Reducible good representations of semisimple Lie algebras $A_r$ and $B_r$ Part I

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Abstract

Given a semisimple (preferably simple) complex Lie algebra $L$, we consider the monoid $\Gamma = \Gamma(L)$ of equivalence classes of the finite dimensional reducible complex representations of $L$. Here $\Gamma$ is identified with the lattice of the corresponding highest weights. (This equips $\Gamma$ with the monoid structure.) For $\pi \in \Gamma$ one considers the symmetric algebra $S(\pi) = \bigoplus_{n=0}^{\infty} S^n(\pi)$ (here regarded as a representation). The elements of $\Gamma$ “occurring” in $S(\pi)$ – i.e., which are the highest weights of some irreducible component of the representation $S(\pi)$ – form a subsemigroup $M(\pi)$ of $\Gamma$. Such a $M(\pi)$ has a naturally defined rank $r(\pi)$ with $1 \leq r(\pi) \leq r = \text{rank of } L$. In this paper we give a classification, for all the simple $L = A_r$ and $L = B_r$ of all the $\pi$ with $r(\pi) < r$.

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1. Introduction

This work deals with the classification of certain finite dimensional representations of simple finite dimensional complex Lie algebras, which we will call good representations in the paper.

Let $L$ be such a Lie algebra and let $r$ be its rank. Let $\{\Pi_1, \Pi_2, \ldots, \Pi_r\}$ be the set of fundamental weights, $\Lambda = \mathbb{Z}\Pi_1 + \mathbb{Z}\Pi_2 + \cdots + \mathbb{Z}\Pi_r$ be the lattice of weights and $\Lambda^+ = N_0\Pi_1 + N_0\Pi_2 + \cdots + N_0\Pi_r$ be the monoid of dominant weights. (The notation is with respect to a fixed Cartan subalgebra in $L$ and a fixed basis of the root system, [4,8].) We consider the $r$-dimensional $\mathbb{Q}$-vector space $\Lambda_\mathbb{Q} = \mathbb{Q}\Pi_1 + \mathbb{Q}\Pi_2 + \cdots + \mathbb{Q}\Pi_r$ ($\mathbb{Q}$ = the field of rational numbers).

We say, that $\lambda \in \Lambda^+$ occurs in a (finite dimensional) representation of $L$, if the representation has an irreducible component with highest weight $\lambda$.

Now let $\rho$ be a finite dimensional representation of $L$ with the representation space $V$. It induces a representation $S^n(\rho)$ of $L$ in the symmetric power $S^n(V)$ of $V$ for all $n = 0, 1, 2, \ldots$. For $\pi = \tau_1 + \tau_2 + \cdots + \tau_t$, $2 \leq t < r$, $S^n(\pi)$ is

$$S^n(\pi) = S^n(\tau_1) \otimes S^0(\tau_2 + \cdots + \tau_1) + S^{n-1}(\tau_1) \otimes S^1(\tau_2 + \cdots + \tau_1) + \cdots + S^0(\tau_1) \otimes S^n(\tau_2 + \cdots + \tau_t).$$

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We define
\[ M(\rho) := \{ \lambda \in \Lambda^+ \mid \text{there is } n \in \mathbb{N}_0, \text{ such that } \lambda \text{ occurs in } S^n(V) \}. \]
Then \( M(\rho) \) is a submonoid of \( \Lambda^+ \) (see [2, page 8]). Let then \( M_\mathbb{Q}(\rho) \) be the \( \mathbb{Q} \)-vector subspace of \( \Lambda_\mathbb{Q} \) generated by \( M(\rho) \). We call the representation \( \rho \) \textit{good}, if \( \dim_\mathbb{Q} M_\mathbb{Q}(\rho) < r \), otherwise we call it \textit{bad}. The dimension \( \dim_\mathbb{Q} M_\mathbb{Q}(\rho) \) is called rank\( \rho \) or rank\( \rho \).

The most reducible representations of the simple Lie algebras \( L = A_r \) and \( L = B_r \) are shown to be bad. In this section of the work the symmetric powers of representations are used, which we have compiled in the \( R \)-list 3 at the end of the work. (The calculations were made with the aid of the Lie calculation package from [9]). In Theorem 3.9 the good reducible representations are listed and they are classified according to the size of their rank. In Corollaries 3.10 and 3.11 the results from Theorems 3.1 and 3.9 are summarized once again.

The numbering of the simple roots in the basis \( B = \{ a_1, a_2, \ldots, a_r \} \) is chosen like in Tits (see [4, 8]). A representation \( \rho : L \to \mathfrak{gl}(V) \) induces in the dual space \( V^* \) of \( V \) a representation \( \rho^* : \mathfrak{gl}(V^*) \) of \( \mathfrak{gl}(V^*) \), which is called the contragradient of \( \rho \). The contragradient of a fundamental representation \( \Pi_i \) is a fundamental representation \( \Pi_i^* \) Since \( (\rho^*)^* = \rho \), this applies
\[ (\Pi_1^{m_1} \cdots \Pi_r^{m_r})^* = (\Pi_1^*)^{m_1} \cdots (\Pi_r^*)^{m_r}. \]
Let as before \( \rho \) be a reducible representation of \( L \). Let \( V \) be the representation space of \( \rho \). Let \( G \) be a simple connected linear algebraic group with a Lie algebra \( L \). A linear algebraic action of \( G \) is then induced on \( V \). We identify \( S(\rho) \) with \( \mathbb{C}[V] = \mathcal{O}(V^*) \), i.e., with the complex algebra of regular functions on the dual space \( V^* \) of \( V \). \( V^* \) is the representation space of the dual representation \( \rho^* \) of \( L \), respectively of \( G \). Note thereby
\[ \rho \text{ is good } \iff \rho^* \text{ is good.} \]
Let \( \rho \) be a representation of \( L \) on the representation space \( V \). Then \( L \) acts also on \( S^n(\rho) \), i.e., on the symmetric powers \( S^n(\rho) \) and finally on the (infinite dimensional) symmetric algebra
\[ S(V) = \bigoplus_{n=0}^{\infty} S^n(V) \quad (S^0(V) = \mathbb{C}). \]
Let \( M(\rho) \) be the following defined subset of \( \Lambda^+ \): A \( \lambda \in \Lambda^+ \) is in \( M(\rho) \), if there is a \( k \geq 0 \), such that \( \lambda \) occurs in \( S^k(\rho) \):
\[ M(\rho) = \{ \lambda \in \Lambda^+ \mid \text{there is a } k \geq 0 \text{ such that } \lambda \text{ occurs in } S^k(\rho) \}. \]

Remark 1.1. \( M(\rho) \) is a submonoid of \( \Lambda^+ \).

