in K_i$. Whence $(f(a), g(a)) \in T^3$. We deduce $g(a) \in T^3[f(a)]$ and $g(a) \in U[f(a)]$. Therefore, $A \subseteq (K, U)[f]$ holds.

Now we show that $\preceq_{\mathcal{U}_k} = \preceq_s$ holds. Let $f, g \in C^{\uparrow}(X, Y)$ and $f \preceq_s g$. Then $f(x) \preceq_Y g(x)$ for all $x \in X$. Since $\preceq_Y = \preceq_{\mathcal{U}}$, we have that $U[g(x)] \subseteq U[f(x)]$ for all $U \in \mathcal{U}$. Let $(K, U) \in \mathcal{U}_k$. It is clear that $(K, U)[g] \subseteq (K, U)[f]$. Then $f \preceq_{\mathcal{U}} g$. Therefore, $\preceq_s \subseteq \preceq_{\mathcal{U}_k}$. Now let $f, g \in C^{\uparrow}(X, Y)$ and $f \preceq_{\mathcal{U}_k} g$. Let $x \in X$ and $U \in \mathcal{U}$. Clearly, $(x, U) \in \mathcal{U}_k$. Since $f \preceq_{\mathcal{U}_k} g$, $U[g(x)] \subseteq U[f(x)]$ holds. Thus, $f \preceq_s g$. Consequently, $\preceq_{\mathcal{U}_k} = \preceq_s$. Second part can be shown similarly to the first part. □

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