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Research Article

Stochastic comparison bounds for an $M_1, M_2/G_1, G_2/1$ retrial queue with two way communication

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Abstract

The main goal in the present paper is to provide a technique that considers the stochastic comparison approach for investigating monotonicity and comparability of an $M_1, M_2/G_1, G_2/1$ retrial queues with two way communication. This approach is developed for comparing a non Markov process to Markov process with many possible stochastic orderings. Particularly, we show the monotonicity of the transition operator of the embedded Markov chain relative to the strong stochastic ordering and convex ordering, as well as the comparability of two transition operators. Bounds are also obtained for the stationary distribution of the number of customers at departure epochs. Additionally, the performance measures of the system considered can be estimated by those of an $M_1, M_2/M_1, M_2/1$ retrial queue with two way communication when the service time distribution is NBUE (respectively NWUE). Finally, we validate stochastic comparison results by presenting a numerical example illustrating the interest of the approach.

Keywords: Retrial queues, Two way communication, Markov chain, Monotonicity, Stochastic comparison, Simulation.

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1. Introduction

Retrial queues arise from various real life situations as well as telecommunication and networks systems, especially in the cognitive radio network and the manufacturing systems [23]. In most publications on retrial queues, the server provides service to the ingoing arrivals made by regular customers [2, 24]. However, there are real-life situations, such as call center, where an operator not only serves incoming calls but it also makes outgoing calls to the outside when the server is free [1]. This type of model is known as retrial queues with two way communication [3, 4, 18].

Given the complexity of this type of model, it becomes difficult to perform quantitative analysis. In fact, using approximation methods is essential to deal with the complexity of these systems. Sakurai and Phung-Duc [18] used a mathematical method based on generating function approach to obtain explicit expressions for the joint stationary distribution of the number of calls in the orbit and the state of the server of a two way communication retrial queues with multiple types of outgoing calls whose durations follow distinct exponential distributions. Ouazine and Abbas [17] proposed a numerical approach based on Taylor series expansion with a statistical aspect for analyzing the stationary performances of the $M_1, M_2/G_1, G_2/1$ retrial queues with two way communication with one type of outgoing calls, while our work is focused on the application of stochastic comparisons of Markov chains, in order to derive bounds for the performance indices of the same model based on results achieved by Boualem [5], Boualem et al. [6, 7, 8, 9, 10, 11, 12, 13], Khalil et Falin [14], Liang [15], Liang and Kulkarni [16] and Stoyan [21].

Qualitative properties of stochastic models constitute an important theoretical basis for approximation methods. Monotonicity properties of performance measures are useful for understanding and solving optimization problems of queueing systems. Concerning the monotonicity properties, few results were derived. This is due to the mathematical complexities of such problems. Liang and Kulkarni [16] studied the monotonicity properties of retrial queues to show how the retrial time distribution affects the behavior of the system. They assume that retrials have a phase type distributions and show that systems with longer retrial times, with respect to the K-dominance create more customers in the system and in orbit. From these results, they obtained the monotonicity properties of several performance measures. Falin and Khalil [14] investigated the monotonicity properties of an $M/G/1$ retrial queue with exponential retrial times and linear retrial rate relatively to stochastic, convex and Laplace orderings. Shin [20] dealt with several multi-server queueing models with exponential retrial times like $A^X/G/c/K$ retrial queue, two nodes tandem retrial queue $A^X/G/c_1/K_1 \rightarrow ./G/c_2/K_2$, MAP_1, $MAP_2/M/c$ retrial queue and $M/M/c/c$ retrial queue with negative arrivals. He showed that the transition operator is monotone if the retrial rates in one system are bounded by the retrial rates in the second one. The monotonicity of results are applied to show the convergence of generalized truncated systems to the original one. Boualem et al. [11] considered an $M/G/1$ retrial queue with vacations and derived several stochastic comparison properties in the sense of strong stochastic ordering and convex ordering. Taleb and Aissani [22] analyzed the monotonicity of the major performance measures with respect to strong stochastic ordering and increasing convex ordering. They discussed the conditions under which the comparison of two unreliable $M/G/1$ retrial queues is performed. The model is compared with a simpler counterpart of unreliable $M/M/1$ retrial queue, while Boualem et al. [12] used this method to get some qualitative approximations for the $M/G/1$ retrial queue with Bernoulli feedback. The stochastic inequalities provide simple insensitive bounds for the arrival rate, service time distributions and retrial parameter and used it in Boualem et al. [13] to investigate various monotonicity properties of a single server retrial queue with First Come First Served (FCFS) orbit and general retrial times. Boualem [5] investigated monotonicity properties of the single server retrial queue with no waiting room and server subject to active breakdowns. These two last works focused mainly on the stochastic bounds for the stationary distribution of the embedded Markov chain related to the model under study.

The remainder of this paper is organized as follows. Section 2 presents the mathematical model formulation. Some useful preliminary results are elaborated in Section 3. Section 4 investigates the monotonicity properties of the embedded Markov chain. Section 5 shows the comparability conditions of stationary distributions of the number of customers in the system. A numerical example illustrating the interest of the approach is provided in Section 6. Finally, a conclusion is presented in the last Section.

2. Mathematical model description

We consider a single server queueing system at which primary ingoing customers arrive according to a Poisson process with rate λ. In addition, the server makes an outgoing call after an exponentially distributed idle time with rate α. We assume that ingoing calls and outgoing calls receive different service times. In the sequel $B_1(x)$ $(B_1(0)=0)$ represents the service time distribution of an ingoing call, while $B_2(x)$ $(B_2(0)=0)$ denotes the service time distribution of an outgoing call. An ingoing call that finds the server busy joins the orbit and it retries to enter the server after an exponentially distributed time with rate μ, so if $(N(t)=j)$, then the current retrial rate is $j\mu$. Also denote the Laplace-Stieltjes transform and the k^{th} moment of $B_l(x)$ as $\beta_l(s)$ and β_l^k respectively, for $l=1,2$ and $k\in\mathbb{Z}_+$, where $\mathbb{Z}_+=\{0,1,2,...\}$. The model under consideration is schematically represented in Figure 1.