Proof. If \( \lambda \) occurs in \( S^n(\rho) \) and \( \mu \) occurs in \( S^m(\rho) \), then \( \lambda + \mu \) occurs in \( S^{n+m}(\rho) \). \( \square \)

We have \( \Lambda_\mathbb{Q} := \mathbb{Q} \Pi_1 + \mathbb{Q} \Pi_2 + \cdots + \mathbb{Q} \Pi_r \subseteq C^* \), where \( C^* \) is the dual space (over the complex numbers) of the Cartan subalgebra \( C \) of \( L \) and \( \mathbb{Q} \) the field of rational numbers. Finally, let \( M_\mathbb{Q}(\rho) := \mathbb{Q}\text{-subspace of } \Lambda_\mathbb{Q} \text{ generated by } M(\rho) \).

Definition 1.2. We call the dimension of \( M_\mathbb{Q}(\rho) \) over \( \mathbb{Q} \) the rank of the representation \( \rho \):
\[ \text{rank } M(\rho) := \text{rank } \rho := \text{dimension of } M_\mathbb{Q}(\rho) \text{ over } \mathbb{Q}. \]

Definition 1.3. A representation \( \rho \) of \( L \) is \textit{good}, if \( \text{rank } M(\rho) < r \). Otherwise, i.e., if \( \text{rank } M(\rho) = r \), \( \rho \) is \textit{bad}. (Note that \( \dim_\mathbb{Q} \Lambda_\mathbb{Q} = r \), because the \( \Pi_1, \ldots, \Pi_r \) are linearly independent.)

Remark 1.4. A reducible representation \( \rho = \tau_1 + \tau_2 + \cdots + \tau_t \) with, \( \tau_1, \tau_2, \ldots, \tau_t \) irreducible representations is good, if the sum \( \tau_1 + \tau_2 + \cdots + \tau_t, t < r \), is good.
2. Reducible bad representations

We considered the irreducible good representations of simple Lie algebras in our doctoral thesis [2]. Here we consider the reducible good representations of simple Lie algebras \( L = A_r \) and \( L = B_r \). The irreducible good representations of simple Lie algebras are given in the following lists (see [2, page 44, Theorem 2.1]).

(i) List(A): \( \Pi_1, \Pi_2, \Pi_{r-1} \) of \( A_r \), \( r \geq 2 \) and \( \Pi_3 \) of \( A_3 \);
(ii) List(B): \( \Pi_1 \) of \( B_r \), \( r \geq 2 \) and \( \Pi_2 \) for \( r = 2, 3, 4 \).

**Theorem 2.1.** (\( R \)-bad) Let \( L \) be a (semi-)simple Lie algebra and \( \rho = \tau_1 + \tau_2 \) be a reducible representation of \( L \) with \( \tau_1 \) and \( \tau_2 \) good. Then \( \rho \) is bad, if \( \rho \) is in the following list:

(i) \( A_r \):
\[ \Pi_1 + \Pi_2, \Pi_1 + \Pi_3, \Pi_1 + \Pi_1, \Pi_1 + \Pi_r, \Pi_r, \Pi_{r-1} + \Pi_r, \Pi_{r-1} + \Pi_r \] of \( A_r \), \( r \geq 4 \);
\( \Pi_1 + \Pi_2, \Pi_1 + \Pi_3, \Pi_2 + \Pi_3, \Pi_2 + \Pi_3, \Pi_3 + \Pi_3, \Pi_3 + \Pi_4, \Pi_3 + \Pi_5 \) of \( A_5 \);

(ii) \( B_r \):
\[ \Pi_1 + \Pi_3, \Pi_3 + \Pi_3, \Pi_1 + \Pi_4, \Pi_4 + \Pi_4 \] of \( B_4 \), \( r \geq 3 \).

**Proof.** According to Lemmas 2.2 and 2.3. \( \square \)

**Lemma 2.2.** (Case \( A_r \)) Let \( \rho = \tau_1 + \tau_2 \) be a reducible representation of \( A_r \), \( r \geq 3 \), where \( \tau_1, \tau_2 \) are from the List(A). Then \( \rho \) is bad if it is not in the list below: \( R \)-list (A-2):

\[ \Pi_1 + \Pi_2, \Pi_1 + \Pi_3, \Pi_1 + \Pi_r, \Pi_r, \Pi_1 + \Pi_1, \Pi_1 + \Pi_2 \] of \( A_r \), \( r \geq 3 \); \( \Pi_2 + \Pi_2 \) of \( A_3 \).

In other words, the following representations are bad:

\( \Pi_1 + \Pi_2, \Pi_1 + \Pi_3, \Pi_1 + \Pi_1, \Pi_1 + \Pi_2, \Pi_2 + \Pi_3, \Pi_3 + \Pi_3, \Pi_2 + \Pi_3, \Pi_3 + \Pi_4, \Pi_3 + \Pi_5 \) of \( A_5 \), \( r \geq 4 \);
\( \Pi_1 + \Pi_2, \Pi_1 + \Pi_3, \Pi_2 + \Pi_3, \Pi_3 + \Pi_3, \Pi_3 + \Pi_4, \Pi_3 + \Pi_5 \) of \( A_5 \).

