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Locally Chainable Sets in Metric Spaces

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ABSTRACT

In this paper, we define locally chainable sets in metric space.

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1. INTRODUCTION

In 1883 Cantor defined connectedness with the help of ε-chains. Beer[1] has characterized compact sets among the connected sets. In 2002, Shrivastava and Agrawal [3] defined ε-chainable sets in metric space. In 2010 Pagey et al. [4] proved that the chainability of points and sets is product invariant property.

In this paper local chainability of two subsets of a metric space has been defined and various examples of such sets have been provided. It is shown that open subsets of two locally chainable sets are themselves locally chainable. Also among open subsets of any metric space X, unions of locally chainable sets at a point are locally chainable at the same point. Local chainability of two sets guarantees the existence of open sets in the ε-neighborhood of these sets such that the open sets are ε-chainable.

Preliminaries [2]: Throughout this paper X will stands for metric space with metric d.

1. Let A be a subset of the metric space X. For $\varepsilon > 0$,

   let$V_\varepsilon(A) = \{x \in X : d(x,A)<\varepsilon\}$ where

   $d(x,A) = \inf \{d(x,a) : a \in A\}$.

2. Let (X,d) be a metric space for each subset A of X, and $\varepsilon > 0$,

   $C_\varepsilon(A)$ is defined to be the set of all points which can be joined to points of A byan ε-chain.

3. C(A) is defined to be $\cap\{ C_\varepsilon(A): \varepsilon > 0 \}$ or equivalently

   $C_\varepsilon(A) = \cup \{V_n(A): n \in N\}$ where

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$V_o(A') = A$, $V_1(A) = \{ x: d(x,A) < \varepsilon \}$

and inductively $V_{n+1}(A) = V_1(V_n(A))$.

4. Let (X,d) be a metric space and A $\subset$ X. Then boundary A, denoted by

Fr(A) is $\bar{A} \cap (\overline{X - A})$.

5. Let (X,d) be a metric space and A $\subset$ X. Then A^o is interior of A in X, A^c will denote the complement of A in X, δ(A) will denote the diameter of A in X, $U_\delta(a)$ will denote the neighborhood of a in X and $\bar{A}$ will denote the closure of A in X.

Definition 1[1]-For points p,q ϵ X an ε-chain of length n from p to q is finite sequence $a_{1,}a_{2,}\ a_3 \ldots \ldots , a_n$ in Xwith $a_0 = p$, $a_n = q$ and d $(a_{i-1}, a_i) < \varepsilon$, $1 \le i \le n$.

We call X ε-chainable if each two points in X can be joined by an ε-chain and X is chainable if X is ε-chainable for each positive ε.

Definition 2[3] – Let A,B $\subset$ X. An ε-chain of length n from A to B is a finite sequence $A_0, A_1, A_2, \ldots \ldots A_n$ of subsets of X with $A = A_0$, $A_n = B$, $A_{i-1} \subset V_\varepsilon(A_i) A_i \subset V_\varepsilon(A_{i-1})$,

$1 \le i \le$ n. If ε-chain exists between A and B we say $\langle A, B \rangle$ is ε-chainable and $\langle A, B \rangle$ is chainable if it is ε-chainable for each positive ε.

Definition 3 – Let A,B $\subset$ X. Then $\langle A, B \rangle$ is locally chainable at (a,b) ∈(A×B) if and only if

for every ε > 0 , ∃ δ > 0 , such that

$\langle U_\delta(a) \cap A^0, U_\delta(b) \cap B^0 \rangle$ is ε-chainable and

$U_\delta(a) \cap A^0 \subset U_\varepsilon(a)$ and $U_\delta(b) \cap B^0 \subset U_\varepsilon(b)$

where $U_\delta(a) \cap A^0$ and $U_\delta(b) \cap B^0$ are non -empty sets. $\langle A, B \rangle$ is locally chainable if $\langle A, B \rangle$ is locally chainable at each point (a,b) ∈(A×B).

Definition 4 – Let A $\subset$ X. Then A is said to be self ε-chainable if every two points of A can be joined by an ε-chain . Also A is said to be self chainable if A is self ε-chainable for every ε > 0.

Definition 5 – Let A, B $\subset$ X. Then $\langle A, B \rangle$ is said to be strongly ε-chainable if and only if any two points of A and B are ε-chainable and $\langle A, B \rangle$ is said to be strongly chainable $\langle A, B \rangle$ is strongly ε-chainable for every ε > 0.