The arrival flows of ingoing and outgoing calls, service times and intervals between successive repeated attempts are assumed to be mutually independent. The state of the

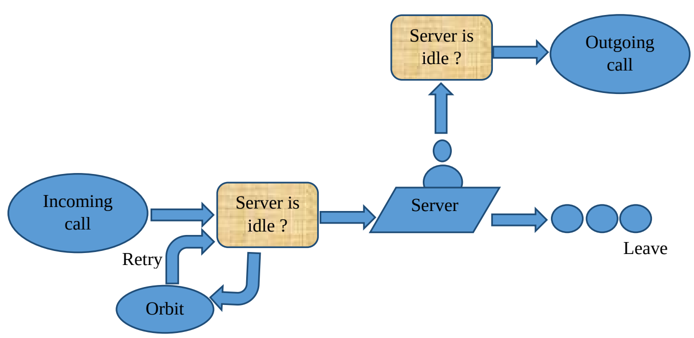

Figure 1. Retrial queue with two way communication.

system at time t can be described by the process
$Y(t)=(C(t),N(t),\xi(t))_{t\geq 0}$, where:

$$C(t)=\begin{cases}0, & \text{if the server is idle at time } t,\\ 1, & \text{if the server is busy with an ingoing service at time } t,\\ 2, & \text{if the server is calling outside at time } t.\end{cases}$$

If $C(t)\in\{1,2\}$, then $\xi(t)$ represents the elapsed time of the (ingoing or outgoing) service in progress. However, the embedded Markov chain $Z_n=N(\xi_n^+)$ associated with this model represents the number of customers in the orbit at the service completion epochs $\{\xi_n, n \in \mathbb{N}\}$ of either an incoming call or an outgoing call.

(2.1) $\quad Z_n = Z_{n-1} - W_n + V_n,$

where V_n is the number of incoming arrivals during the n^{th} service time,

$$W_n = \begin{cases} 1, & \text{if the } n^{th} \text{ customer in service proceeds from the orbit,} \\ 0, & \text{otherwise.} \end{cases}$$

We assume that $\rho = \lambda\beta_1^1 < 1$ which is the necessary and sufficient condition for the stability of $\{Z_n, n \in \mathbb{Z}_+\}$ [3].

The one step transition probabilities of the embedded Markov chain are given by:

$$p_{n,m} = \begin{cases} \frac{n\mu}{\lambda+\alpha+n\mu} k_0^1, & \text{if } m = n-1, n \geq 1, \\ \frac{\lambda}{\lambda+\alpha+n\mu} k_{m-n}^1 + \frac{\alpha}{\lambda+\alpha+n\mu} k_{m-n}^2 + \frac{n\mu}{\lambda+\alpha+n\mu} k_{m-n+1}^1, & \text{if } 0 \leq n \leq m, \end{cases}$$

where

$$k_j^l = \int_0^\infty e^{-\lambda x} \frac{(\lambda x)^j}{j!} dB_l(x), \; l = 1, 2, \; j \in \mathbf{Z}_+,$$

expresses the probability that there are j incoming calls that arrive during the service time of an incoming or an outgoing call.

3. Preliminary results

3.1. Definition. Let X and Y be two random variables with distribution function F and G respectively.

(1) X is said to be smaller than Y with respect to usual stochastic order (written $X \leq_{st} Y$ or $F \leq_{st} G$) if and only if $F(t) \geq G(t)$, for all real t.

(2) X is less than Y in convex order (written $X \leq_v Y$ or $F \leq_v G$)) if and only if $\int_x^{+\infty} (1 - F(t))dt \leq \int_x^{+\infty} (1 - G(t))dt$, for all real t.

(3) If X and Y are discret random variables taking values in $\mathbb{N}$, with distributions $p_i = P(X = i)$ and $q_i = P(Y = i)$, $i \in \mathbb{N}$, then
- $X \leq_{st} Y$ if and only if $\overline{p}_i \leq \overline{q}_i$, for all $i \in \mathbb{N}$,
- $X \leq_v Y$ if and only if $\overline{\overline{p}}_i \leq \overline{\overline{q}}_i$, for all $i \in \mathbb{N}$,

where $\overline{p}_i = \sum_{j \geq i} p_j$ and $\overline{\overline{p}}_i = \sum_{j \geq i} \overline{p}_j$.

(4) F is NBUE (New Better than Used in Expectation) (respectively, NWUE- New Worse than Used in Expectation) if and only if $F_e \leq_{st} F$ (respectively, $F_e \geq_{st} F$),

where $F_e(x) = \frac{1}{m} \int_0^x \overline{F}(t)dt, x \geq 0.$

For a comprehensive discussion on stochastic orders and their applications, one may refer to [19, 21] and references therein.

Now, we consider two $M_1, M_2/G_1, G_2/1$ retrial queues with two way communication Σ_1 and Σ_2 with parameters $\lambda^{(i)}$, $\mu^{(i)}$, $\alpha^{(i)}$, $B_1^{(i)}(x)$, $B_2^{(i)}(x)$, $k_n^{(i)}$, and $\pi_n^{(i)}$ (the stationary distribution of the number of customers in the orbit in Σ_i), $i = 1, 2$.

The following Lemma gives the conditions under which the probabilities of the number of incoming arrivals during the service of an incoming or outgoing call of two $M_1, M_2/G_1, G_2/1$ retrial queues with two way communication $\{k_n^{(i)}, \; i = 1, 2 \text{ and } n \in \mathbb{N}\}$ are comparable relatively to stochastic and convex ordering.

3.2. Lemma. *Let Σ_1 and Σ_2 be two $M_1, M_2/G_1, G_2/1$ retrial queues with two way communication.*

(1) *If* $\lambda^{(1)} \leq \lambda^{(2)}$ *and* $B_l^{(1)} \leq_{st} B_l^{(2)}$, $l = 1, 2$, *then* $\{k_n^{(1)}\} \leq_{st} \{k_n^{(2)}\}$.

(2) *If* $\lambda^{(1)} \leq \lambda^{(2)}$ *and* $B_l^{(1)} \leq_v B_l^{(2)}$, $l = 1, 2$, *then* $\{k_n^{(1)}\} \leq_v \{k_n^{(2)}\}$.