**Proof.** Let \( n = 3 \). The reducible representations \( \Pi_1 + \Pi_2, \Pi_2 + \Pi_3 \) of \( A_3 \) are bad according to \( R \)-list 3.1.1.

Let \( n = 4 \).

The reducible representations \( \Pi_1 + \Pi_2, \Pi_1 + \Pi_3, \Pi_2 + \Pi_2, \Pi_2 + \Pi_3, \Pi_2 + \Pi_4, \Pi_3 + \Pi_3, \Pi_3 + \Pi_4 \) are bad according to \( R \)-list 3.1.2.

Let \( n = 5 \).

The reducible representations \( \Pi_1 + \Pi_2, \Pi_1 + \Pi_3, \Pi_1 + \Pi_4, \Pi_2 + \Pi_2, \Pi_2 + \Pi_3, \Pi_2 + \Pi_4, \Pi_2 + \Pi_5, \Pi_3 + \Pi_3, \Pi_3 + \Pi_4, \Pi_3 + \Pi_5, \Pi_4 + \Pi_4, \Pi_4 + \Pi_5 \) of \( A_5 \) are bad according to \( R \)-list 3.1.3.

Let \( n \geq 6 \). Proof by induction.

There are regular subalgebras \( A_{r-1} \) and \( A'_{r-1} \) in \( A_r \) as in the picture. The representations \( \Pi_1 + \Pi_2, \Pi_1 + \Pi_1, \Pi_2 + \Pi_2, \Pi_2 + \Pi_1, \Pi_2 + \Pi_2 + \Pi_1, \Pi_2 + \Pi_1 + \Pi_1, \Pi_2 + \Pi_1 + \Pi_1 + \Pi_1, \Pi_2 + \Pi_1 + \Pi_1 + \Pi_1 + \Pi_1, \Pi_2 + \Pi_1 + \Pi_1 + \Pi_1 + \Pi_1 + \Pi_1 \) are bad in \( A_{r-1} \) and \( A'_{r-1} \).

\( \rho = \Pi_1 + \Pi_2 \)

a) \( L = A_{r-1} = A_{2m-1} \).

The trivial component \( \Pi_0 \) occurs with multiplicity 1 in \( S^m(\Pi_2) \) of \( A_{2m-1} \) (see [2, page 71, Behauptung 1]).
Reducible good representations of semisimple Lie algebras $A_r$ and $B_r$

$\rho = \Pi_1 + \Pi_2$ of $A_{r-1}$ is also bad in $A_r$ according to “Kriterium 1” (see [2, page 13]).

b) $L = A_{r-1} = A_{2m}$.

$$S^{m+1}(\Pi_1 + \Pi_2) = S^{m+1}(\Pi_1) + S^m(\Pi_1) \otimes \Pi_2 + \cdots + \Pi_1 \otimes S^m(\Pi_2) + S^{m+1}(\Pi_2).$$

The component $\Pi_{2m}$ occurs in $S^m(\Pi_2)$ of $A_{2m}$ (see [2, page 79, Behauptung 2]).

$\Pi_1 \otimes S^m(\Pi_2)$ contains the component $\Pi_1 \otimes \Pi_{2m}$ and $\Pi_1 \otimes \Pi_{2m} = \Pi_1 \Pi_{2m} + \Pi_0$. The trivial component $\Pi_0$ occurs in $S^{m+1}(\Pi_1 + \Pi_2)$. According to Kriterium 1 $\rho = \Pi_1 + \Pi_2$ of $A_{r-1}$ is bad in $A_r$.

$\rho = \Pi_2 + \Pi_2$.

a) $L = A_r = A_{2m-1}$. The case is similar to $\rho = \Pi_1 + \Pi_2$ of $A_{2m}$.

b) $L = A_r = A_{2m}$.

The null vector $\delta$ occurs in $A_{2m-1}$. Hence it has $\Delta$ as its support in the root diagram according to Facts 1.3 (see [2, page 11]). Therefore $\rho = \Pi_2 + \Pi_2$ of $A_{2m}$ is bad.

The proof of the cases $\rho = \Pi_1 + \Pi_{r-1}, \Pi_2 + \Pi_{r-1}, \Pi_{r-1}, \Pi_{r-1}, \Pi_r, \Pi_r + \Pi_{r-1}$ is similar to the proof for $\rho = \Pi_1 + \Pi_2$ or $\rho = \Pi_2 + \Pi_2$ ($\Pi'_1 = \Pi_r, \Pi'_2 = \Pi_{r-1}$).

\[ \square \]

Lemma 2.3. (Case $B_r$) Consider the following list of reducible representations of $B_r$.

R-list (B-2): $\Pi_1 + \Pi_1$ of $B_r$, $r \geq 3$.

All other reducible representations $\rho = \tau_1 + \tau_2$ are bad, where $\tau_1, \tau_2$ are from List(B) (i.e., $\Pi_1 + \Pi_3, \Pi_3 + \Pi_3$ of $B_3$, $\Pi_1 + \Pi_4, \Pi_4 + \Pi_4$ of $B_4$).

Proof. Let $n = 3$.

The reducible representations $\Pi_1 + \Pi_3, \Pi_3 + \Pi_3$ of $B_3$ are bad according to the R-list 3.2.1.

Let $n = 4$.

The reducible representations $\Pi_1 + \Pi_4, \Pi_4 + \Pi_4$ of $B_4$ are bad according to the R-list 3.2.2.

\[ \square \]

3. Reducible good representations

Theorem 3.1. (R-good, 2-sum) Let $L$ be a simple Lie algebra and let $\rho = \tau_1 + \tau_2$ be a reducible representation with $\tau_1, \tau_2$ good. Then $\rho$ is good, if $\rho$ is in the following list:

(i) $A_r$: $\Pi_1 + \Pi_1, \Pi_1 + \Pi_r, \Pi_r + \Pi_r$ of $A_r$, $r \geq 3$ and $\Pi_2 + \Pi_2$ of $A_3$.