Where,

$$k_j^{(i)} = \int_0^{+\infty} \frac{(\lambda^{(i)} x)^j}{j!} e^{-\lambda^{(i)} x} dB_l^{(i)}(x), \;\; i = 1, 2, \;\; l = 1, 2.$$

Proof. Suppose that $\lambda^{(1)} \leq \lambda^{(2)}$ and $B_l^{(1)} \leq_{st} B_l^{(2)}$, $l = 1, 2$. By definition, we have

$$\begin{aligned} \overline{k}_n^{(i)} &= \sum_{j=n}^{+\infty} k_j^{(i)} = \int_0^{+\infty} \sum_{j=n}^{+\infty} \frac{(\lambda^{(i)} x)^j}{j!} \exp\{-\lambda^{(i)} x\} dB_l^{(i)}(x) \\ &= \int_0^{+\infty} f_n(x,\ \lambda) dB_l^{(i)}(x), \;\; l = 1, 2. \end{aligned}$$

The function $f_n(x,\ \lambda) = \sum\limits_{j=n}^{+\infty} \dfrac{(\lambda^{(i)} x)^j}{j!} \exp\{-\lambda^{(i)} x\}$ is increasing with respect to λ and x. In fact,

$$\begin{aligned} \frac{\partial}{\partial x} f_n(x,\ \lambda) &= \lambda \frac{(\lambda x)^{n-1}}{(n-1)!} \exp\{-\lambda x\} > 0, \quad \forall\, x \geq\ 0. \\ \frac{\partial}{\partial \lambda} f_n(x,\ \lambda) &= x \exp\{-\lambda x\} \frac{(\lambda x)^{n-1}}{(n-1)!} > 0. \end{aligned}$$

Since $B_l^{(1)} \leq_{st} B_l^{(2)}$ $(l = 1, 2)$ and $\lambda^{(1)} \leq \lambda^{(2)}$, then

$$\int_0^{+\infty} f_n(x,\ \lambda^{(1)}) dB_l^{(1)}(x) \leq \int_0^{+\infty} f_n(x,\ \lambda^{(1)}) dB_l^{(2)}(x) \leq \int_0^{+\infty} f_n(x,\ \lambda^{(2)}) dB_l^{(2)}(x).$$

In other words, to prove that $\{k_n^{(1)}\} \leq_v \{k_n^{(2)}\}$, we have to establish the usual numerical inequality

$$\overline{\overline{k}}_n^{(1)} = \sum_{m=n}^{+\infty} \overline{k}_m^{(1)} \leq \sum_{m=n}^{+\infty} \overline{k}_m^{(2)} = \overline{\overline{k}}_n^{(2)}.$$

On the other hand, we have

$$\begin{aligned} \{k_n^{(1)}\} \leq_v \{k_n^{(2)}\} \quad &\Leftrightarrow \quad \overline{\overline{k}}_n^{(1)} = \sum_{m=n}^{+\infty} \overline{k}_m^{(1)} \leq \sum_{m=n}^{+\infty} \overline{k}_m^{(2)} = \overline{\overline{k}}_n^{(2)} \\ &\Leftrightarrow \quad \int_0^{+\infty} \sum_{m=n}^{+\infty} \sum_{l=m}^{+\infty} \frac{(\lambda^{(1)} x)^l}{l!} \exp\{-\lambda^{(1)} x\} dB_l^{(1)}(x) \\ &\qquad \leq \int_0^{+\infty} \sum_{m=n}^{+\infty} \sum_{l=m}^{+\infty} \frac{(\lambda^{(2)} x)^l}{l!} \exp\{-\lambda^{(2)} x\} dB_l^{(2)}(x) \\ &\Leftrightarrow \quad \int_0^{+\infty} \sum_{m=n}^{+\infty} f_m(x,\ \lambda^{(1)}) dB_l^{(1)}(x) \\ &\qquad \leq \int_0^{+\infty} \sum_{m=n}^{+\infty} f_m(x,\ \lambda^{(2)}) dB_l^{(2)}(x), \;\; l = 1, 2, \end{aligned} \tag{3.1}$$

with, $f_m(x,\ \lambda^{(i)}) = \sum_{l=m}^{+\infty} \frac{(\lambda^{(i)}x)^l}{l!} \exp\{-\lambda^{(i)}x\}$.

The function $\overline{f}_n(x,\ \lambda) = \sum_{m=n}^{+\infty} f_m(x,\ \lambda)$ is increasing with respect to λ and is increasing and convex with respect to x. Indeed,

$$\frac{\partial^2}{\partial x^2}\overline{f}_n(x,\ \lambda) = \lambda \frac{\partial}{\partial x} f_{n-1}(x,\ \lambda) = \lambda^2 \left(\frac{(\lambda x)^{n-2}}{(n-2)!}\right) \exp\{-\lambda x\} > 0.$$

So, with the help of Theorem 1.3.1 given in [21] and by monotonicity of $\overline{f}_n(x,\ \lambda)$ with respect to λ, we obtain the result. $\square$

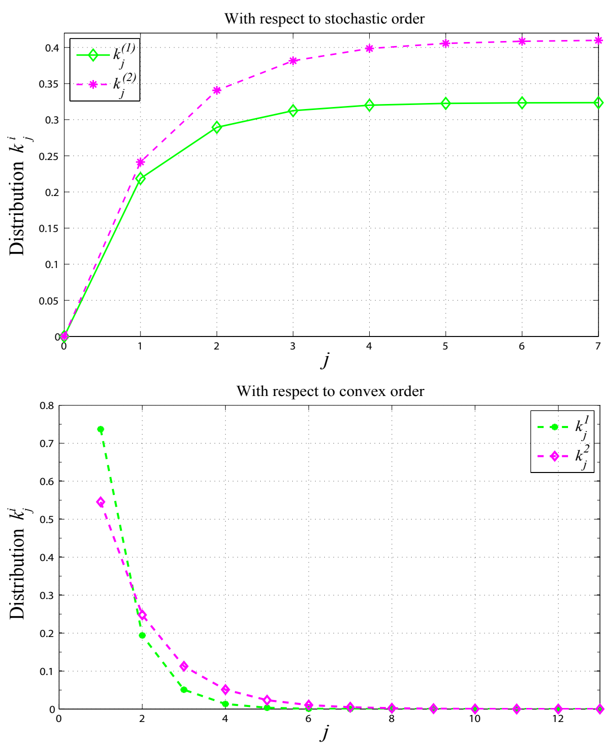

Figure 2. Comparison of $\{k_n^{(i)}, i = 1, 2\}$ with respect to the stochastic and convex orders.

Figure 2 shows that the probability of the number of incoming arrivals during the service times of an incoming or outgoing call is stochastically (with respect to the convex order respectively) decreasing, with increasing service time distribution and increasing the rate parameter of the incoming call.

4. Monotonicity properties of the embedded Markov chain

To every distribution $p = (p_n)_{n\geq 0}$, the transition operator τ of the embedded Markov chain associates a distribution $\tau_p = q = (q_m)_{m\geq 0}$ such that

$$q_m = \sum_{n\geq 0} p_n p_{n,m}.$$

The following Theorems give the monotonicity conditions of the transition operator τ of the embedded Markov chain relatively to stochastic and convex ordering.

4.1. Theorem. *Under the condition $B_2 \leq_{st} B_1$, the transition operator τ is monotone with respect to the stochastic order $\leq_{st}$, i.e. for any two distributions $p^{(1)}$ and $p^{(2)}$, the inequality $p^{(1)} \leq_{st} p^{(2)} \Rightarrow \tau p^{(1)} \leq_{st} \tau p^{(2)}$.*

Proof. The operator τ is monotone with respect to $\leq_{st}$ if and only if (see [21])

$$\overline{p}_{n-1,m} \leq \overline{p}_{n,m} = \sum_{k=m}^{+\infty} p_{n,k}, \quad \forall\ n,\ m, \tag{4.1}$$

with,

$$\begin{aligned}
\overline{p}_{n,m} &= \sum_{k=m}^{+\infty}\left[\frac{\lambda}{\lambda+\alpha+n\mu}k^1_{k-n} + \frac{\alpha}{\lambda+\alpha+n\mu}k^2_{k-n} + \frac{n\mu}{\lambda+\alpha+n\mu}k^1_{k-n+1}\right]\\
&= \frac{\lambda}{\lambda+\alpha+n\mu}\overline{k}^1_{m-n} + \frac{\alpha}{\lambda+\alpha+n\mu}\overline{k}^2_{m-n} + \frac{n\mu}{\lambda+\alpha+n\mu}\overline{k}^1_{m-n+1}\\
&= \frac{\lambda+n\mu}{\lambda+\alpha+n\mu}\overline{k}^1_{m-n} + \frac{\alpha}{\lambda+\alpha+n\mu}\overline{k}^2_{m-n} - \frac{n\mu}{\lambda+\alpha+n\mu}k^1_{m-n}\\
&= \frac{\lambda+n\mu}{\lambda+\alpha+n\mu}\overline{k}^1_{m-n+1} + \frac{\alpha}{\lambda+\alpha+n\mu}\overline{k}^2_{m-n} + \frac{\lambda}{\lambda+\alpha+n\mu}k^1_{m-n},
\end{aligned}$$

and,

$$\begin{aligned}
\overline{p}_{n-1,m} &= \frac{\lambda+(n-1)\mu}{\lambda+\alpha+(n-1)\mu}\overline{k}^1_{m-n+1} + \frac{\alpha}{\lambda+\alpha+(n-1)\mu}\overline{k}^2_{m-n+1}\\
&\quad -\frac{(n-1)\mu}{\lambda+\alpha+(n-1)\mu}k^1_{m-n+1}.
\end{aligned}$$

So,

$$\begin{aligned}
\overline{p}_{n,m} - \overline{p}_{n-1,m} &= \frac{\alpha\mu}{(\lambda+\alpha+n\mu)(\lambda+\alpha+(n-1)\mu)}[\overline{k}^1_{m-n+1} - \overline{k}^2_{m-n+1}]\\
&\quad +\frac{\lambda}{\lambda+\alpha+n\mu}k^1_{m-n} + \frac{(n-1)\mu}{\lambda+\alpha+(n-1)\mu}k^1_{m-n+1}\\
&\quad +\frac{\alpha}{\lambda+\alpha+n\mu}k^2_{m-n} \geq 0.
\end{aligned}$$

Consequently, since $B_2 \leq_{st} B_1$, then inequality (4.1) is verified. Finally, the operator τ is monotone with respect to the stochastic order ($\leq_{st}$). □

4.2. Theorem. *Under the condition $B_1 \equiv_v B_2$, the transition operator τ is monotone with respect to the convex order $\leq_v$, i.e. for any two distributions $p^{(1)}$ and $p^{(2)}$, the following inequality holds*

$$p^{(1)} \leq_v p^{(2)} \Rightarrow \tau p^{(1)} \leq_v \tau p^{(2)}.$$

Proof. The operator τ is monotone with respect to the convex order if and only if (see [21]):

$$2\overline{\overline{p}}_{n,m} \leq \overline{\overline{p}}_{n-1,m} + \overline{\overline{p}}_{n+1,m},\ \forall\ n,\ m, \tag{4.2}$$

where

$$\overline{\overline{p}}_{n,m} = \sum_{k=m}^{+\infty} \overline{p}_{n,k}.$$

In other words, we obtain

$$\begin{aligned}
\overline{\overline{p}}_{n,m} &= \sum_{k=m}^{+\infty}\left[\frac{\lambda}{\lambda+\alpha+n\mu}\overline{k}^{1}_{k-n} + \frac{\alpha}{\lambda+\alpha+n\mu}\overline{k}^{2}_{k-n} + \frac{n\mu}{\lambda+\alpha+n\mu}k^{1}_{k-n+1}\right]\\
&= \frac{\lambda}{\lambda+\alpha+n\mu}\overline{\overline{k}}^{1}_{m-n} + \frac{\alpha}{\lambda+\alpha+n\mu}\overline{\overline{k}}^{2}_{m-n} + \frac{n\mu}{\lambda+\alpha+n\mu}\overline{\overline{k}}^{1}_{m-n+1}\\
&= \frac{\lambda+n\mu}{\lambda+\alpha+n\mu}\overline{\overline{k}}^{1}_{m-n} + \frac{\alpha}{\lambda+\alpha+n\mu}\overline{\overline{k}}^{2}_{m-n} - \frac{n\mu}{\lambda+\alpha+n\mu}\overline{k}^{1}_{m-n}\\
&= \frac{\lambda+n\mu}{\lambda+\alpha+n\mu}\overline{\overline{k}}^{1}_{m-n+1} + \frac{\alpha}{\lambda+\alpha+n\mu}\overline{\overline{k}}^{2}_{m-n} + \frac{\lambda}{\lambda+\alpha+n\mu}\overline{k}^{1}_{m-n},
\end{aligned}$$