(ii) $B_r$: $\Pi_1 + \Pi_1$ of $B_r$, $r \geq 3$.

Proof. The statement of the theorem follows in the Lemmas 3.2 and 3.3.

\[ \square \]

Lemma 3.2. (Case $A_r$) Let $L \cong A_r$, $r \geq 3$, and let $\rho = \tau_1 + \tau_2$ be a reducible representation of $L$ with $\tau_1, \tau_2$ good.

(1) The irreducible components $\lambda$ which occur in $S^n(\Pi_1 + \Pi_1)$ are exactly those from the type $\lambda = \Pi_1^{m_1} \Pi_2^{m_2}$, $m_1, m_2 \in \mathbb{N}_0$. Hence $\text{rank}(\Pi_1 + \Pi_1, A_r) = 2$ and $\Pi_1 + \Pi_1$ of $A_r$ is good.

(2) The irreducible components $\lambda$ which occur in $S^n(\Pi_1 + \Pi_r)$ are exactly those from the type $\lambda = \Pi_1^{m_1} \Pi_r^{m_2}$, $m_1, m_2 \in \mathbb{N}_0$. Hence $\text{rank}(\Pi_1 + \Pi_r, A_r) = 2$ and $\Pi_1 + \Pi_r$ of $A_r$ is good.

(3) The irreducible components $\lambda$ which occur in $S^n(\Pi_r + \Pi_r)$ are exactly those from the type $\lambda = \Pi_1^{m_1} \Pi_r^{m_2}$, $m_1, m_2 \in \mathbb{N}_0$. Hence $\text{rank}(\Pi_r + \Pi_r, A_r) = 2$ and $\Pi_r + \Pi_r$ of $A_r$ is good.

Proof. The case (1): According to Panyushev (see [2, page 47], [7]) $\text{rank}(\Pi_1 + \Pi_1, A_r) = \text{rank}A_r - \text{rank}K$, and $K = A_{r-2}$ (see [5],[6]).

$$\text{rank}(\Pi_1 + \Pi_1, A_r) = \text{rank}A_r - \text{rank}A_{r-2} = r - (r-2) = 2, \quad (3.1)$$

$$S^2(\Pi_1 + \Pi_1) = S^2(\Pi_1) + S^1(\Pi_1) \otimes S^1(\Pi_1) = 2S^2(\Pi_1) + \Pi_1 \otimes \Pi_1,$$

$$\Pi_1 \otimes \Pi_1 = \Pi_1^2 + \Pi_2 \text{ with deg } \Pi_2 = 2, \quad (3.2)$$
According to (1) and (2) the irreducible components \( \lambda \) which occur in \( S^n(\Pi_1 + \Pi_1) \) are from the type \( \lambda = \Pi^{m_1}_1 \Pi^{m_2}_3 \).

The case (2): According to Panyushev \( \text{rank}(\Pi_1 + \Pi_r, A_r) = \text{rank}A_r - \text{rank}K \) and \( \Pi^1_1 = \Pi_r.

\[
\text{rank}(\Pi_1 + \Pi_r, A_r) = \text{rank}A_r - \text{rank}A_{r-2} = r - (r - 2) = 2.
\]

(3.3)

According to (3) the irreducible components \( \lambda \) which occur in \( S^n(\Pi_1 + \Pi_r) \) are from the type \( \lambda = \Pi^{m_1}_1 \Pi^{m_2}_3 \).

The case (3): According to Panyushev

\[
\text{rank}(\Pi_1 + \Pi_r, A_r) = \text{rank}A_r - \text{rank}K \quad \text{and} \quad \Pi^1_1 = \Pi_r.
\]

\[
\text{rank}(\Pi_r + \Pi_r, A_r) = \text{rank}A_r - \text{rank}A_{r-2} = r - (r - 2) = 2,
\]

(3.4)

\[
S^2(\Pi_r + \Pi_r) = S^2(\Pi_r) + S^1(\Pi) \otimes S^1(\Pi_r) + S^2(\Pi_r) + \Pi_r \otimes \Pi_r
\]

(3.5)

According to (4) and (5) the irreducible components \( \lambda \) which occur in \( S^n(\Pi_r + \Pi_r) \) are from the type \( \lambda = \Pi^{m_1}_1 \Pi^{m_2}_3 \).

Lemma 3.3. (Case \( B_r \)) Let \( L \cong B_r, r \geq 3 \), and let \( \rho = \tau_1 + \tau_2 \) be a reducible representation with \( \tau_1, \tau_2 \) good. Then the irreducible components \( \lambda \) which occur in \( S^n(\Pi_1 + \Pi_1) \) are exactly those from the type \( \lambda = \Pi^{m_1}_1 \Pi^{m_2}_3, m_1, m_2 \in \mathbb{N}_0 \). Hence \( \text{rank}(\Pi_1 + \Pi_1, B_r) = 2 \) and \( \Pi_1 + \Pi_1 \) of \( B_r \) is good.

**Proof.** The statement follows similarly to the case (1) of Lemma 3.2 for \( A_r \) with \( K = B_r \).

Theorem 3.4. (R-good, 3-sum) Let \( L \) be a simple Lie algebra and let \( \rho = \tau_1 + \tau_2 + \tau_3 \) be a reducible representation with \( \tau_1, \tau_2, \tau_3 \) good. Then \( \rho \) is good, if it is in the following list:

(i) \( A_r \): \( \Pi_1 + \Pi_1 + \Pi_1, \Pi_1 + \Pi_1 + \Pi_r, \Pi_1 + \Pi_r + \Pi_r, \Pi_r + \Pi_r + \Pi_r \) of \( A_r, r \geq 4 \).

(ii) \( B_r \): \( \Pi_1 + \Pi_1 + \Pi_1 \) of \( B_r, r \geq 4 \).