$$\begin{aligned}
\overline{\overline{p}}_{n-1,m} &= \frac{\lambda+(n-1)\mu}{\lambda+\alpha+(n-1)\mu}\overline{\overline{k}}^{1}_{m-n} + \frac{\alpha}{\lambda+\alpha+(n-1)\mu}\overline{\overline{k}}^{2}_{m-n}\\
&\quad -\left(\frac{\lambda+(n-1)\mu}{\lambda+\alpha+(n-1)\mu}\right)\overline{k}^{1}_{m-n} + \frac{(n-1)\mu}{\lambda+\alpha+(n-1)\mu}k^{1}_{m-n}\\
&\quad -\frac{\alpha}{\lambda+\alpha+(n+1)\mu}\overline{k}^{2}_{m-n},
\end{aligned}$$

and

$$\begin{aligned}
\overline{\overline{p}}_{n+1,m} &= \frac{\lambda+(n+1)\mu}{\lambda+\alpha+(n+1)\mu}\overline{\overline{k}}^{1}_{m-n} + \frac{\alpha}{\lambda+\alpha+(n+1)\mu}\overline{\overline{k}}^{2}_{m-n}\\
&\quad +\frac{\lambda}{\lambda+\alpha+(n+1)\mu}\overline{k}^{1}_{m-n} + \frac{\alpha}{\lambda+\alpha+(n+1)\mu}\overline{k}^{2}_{m-n}\\
&\quad +\frac{\lambda}{\lambda+\alpha+n\mu}k^{1}_{m-n-1} + \frac{\alpha}{\lambda+\alpha+(n+1)\mu}k^{2}_{m-n-1}.
\end{aligned}$$

After some algebraic manipulation, we obtain

$$\begin{aligned}
&\overline{\overline{p}}_{n-1,m} + \overline{\overline{p}}_{n+1,m} - 2\overline{\overline{p}}_{n,m} =\\
= &\frac{2\alpha\mu^2}{(\lambda+\alpha+(n+1)\mu)(\lambda+\alpha+n\mu)(\lambda+\alpha+(n-1)\mu)}\left[\overline{\overline{k}}^{2}_{m-n+1} - \overline{\overline{k}}^{1}_{m-n+1}\right]\\
+ &\frac{2\alpha\mu^2}{(\lambda+\alpha+(n+1)\mu)(\lambda+\alpha+n\mu)(\lambda+\alpha+(n-1)\mu)}\left[\overline{k}^{2}_{m-n} - \overline{k}^{1}_{m-n}\right]\\
+ &\frac{2\alpha\mu(\lambda+\alpha+n\mu)}{(\lambda+\alpha+(n+1)\mu)(\lambda+\alpha+(n-1)\mu)(\lambda+\alpha+n\mu)}\left[\overline{k}^{1}_{m-n} - \overline{k}^{2}_{m-n}\right]\\
+ &\frac{2\mu^2(\lambda+\alpha)}{(\lambda+\alpha+(n+1)\mu)(\lambda+\alpha+n\mu)}\overline{k}^{1}_{m-n} + \frac{(n-1)\mu}{\lambda+\alpha+(n-1)\mu}k^{1}_{m-n}\\
+ &\frac{\lambda}{\lambda+\alpha+(n+1)\mu}k^{1}_{m-n-1} + \frac{\alpha}{\lambda+\alpha+(n+1)\mu}k^{2}_{m-n-1} \geq 0.
\end{aligned}$$

Finally, we find that for an $M_1, M_2/G_1, G_2/1$ retrial queue with two way communication, the transition operator τ is monotone with respect to the convex order ($\leq_v$) under the condition $B_1 \equiv_v B_2$. □

In the following Theorems, we give comparability conditions of two transition operators relatively to stochastic order ($\leq_{st}$) and convex order ($\leq_v$).

4.3. Theorem. *Let $\tau^{(1)}$, $\tau^{(2)}$ be the transition operators of the embedded Markov chains added to each model Σ_1 and Σ_2. If $\lambda^{(1)} \leq \lambda^{(2)}$, $\mu^{(1)} \geq \mu^{(2)}$, $\alpha^{(1)} \leq \alpha^{(2)}$, $B_1^{(1)} \leq_{st} B_1^{(2)}$ and $B_2^{(1)} \leq_{st} B_2^{(2)}$, then $\tau^{(1)} \leq_{st} \tau^{(2)}$, i.e. for any distribution p, we have $\tau^{(1)}p \leq_{st} \tau^{(2)}p$.*

Proof. We must prove that (see Stoyan [21])

$$\overline{p}_{nm}^{(1)} \quad \leq \quad \overline{p}_{nm}^{(2)}, \ \forall\ 0 \leq n \leq m.$$

This is equivalent to

$$\begin{aligned} & \frac{\lambda^{(1)} + n\mu^{(1)}}{\lambda^{(1)} + \alpha^{(1)} + n\mu^{(1)}} \overline{k^1}^{(1)}_{m-n+1} + \frac{\alpha^{(1)}}{\lambda^{(1)} + \alpha^{(1)} + n\mu^{(1)}} \overline{k^2}^{(1)}_{m-n} \\ - \quad & \frac{n\mu^{(1)}}{\lambda^{(1)} + \alpha^{(1)} + n\mu^{(1)}} k^{1\,(1)}_{\ m-n} \leq \frac{\lambda^{(2)} + n\mu^{(2)}}{\lambda^{(2)} + \alpha^{(2)} + n\mu^{(2)}} \overline{k^1}^{(2)}_{m-n+1} \\ + \quad & \frac{\alpha^{(2)}}{\lambda^{(2)} + \alpha^{(2)} + n\mu^{(2)}} \overline{k^2}^{(2)}_{m-n} - \frac{n\mu^{(2)}}{\lambda^{(2)} + \alpha^{(2)} + n\mu^{(2)}} k^{1\,(2)}_{\ m-n}. \end{aligned}$$

By Lemma 3.2, we have $\{k_n^{l\,(1)}\} \leq_{st} \{k_n^{l\,(2)}\}$.