**Proof.** The statement of the theorem follows from Lemmas 3.5 and 3.6.

Lemma 3.5. (Case \( A_r \)) Let \( L \cong A_r, r \geq 4 \), and let \( \rho = \tau_1 + \tau_2 + \tau_3 \) be a reducible representation of \( L \) with \( \tau_1, \tau_2, \tau_3 \) good.

(1) The irreducible components \( \lambda \) which occur in \( S^n(\Pi_1 + \Pi_1 + \Pi_1) \) are exactly those from the type \( \lambda = \Pi^{m_1}_1 \Pi^{m_2}_3 \Pi^{m_3}_3, m_1, m_2, m_3 \in \mathbb{N}_0 \). Hence \( \text{rank}(\Pi_1 + \Pi_1 + \Pi_1, A_r) = 3 \) and \( \Pi_1 + \Pi_1 + \Pi_1 \) of \( A_r \) is good.

(2) The irreducible components \( \lambda \) which occur in \( S^n(\Pi_r + \Pi_r + \Pi_r) \) are exactly those from the type \( \lambda = \Pi^{m_1}_1 \Pi^{m_2}_3 \Pi^{m_3}_3, m_1, m_2, m_3 \in \mathbb{N}_0 \). Hence \( \text{rank}(\Pi_r + \Pi_r + \Pi_r, A_r) = 3 \) and \( \Pi_r + \Pi_r + \Pi_r \) of \( A_r \) is good.

(3) The irreducible components \( \lambda \) which occur in \( S^n(\Pi_1 + \Pi_1 + \Pi_r) \) are exactly those from the type \( \lambda = \Pi^{m_1}_1 \Pi^{m_2}_3 \Pi^{m_3}_3, m_1, m_2, m_3 \in \mathbb{N}_0 \). Hence \( \text{rank}(\Pi_1 + \Pi_1 + \Pi_r, A_r) = 3 \) and \( \Pi_1 + \Pi_1 + \Pi_r \) of \( A_r \) is good.

(4) The irreducible components \( \lambda \) which occur in \( S^n(\Pi_r + \Pi_r + \Pi_r) \) are exactly those from the type \( \lambda = \Pi^{m_1}_1 \Pi^{m_2}_3 \Pi^{m_3}_3, m_1, m_2, m_3 \in \mathbb{N}_0 \). Hence \( \text{rank}(\Pi_r + \Pi_r + \Pi_r, A_r) = 3 \) and \( \Pi_r + \Pi_r + \Pi_r \) of \( A_r \) is good.

**Proof.** The case (1): According to Panyushev (see [2, page 47], [7]) \( \text{rank}(\Pi_1 + \Pi_1 + \Pi_1, A_r) = \text{rank}A_r - \text{rank}K \), and \( K = A_r \) (see [3, 6]).

\[
\text{rank}(\Pi_1 + \Pi_1 + \Pi_1, A_r) = \text{rank}A_r - \text{rank}A_{r-3} = r - (r - 3) = 3.
\]

(3.6)

In \( S^2(\Pi_1 + \Pi_1 + \Pi_1) = S^2(\Pi_1 + \Pi_1) + S^1(\Pi_1 + \Pi_1) \otimes S^1(\Pi_1) + S^2(\Pi_1) \)

\[
S^3(\Pi_1 + \Pi_1 + \Pi_1) = S^3(\Pi_1 + \Pi_1) + S^3(\Pi_1 + \Pi_1) \otimes S^1(\Pi_1 + \Pi_1) + S^1(\Pi_1 + \Pi_1) \otimes S^3(\Pi_1) + S^3(\Pi_1, S^2(\Pi_1 + \Pi_1) \otimes S^1(\Pi_1) = (3 \Pi^2_1 + \Pi_2) \otimes \Pi_1 = 3 \Pi^2_1 \otimes \Pi_1 + \Pi_2 \otimes \Pi_1 + \Pi_2 \otimes \Pi_1
\]

(3.7)

\[
S^3(\Pi_1 + \Pi_1 + \Pi_1) = S^3(\Pi_1 + \Pi_1) + S^3(\Pi_1 + \Pi_1) \otimes S^3(\Pi_1) + S^3(\Pi_1 + \Pi_1) \otimes S^3(\Pi_1) + S^3(\Pi_1, S^3(\Pi_1)
\]

(3.8)
According to (6), (7), and (8) the irreducible components $\lambda$ which occur in $S^n(\Pi_1 + \Pi_1 + \Pi_1)$ are from the type $\lambda = \Pi_1^{m_1} \Pi_2^{m_2} \Pi_3^{m_3}$.

The proof for $(\Pi_1 + \Pi_1 + \Pi_1, A_r)$ is similar to $(\Pi_1 + \Pi_1 + \Pi_1, A_r)$, since $\Pi_1 = \Pi_r$. The irreducible components $\lambda$ which occur in $S^n(\Pi_1 + \Pi_1 + \Pi_1)$ are from the type $\lambda = \Pi_1^{m_1} \Pi_2^{m_2} \Pi_3^{m_3}$.

The proofs of the cases (3) and (4) are similar to the proofs of the cases (1) and (2). □

**Lemma 3.6.** (Case $B_r$) Let $L \cong B_r$, $r \geq 4$, and let $\rho = \tau_1 + \tau_2 + \tau_3$ be a reducible representation with $\tau_1, \tau_2, \tau_3$ good. The irreducible components $\lambda$ which occur in $S^n(\Pi_1 + \Pi_1 + \Pi_1)$ are exactly those from the type $\lambda = \Pi_1^{m_1} \Pi_2^{m_2} \Pi_3^{m_3}$, $m_1, m_2, m_3 \in \mathbb{N}_0$. Hence $\text{rank}(\Pi_1 + \Pi_1 + \Pi_1, B_r) = 3$ and $\Pi_1 + \Pi_1 + \Pi_1$ of $B_r$ is good.

**Proof.** The arguments for the proof are similar to those for the proof for $(\Pi_1 + \Pi_1 + \Pi_1, A_r)$ with $K = B_{r-3}$. □

**Theorem 3.7.** (R-good, t-sum) Let $L$ be a simple Lie algebra and let $\rho = \tau_1 + \tau_2 + \cdots + \tau_t$ be a reducible representation with $\tau_1, \tau_2, \ldots, \tau_t$ good. Then $\rho$ is good, if it is in the following list:

(i) $A_r$: $\Pi_1 + \Pi_1 + \cdots + \Pi_1, \Pi_r + \Pi_r + \cdots + \Pi_r$ (t-sum), $t < r \geq 3$, and

\[
\begin{array}{c}
\Pi_1 + \Pi_1 + \cdots + \Pi_1 + \Pi_r + \Pi_r + \cdots + \Pi_r, \\
(\text{t times}) \quad (\text{t times})
\end{array}
\]

(ii) $B_r$: $\Pi_1 + \Pi_1 + \cdots + \Pi_1$ (t-sum), $t < r \geq 3$.

**Proof.** The statement follows from Lemma 3.8. □

**Lemma 3.8.** (Case $A_r$) Let $L$ be a simple Lie algebra, let $r \geq 5$ and let $\rho = \tau_1 + \tau_2 + \cdots + \tau_t$ be a reducible representation of $L$ with $\tau_1, \tau_2, \ldots, \tau_t$ good, $t < r$.

(1) $L \cong A_r$ (or $L \cong B_r$)

The irreducible components $\lambda$ which occur in $S^n(\Pi_1 + \Pi_1 + \cdots + \Pi_1)$ (t-sum) are exactly those from the type $\lambda = \Pi_1^{m_1} \Pi_2^{m_2} \cdots \Pi_k^{m_k}$ with $m_1, m_2, \ldots, m_t \in \mathbb{N}_0$. Hence $\text{rank}(\Pi_1 + \Pi_1 + \cdots + \Pi_1, A_r(B_r)) = t < r$

and $\Pi_1 + \Pi_1 + \cdots + \Pi_1$ of $A_r$ (or of $B_r$) is good.

(2) (a) $L \cong A_r$

The irreducible components $\lambda$ which occur in $S^n(\Pi_r + \Pi_r + \cdots + \Pi_r)$ (t-sum) are exactly those from the type $\lambda = \Pi_1^{m_1} \Pi_2^{m_2} \cdots \Pi_{r-1}^{m_{r-1}} \Pi_{r+1}^{m_{r+1}} \cdots \Pi_k^{m_k}$ with $m_1, m_2, \ldots, m_t \in \mathbb{N}_0$. Hence $\text{rank}(\Pi_r + \Pi_r + \cdots + \Pi_r, A_r) = t < r$ and $\Pi_r + \Pi_r + \cdots + \Pi_r$ of $A_r$ is good.

(b) The irreducible components $\lambda$ which occur in $S^n(\Pi_1 + \Pi_1 + \cdots + \Pi_1) + \Pi_1 + \Pi_r + \cdots + \Pi_r + \cdots + \Pi_r)$ (k+s)-sum are exactly those from the type $\lambda = \Pi_1^{m_1} \Pi_2^{m_2} \cdots \Pi_k^{m_k} \Pi_{r-1}^{m_{r-1}} \Pi_{r+1}^{m_{r+1}} \cdots \Pi_{r+s}^{m_r}$ with $m_1, m_2, \ldots, m_k$ and $n_1, n_2, \ldots, n_s \in \mathbb{N}_0$. Hence $\text{rank}(\Pi_1 + \Pi_1 + \cdots + \Pi_1) + \Pi_1 + \Pi_r + \cdots + \Pi_r, A_r) = k + s = t < r$. 


and \( \Pi_1 + \Pi_1 + \cdots + \Pi_1 \) \( k \) times \( + \Pi_t + \Pi_r + \cdots + \Pi_r \) \( s \) times of \( A_r \) is good.

**Proof.** The case (1): According to Panyushev (see [2, page 47], [7]) rank(\( \Pi_1 + \Pi_2 + \cdots + \Pi_t, A_r \)) = rankA_r - rankK, and \( K = A_r - t \) (see [1,6]).

\[
\text{rank}(\Pi_1 + \Pi_2 + \cdots + \Pi_t, A_r) = \text{rank}A_r - \text{rank}A_r - t = r - (r - t) = t,
\]

(3.9)

\[
\pi_i \otimes \pi_j = \pi_i \pi_j + \pi_{i-1} \pi_{j+1} + \cdots + \begin{cases} \pi_{i+j}, & r + 1 - j - i \geq 0, \\ \pi_{j+i-r-1}, & r + 1 - j - i \leq 0, \end{cases}
\]

(3.10)

\( r \geq 2, i \leq j \). According to (9) and (10) the irreducible components which occur in \( S^n(\Pi_1 + \Pi_1 + \cdots + \Pi_1) \) are from the type \( \lambda = \Pi_1^{m_1} \Pi_2^{m_2} \cdots \Pi_r^{m_r} \).

The proof of the case (2a) is similar to (1) (since \( \Pi_i^s = \Pi_{r+1-i}, i = 1, 2, \ldots, r \)).

The case (2b):

\[
\text{rank}(\Pi_1 + \Pi_1 + \cdots + \Pi_1 + \Pi_r + \Pi_r + \cdots + \Pi_r, A_r) = \text{rank}A_r - \text{rank}A_r - (k+s) = k+s = t < r.
\]

(3.11)

According to (10) and (11) the irreducible components which occur in \( S^n(\Pi_1 + \Pi_1 + \cdots + \Pi_1 + \Pi_r + \Pi_r + \cdots + \Pi_r) \) are from the type \( \lambda = \Pi_1^{m_1} \Pi_2^{m_2} \cdots \Pi_k^{m_k} \Pi_r^{m_r} \Pi_{r-1}^{m_{r-1}} \cdots \Pi_{r+1-s}^{m_{r+1-s}} \). \( \square \)

**Theorem 3.9.** (Classification) The following lists of \( \rho \) and for \( L = A_r, L = B_r \) contain all reducible good representations and their ranks.