As $\lambda^{(1)} \leq \lambda^{(2)}$, $\alpha^{(1)} \leq \alpha^{(2)}$ and $\mu^{(1)} \geq \mu^{(2)}$, we have $\frac{\lambda^{(1)}+\alpha^{(1)}}{\mu^{(1)}} \leq \frac{\lambda^{(2)}+\alpha^{(2)}}{\mu^{(2)}}$, and since the function $\frac{m}{x+m}$ is decreasing, so

$$\frac{n\mu^{(1)}}{\lambda^{(1)} + \alpha^{(1)} + n\mu^{(1)}} \geq \frac{n\mu^{(2)}}{\lambda^{(2)} + \alpha^{(2)} + n\mu^{(2)}}. \tag{4.3}$$

Further, as the function $\frac{x}{x+m}$ is increasing and $\frac{\lambda^{(1)}}{\mu^{(1)}} \leq \frac{\lambda^{(2)}}{\mu^{(2)}}$ so,

$$\frac{\lambda^{(1)}}{\lambda^{(1)} + n\mu^{(1)}} \leq \frac{\lambda^{(2)}}{\lambda^{(2)} + n\mu^{(2)}},$$

hence,

$$\lambda^{(1)} + n\mu^{(1)} \geq \lambda^{(2)} + n\mu^{(2)}.$$

Consequently, $\alpha^{(1)} \leq \alpha^{(2)}$ and $\lambda^{(1)} + n\mu^{(1)} \geq \lambda^{(2)} + n\mu^{(2)}$ implies that

$$\frac{\lambda^{(1)} + n\mu^{(1)}}{\alpha^{(1)}} \geq \frac{\lambda^{(2)} + n\mu^{(2)}}{\alpha^{(2)}}.$$

Indeed,

$$\frac{\lambda^{(1)} + n\mu^{(1)}}{\lambda^{(1)} + \alpha^{(1)} + n\mu^{(1)}} \geq \frac{\lambda^{(2)} + n\mu^{(2)}}{\lambda^{(2)} + \alpha^{(2)} + n\mu^{(2)}}. \tag{4.4}$$

From Lemma 3.2 and inequalities (4.3) and (4.4), we get the following result

$$\begin{aligned} & \frac{\lambda^{(1)} + n\mu^{(1)}}{\lambda^{(1)} + \alpha^{(1)} + n\mu^{(1)}} \overline{k^1}^{(1)}_{m-n} - \frac{n\mu^{(1)}}{\lambda^{(1)} + \alpha^{(1)} + n\mu^{(1)}} k^{1\,(1)}_{\ m-n} \\ - \quad & \frac{\lambda^{(2)} + n\mu^{(2)}}{\lambda^{(2)} + \alpha^{(2)} + n\mu^{(2)}} \overline{k^1}^{(2)}_{m-n} + \frac{n\mu^{(2)}}{\lambda^{(2)} + \alpha^{(2)} + n\mu^{(2)}} k^{1\,(2)}_{\ m-n} \\ \leq \quad & \frac{\lambda^{(1)} + n\mu^{(1)}}{\lambda^{(1)} + \alpha^{(1)} + n\mu^{(1)}} \overline{k^1}^{(1)}_{m-n} - \frac{n\mu^{(2)}}{\lambda^{(2)} + \alpha^{(2)} + n\mu^{(2)}} k^{1\,(2)}_{\ m-n} \\ - \quad & \frac{\lambda^{(1)} + n\mu^{(1)}}{\lambda^{(1)} + \alpha^{(1)} + n\mu^{(1)}} \overline{k^1}^{(1)}_{m-n} + \frac{n\mu^{(2)}}{\lambda^{(2)} + \alpha^{(2)} + n\mu^{(2)}} k^{1\,(2)}_{\ m-n} = 0. \end{aligned}$$

Furthermore,

$$\frac{\lambda+\alpha+n\mu}{\lambda+\alpha+n\mu}=1\Rightarrow\frac{\lambda+n\mu}{\lambda+\alpha+n\mu}=1-\frac{\alpha}{\lambda+\alpha+n\mu},$$

so,

$$\frac{\lambda^{(1)}+n\mu^{(1)}}{\lambda^{(1)}+\alpha^{(1)}+n\mu^{(1)}}\geq\frac{\lambda^{(2)}+n\mu^{(2)}}{\lambda^{(2)}+\alpha^{(2)}+n\mu^{(2)}},$$

thus

$$1-\frac{\alpha^{(1)}}{\lambda^{(1)}+\alpha^{(1)}+n\mu^{(1)}}\geq 1-\frac{\alpha^{(2)}}{\lambda^{(2)}+\alpha^{(2)}+n\mu^{(2)}},$$

which implies

$$\frac{\alpha^{(1)}}{\lambda^{(1)}+\alpha^{(1)}+n\mu^{(1)}}\leq\frac{\alpha^{(2)}}{\lambda^{(2)}+\alpha^{(2)}+n\mu^{(2)}}. \tag{4.5}$$

Consequently, if $B_1^{(1)}\leq_{st}B_1^{(2)}$ and $B_2^{(1)}\leq_{st}B_2^{(2)}$, we obtain the desired result. □

4.4. Theorem. *Let $\tau^{(1)}$ and $\tau^{(2)}$ be the transition operators of the embedded Markov chains added to each model Σ_1 and Σ_2, respectively. If we have $\lambda^{(1)}\leq\lambda^{(2)}$, $\mu^{(1)}\geq\mu^{(2)}$, $\alpha^{(1)}\leq\alpha^{(2)}$, $B_1^{(1)}\leq_v B_1^{(2)}$ and $B_2^{(1)}\leq_v B_2^{(2)}$, then $\tau^{(1)}\leq_v\tau^{(2)}$, i.e. for any distribution p we have $\tau^{(1)}p\leq_v\tau^{(2)}p$.*

Proof. To prove that

$$\overline{\overline{p}}_{nm}^{(1)}\quad\leq\quad\overline{\overline{p}}_{nm}^{(2)},\quad\forall\ 0\leq n\leq m,$$

we have to establish the following stochastic inequality

$$\begin{aligned}
&\frac{\lambda^{(1)}+n\mu^{(1)}}{\lambda^{(1)}+\alpha^{(1)}+n\mu^{(1)}}\overline{\overline{k^1}}^{(1)}_{m-n+1}+\frac{\alpha^{(1)}}{\lambda^{(1)}+\alpha^{(1)}+n\mu^{(1)}}\overline{\overline{k^2}}^{(1)}_{m-n}\\
-\quad&\frac{n\mu^{(1)}}{\lambda^{(1)}+\alpha^{(1)}+n\mu^{(1)}}\overline{k^1}^{(1)}_{m-n}\leq\frac{\lambda^{(2)}+n\mu^{(2)}}{\lambda^{(2)}+\alpha^{(2)}+n\mu^{(2)}}\overline{\overline{k^1}}^{(2)}_{m-n+1}\\
+\quad&\frac{\alpha^{(2)}}{\lambda^{(2)}+\alpha^{(2)}+n\mu^{(2)}}\overline{\overline{k^2}}^{(2)}_{m-n}-\frac{n\mu^{(2)}}{\lambda^{(2)}+\alpha^{(2)}+n\mu^{(2)}}\overline{k^1}^{(2)}_{m-n}.
\end{aligned}$$

The rest of proof is similar to that of Theorem 4.3. □

5. Stochastic inequalities for the stationary number of customers in the system

The following Theorems give comparability conditions of stationary distributions of the number of customers in the orbit for two systems Σ_1 and Σ_2, with respect to stochastic and convex orders.