1. \( L = A_r \) and \( \rho = \Pi_1 + \Pi_1, \rho = \Pi_1 + \Pi_r, \rho = \Pi_r + \Pi_r, r \geq 3, \)
   
   (i) \( L = A_3 \) and \( \rho = \Pi_2 + \Pi_2, \)
   
   (ii) \( L = A_r \) and \( \rho = \Pi_1 + \Pi_1, r \geq 3. \)
   
   Then \( \text{rank}M(\rho) = 2. \)

2. \( L = B_r \) and \( \rho = \Pi_1 + \Pi_1 + \Pi_1, \rho = \Pi_1 + \Pi_1 + \Pi_r, \rho = \Pi_1 + \Pi_r + \Pi_r, \rho = \Pi_r + \Pi_r + \Pi_r, \)
   
   (i) \( L = B_4 \) and \( \rho = \Pi_1 + \Pi_1 \).
   
   Then \( \text{rank}M(\rho) = 3. \)

3. \( L = A_r \) and \( \rho = \Pi_1 + \Pi_1 + \Pi_1 + \Pi_1, \rho = \Pi_1 + \Pi_1 + \Pi_1 + \Pi_1, \rho = \Pi_1 + \Pi_1 + \Pi_1 + \Pi_r, \rho = \Pi_1 + \Pi_r + \Pi_r + \Pi_r, \rho = \Pi_r + \Pi_r + \Pi_r + \Pi_r, \)
   
   (i) \( L = B_r \) and \( \rho = \Pi_1 + \Pi_1 + \Pi_1 \).
   
   Then \( \text{rank}M(\rho) = 4. \)

4. \( L = A_r \) or \( L = B_r, \rho = \Pi_1 + \Pi_1 + \cdots + \Pi_1 \) \( t \)-times, \( t < r, \)
   
   (i) \( L = A_r \) and \( \rho = \Pi_r + \Pi_r + \cdots + \Pi_r \) \( t \)-times, \( t < r, \)
   
   (ii) \( L = A_r, r \geq 6, \rho = \Pi_r + \Pi_r + \cdots + \Pi_r \) \( t \)-times, \( t < r, \)
\[ \rho = \Pi_1 + \Pi_1 + \cdots + \Pi_1 + \Pi_r + \Pi_r + \cdots + \Pi_r \text{ \((k + s)\)-sum, } k + s = t < r. \]

Then \( \text{rank} M(\rho) = t. \)

**Proof.** Everything follows from Lemmas 2.2, 2.3, 3.2, 3.3, 3.5, 3.6, 3.8. \( \square \)

Here we summarize our results again. We order the reducible good representations but on a different principle, namely after rank.

**Corollary 3.10.** Let \( L \cong A_r \) and \( \rho = \tau_1 + \tau_2 \) be a reducible representation of \( A_r \) with \( \tau_1, \tau_2 \) good.

1. For \( A_r, r \geq 3: \)
   a. \( S^n(\Pi_1 + \Pi_1) = \sum_{m_1, m_2} (m_1 + 1)\Pi_1^{m_1}\Pi_2^{m_2} \) with \( m_1 + 2m_2 = n, m_1, m_2 \in \mathbb{N}_0, \)
   b. \( S^n(\Pi_1 + \Pi_r) = \sum_{m_1, m_2} \Pi_1^{m_1}\Pi_r^{m_2} \) with \( m_1 + m_2 = n - 2k, k = 0, 1, \ldots, \left\lceil \frac{n}{2} \right\rceil, \)
   c. \( S^n(\Pi_r + \Pi_r) = \sum_{m_1, m_2} (m_1 + 1)\Pi_r^{m_1}\Pi_r^{m_2} \) with \( m_1 + 2m_2 = n, m_1, m_2 \in \mathbb{N}_0, \)

2. For \( A_3: \)
   \[ S^n(\Pi_2 + \Pi_2) = \frac{(k + 1)(k + 2)}{2} \sum_{m_1, m_2} (m_1 + 1)\Pi_2^{m_1}(\Pi_1\Pi_3)^{m_2} \]
   with \( m_1 + 2m_2 = n - 2k, k = 0, 1, \ldots, \left\lceil \frac{n}{2} \right\rceil. \)

**Corollary 3.11.** Let \( L \cong B_r \) and \( \rho = \tau_1 + \tau_2 \) be a reducible representation of \( B_r, r \geq 3, \) with \( \tau_1, \tau_2 \) good. Then

\[ S^n(\Pi_1 + \Pi_1) = \frac{(k + 1)(k + 2)}{2} \sum_{m_1, m_2} (m_1 + 1)\Pi_1^{m_1}\Pi_2^{m_2} \]

with \( m_1 + 2m_2 = n - 2k, k = 0, 1, \ldots, \left\lceil \frac{n}{2} \right\rceil. \)

4. R-List 3

**R-List 3.1 A_r:**

**R-List 3.1.1 A_3:**

\[ S^2(\pi_1 + \pi_2) = \pi_1^2 + \pi_2^2 + \pi_3 + \pi_1\pi_2 + \pi_0, \]
\[ S^2(\pi_2 + \pi_3) = \pi_2^2 + \pi_3^2 + \pi_1 + \pi_2\pi_3 + \pi_0; \]