5.1. Theorem. *Let $\pi_n^{(1)}$ and $\pi_n^{(2)}$ be the stationary distributions of the number of customers in Σ_1 and Σ_2. If $\lambda^{(1)}\leq\lambda^{(2)}$,$\mu^{(1)}\geq\mu^{(2)}$, $\alpha^{(1)}\leq\alpha^{(2)}$, $B_1^{(1)}\leq_{so}B_1^{(2)}$, $B_2^{(1)}\leq_{so}B_2^{(2)}$ and $B_2^{(2)}\leq_{st}B_1^{(2)}$ (resp. $B_1^{(2)}\leq_v B_2^{(2)}$), then $\{\pi_n^{(1)}\}\leq_{so}\{\pi_n^{(2)}\}$, where $so=(st$ or $v)$.*

Proof. It is well known that the distribution of the number of customers in the system at steady state coincides with that of the system at departure epoch. The stationary distribution coincides with the limit distribution, since the corresponding embedded Markov chain $\{Z_n,n\geq 1\}$ is ergodic. So, using Theorems 4.3 and 4.4, we obtain by induction

$$\tau^{(1)}p^{(1)}\leq_{so}\tau^{(2)}p^{(2)}, \tag{5.1}$$

for any two distributions $p^{(1)}$, $p^{(2)}$

$$\tau^{(1)}p_n^{(1)}=P(Z_k^{(1)}=(C,n))\leq_{so}P(Z_k^{(2)}=(C,n))=\tau^{(2)}p_n^{(2)}.$$

When $k \longrightarrow \infty$, we have $\{\pi_n^{(1)}\} \leq_{so} \{\pi_n^{(2)}\}$, $so = (st \text{ or } v)$. $\square$

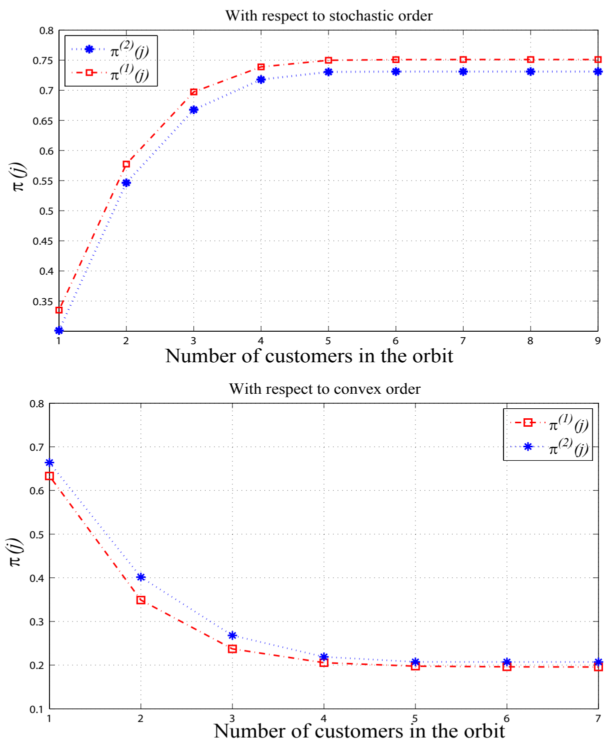

Figure 3. Comparison of the stationary probabilities of two systems Σ_1 and Σ_2

From [3], we can exactly compute the stationary probabilities. According to the conditions cited in Theorem 5.1, Figure 3 presents the exact values of the stationary distribution of the two queueing systems Σ_1 and Σ_2. It is well observed that the cuves become consistent along the increasing of j. Further, while comparing the two stationary probabilities, we see that $\pi^{(1)}(j)$ is greater than $\pi^{(2)}(j)$ with respect to stochastic order (respectively $\pi^{(1)}(j)$ is lower than $\pi^{(2)}(j)$ with respect to convex order).

5.2. Theorem. *If the service time distributions of ingoing and outgoing calls are NBUE (NWUE) in the $M_1, M_2/G_1, G_2/1$ retrial queue with two way communication, and if $B_1^{(1)} \leq_v B_1^{(2)} \equiv B_1^*$, $B_2^{(1)} \leq_v B_2^{(2)} \equiv B_2^*$ and $B_1^{(2)} \leq_v B_2^{(2)}$, then $(\pi_n) \leq_v (\pi_n^*)$ (greater relative to the convex ordering), where (π_n^*) is the stationary distribution of the number of customers in the orbit for the $M_1, M_2/M_1, M_2/1$ retrial queue with two way communication.*

Proof. Consider an $M_1, M_2/M_1, M_2/1$ retrial queue with two way communication with the same parameters as in $M_1, M_2/G_1, G_2/1$ retrial queue with two way communication: arrival rate (ingoing call) λ, retrial rate μ, outgoing call rate α, mean service times β_1^1

and β_2^1, but with exponentially distributed service times, $\theta_1 = \frac{1}{\beta_1^1}$ and $\theta_2 = \frac{1}{\beta_2^1}$.

$$B_1^*(x) = \begin{cases} 1 - e^{-\frac{x}{\beta_1^1}}, & \text{if } x \geq 0, \\ 0, & \text{if } x < 0. \end{cases}$$

$$B_2^*(x) = \begin{cases} 1 - e^{-\frac{x}{\beta_2^1}}, & \text{if } x \geq 0, \\ 0, & \text{if } x < 0. \end{cases}$$

From Stoyan [21], if $B_l(x)$, $l = 1, 2$ are NBUE (respectively, NWUE), then

$$B_l(x) \leq_v B_l^*(x), \ (\text{respectively } B_l(x) \geq_v B_l^*(x)), \ l = 1, 2.$$

Since $B_2^{(1)} \leq_v B_2^{(2)}$ and $B_1^* \leq_v B_2^{(2)}$, then using Theorem 5.1, we deduce that the stationary distribution of the number of customers in the orbit of $M_1, M_2/G_1, G_2/1$ retrial queue with two way communication is less (resp. greater) than the stationary distribution of the number of customers in the orbit of an $M_1, M_2/M_1, M_2/1$ retrial queue with two way communication. $\square$

6. Numerical example

In this section, we give a numerical illustration concerning Theorem 5.2. First, we developed a simulator, with Matlab environment, describing the behavior of the $M_1, M_2/G_1, G_2/1$ retrial queue with two way communication. Then, we estimated the stationary probabilities of this system when the service time distribution is NBUE (respectively NWUE), which we compared to those of the $M_1, M_2/M_1, M_2/1$ retrial queue with two way communication. Thus, we set the incoming call rate $\lambda = 0.3$, the outgoing call rate $\alpha = 0.2$, retrial rate $\mu = 1$, the simulation time $Tmax = 1000$ time units and $n = 100$ (the number of replications).

We chose one type of probability distributions NBUE (a Weibull distribution ($Wbl(a,\ b)$ with $a > 1$)), two other laws type NWUE (a Weibull distribution ($Wbl(a,\ b)$ with $a \leq 1$) and a Gamma distribution ($\Gamma(a, b)$ with $0 \leq a < 1$)) for service times of incoming and outgoing call ($B_1(x), B_2(x)$) with different parameters (see Table 1).

Table 1. Different distributions with respective parameters

	$B_1(x)$		$B_2(x)$	
	(a)	**(b)**	**(a)**	**(b)**
NBUE	$Wbl(3.11, 2)$	$Wbl(2, 4)$	$Wbl(2.5, 2)$	$Wbl(1.78, 2)$
Exp	$Exp(0.65)$	$Exp(1.00)$	$Exp(0.61)$	$Exp(0.55)$
NWUE	$Wbl(0.5, 0.46)$	$Wbl(0.66, 0.46)$	$Wbl(0.5, 0.3333)$	$Wbl(0.42, 0.3333)$
	$\Gamma(0.6, 4)$	$\Gamma(0.56, 4)$	$\Gamma(0.53, 2)$	$\Gamma(0.48, 2)$

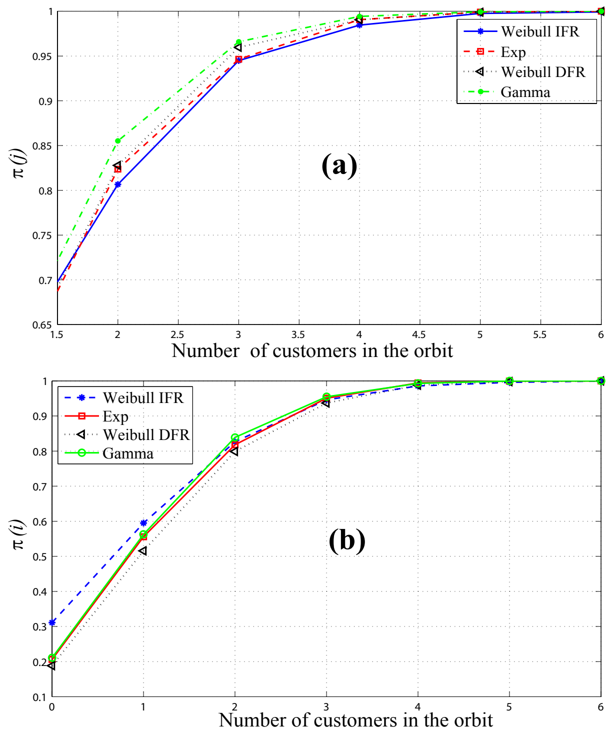

Figure 4. Comparison of the stationary probabilities with respect to stochastic order for different laws under the parameter setting specified in Table 1.

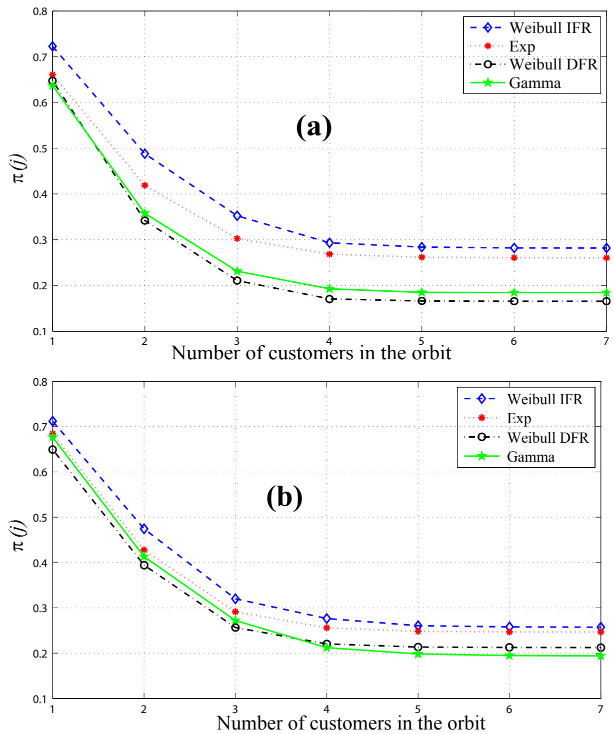

Figure 5. Comparison of the stationary probabilities with respect to the convex order for different laws under the parameter setting specified in Table 1.

Figure 4 and Figure 5, reflecting all the cases studied in Table 1, show that the theoretical results obtained are confirmed by the simulation ones (a good agreement between the analytical results and those of simulation). Consequently, performance measures of the system considered can be estimated by those of the $M_1, M_2/M_1, M_2/1$ retrial queue with two way communication.

7. Conclusion

In this work, we used the general theory of stochastic orderings to investigate the monotonicity properties of the $M_1, M_2/G_1, G_2/1$ retrial queue with two way communication. Particularly, the main result of this paper consisted in giving insensitive stochastic bounds for the stationary distribution of the embedded Markov chain. The proposed approach is quite different from those given in [3, 17], in the sense that it provides from the fact that we can come to a compromise between the role of these qualitative bounds and the complexity of resolution of some complicated systems where some parameters are not perfectly known. Besides, the obtained bounds (lower and upper) are easy to calculate and seem to be good approximations for performance measures of the considered system. Finally, we discussed the conditions under which the comparison of this model with an $M_1, M_2/M_1, M_2/1$ retrial queue with two way communication is valid, and hence bounds of performance measures are derived. An illustrative numerical example is presented to support theoretical results.

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