**R-List 3.1.2 A_4:**

\[ S^2(\pi_1 + \pi_2) = \pi_1^2 + \pi_2^2 + \pi_3 + \pi_4 + \pi_1\pi_2, \]
\[ S^2(\pi_1 + \pi_3) = \pi_1^2 + \pi_3^2 + \pi_1 + \pi_1\pi_3, \]
\[ S^3(\pi_1 + \pi_3) = \pi_1^3 + \pi_3^3 + \pi_1\pi_3^2 + \pi_1^2\pi_3 + \pi_1\pi_3 + \pi_3\pi_4 + \pi_3^2\pi_4 + \pi_1^2 + \pi_2, \]
\[ S^2(\pi_2 + \pi_3) = 3\pi_2^2 + 3\pi_4 + \pi_1\pi_3, \]
\[ S^3(\pi_2 + \pi_3) = 4\pi_3^2 + 6\pi_2\pi_4 + 2\pi_1\pi_3\pi_4 + 2\pi_1, \]
\[ S^4(\pi_2 + \pi_3) = 5\pi_3^4 + 9\pi_2^2\pi_4 + 6\pi_2\pi_3^2 + 3\pi_1\pi_2\pi_3\pi_4 + 3\pi_3^3 + 3\pi_1\pi_3\pi_4 + 4\pi_1\pi_2 + \pi_1^2\pi_3^2, \]
\[ S^2(\pi_2 + \pi_3) = \pi_2^2 + \pi_3^2 + \pi_1 + \pi_4 + \pi_2\pi_3 + \pi_0, \]
\[ S^2(\pi_2 + \pi_4) = \pi_2^2 + \pi_4^2 + \pi_1 + \pi_4 + \pi_2\pi_4, \]
\[ S^3(\pi_2 + \pi_4) = \pi_2^3 + \pi_4^3 + \pi_2^2\pi_4 + \pi_2\pi_4 + \pi_1\pi_2 + \pi_1\pi_4 + \pi_2\pi_4 + \pi_4^2, \]
\[ S^2(\pi_3 + \pi_3) = \pi_3^2 + 3\pi_1 + \pi_2\pi_4, \]
\[ S^3(\pi_3 + \pi_3) = 4\pi_3^3 + 6\pi_1\pi_3 + 2\pi_2\pi_3\pi_4 + 2\pi_4, \]
\[ S^4(\pi_3 + \pi_3) = 5\pi_3^4 + 9\pi_1\pi_3^2 + 6\pi_1^2\pi_3 + 3\pi_2\pi_3^2\pi_4 + 3\pi_2 + 3\pi_1\pi_2\pi_4 + 4\pi_3\pi_4 + \pi_2^2\pi_4, \]
\[ S^2(\pi_3 + \pi_4) = \pi_3^2 + \pi_4^2 + \pi_2 + \pi_1 + \pi_3\pi_4; \]

**R-List 3.1.3 A_5:**

Reductive good representations of semisimple Lie algebras \( A_r \) and \( B_r \)}

1153
\[ S^2(\pi_1 + \pi_2) = \pi_1^2 + \pi_2^2 + \pi_3 + \pi_4 + \pi_1 \pi_2, \]
\[ S^3(\pi_1 + \pi_2) = \pi_1^3 + \pi_2^3 + \pi_1 \pi_2 + \pi_1 \pi_3 + \pi_2 \pi_3 + \pi_1 \pi_4 + \pi_2 \pi_4 + \pi_1 \pi_2 \pi_3 + \pi_1 \pi_2 \pi_4 + \pi_1 \pi_3 \pi_2 + \pi_1 \pi_3 \pi_4 + \pi_2 \pi_3 \pi_4 + \pi_1 \pi_2 \pi_3 \pi_4, \]
\[ S^2(\pi_1 + \pi_3) = \pi_1^2 + \pi_3^2 + \pi_1 \pi_3 + \pi_1 \pi_4 + \pi_3 \pi_4, \]
\[ S^3(\pi_1 + \pi_3) = \pi_1^3 + \pi_3^3 + \pi_1 \pi_3 + \pi_1 \pi_2 \pi_3 + \pi_2 \pi_3 \pi_4 + \pi_1 \pi_4 + \pi_2 \pi_4 + \pi_1 \pi_2 \pi_3 \pi_4, \]
\[ S^2(\pi_2 + \pi_3) = \pi_2^2 + \pi_3^2 + \pi_1 \pi_2 + \pi_1 \pi_3 + \pi_2 \pi_3, \]
\[ S^3(\pi_2 + \pi_3) = \pi_2^3 + \pi_3^3 + \pi_1 \pi_2 + \pi_1 \pi_3 + \pi_2 \pi_3 + \pi_1 \pi_2 \pi_3 \pi_4, \]
\[ S^2(\pi_3 + \pi_4) = \pi_3^2 + \pi_4^2 + \pi_1 \pi_3 + \pi_1 \pi_4 + \pi_3 \pi_4, \]
\[ S^3(\pi_3 + \pi_4) = \pi_3^3 + \pi_4^3 + \pi_1 \pi_3 + \pi_1 \pi_4 + \pi_3 \pi_4 + \pi_1 \pi_2 \pi_3 \pi_4, \]
\[ S^2(\pi_4 + \pi_1) = \pi_4^2 + \pi_1^2 + \pi_2 + \pi_3 + \pi_1 \pi_2 + \pi_1 \pi_3 + \pi_2 \pi_3 + \pi_1 \pi_4 + \pi_2 \pi_4 + \pi_1 \pi_2 \pi_3 + \pi_1 \pi_2 \pi_4 + \pi_1 \pi_3 \pi_2 + \pi_1 \pi_3 \pi_4 + \pi_2 \pi_3 \pi_4 + \pi_1 \pi_2 \pi_3 \pi_4, \]
\[ S^3(\pi_4 + \pi_1) = \pi_4^3 + \pi_1^3 + \pi_2 + \pi_3 + \pi_1 \pi_2 + \pi_1 \pi_3 + \pi_2 \pi_3 + \pi_1 \pi_4 + \pi_2 \pi_4 + \pi_1 \pi_2 \pi_3 + \pi_1 \pi_2 \pi_4 + \pi_1 \pi_3 \pi_2 + \pi_1 \pi_3 \pi_4 + \pi_2 \pi_3 \pi_4 + \pi_1 \pi_2 \pi_3 \pi_4. \]
Reducible good representations of semisimple Lie algebras $A_r$ and $B_r$.