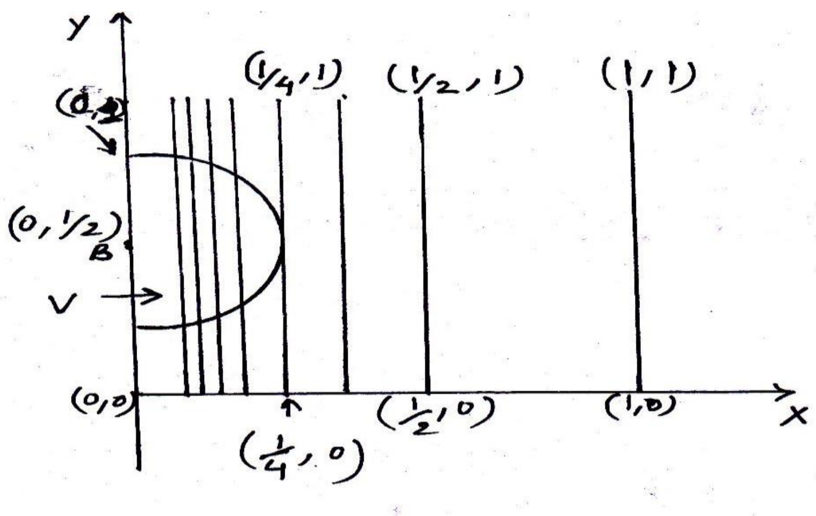
Examples of Locally Chainable Sets

(1) Let X be the set B$\cup A$ where

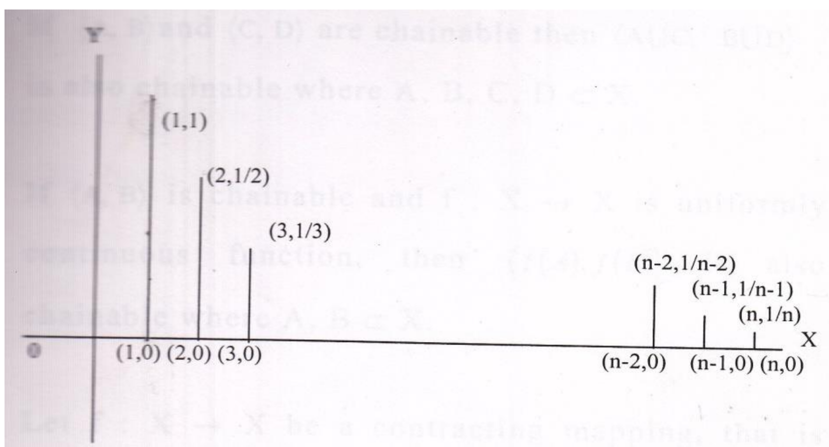
B = {(x,0) ∈ R^2 ,$0 \le x \le 1$} and

A = {(x,y)

$\in R^2 : 0 \le y \le 1, x = 0 \ or \ x = \frac{1}{n} \ for \ some \ n \in N\}$



X consists of infinitely many vertical segments of unit length,including a segment on the y- axis and a horizontal segment along the x- axis. Given X the relative topology induced by the usual topology on the plane; then any two vertical segments of X are locally chainable except if one of one the vertical segment is {(0,y) ∈ R^2} ⊂ X.



X = { (n, y): 0≤y ≤ $\frac{1}{n}$, n∈ N }

Then$\langle A,B\rangle$ is chainable in the subspace (X,d) of (R×R, d) however $\langle A,B\rangle$ is not locally chainable at any of the points ((n,0), (n,$\frac{1}{n}$)) n∈N, since $A^0 = \emptyset$ and $B^0 = \emptyset$. This is an example of two sets which are chainable but not locally chainable.

(2) Let A= {(n, $\frac{1}{n}$): n∈ N}

B = { (n,0):n ∈ N }

(3) Consider the discrete metric space (X,d). Then for any two subsets A and B of X , $\langle A,B\rangle$ is chainable for ε > 1, but not locally chainable at any of the points of (A×B) provided A$\cap B = \emptyset$.

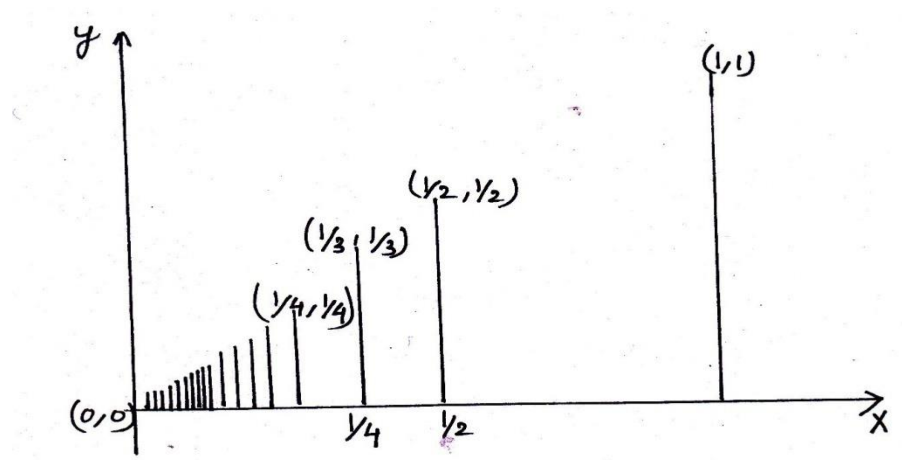
Note : If A$\cap B \neq \emptyset$, then $\langle A,B\rangle$ is locally chainable.

(4)Let Y be the subspace $\{0\}\cup\{\frac{1}{n} : n = 1,2 \ldots\}$ of R. Let A = $\{\frac{1}{n}\}$ and B = $\{\frac{1}{m}\}$, n≠m then $\langle A, B\rangle$ is neither chainable nor locally chainable in Y.

(5) Let $(B\cup A \cup C)$ be a subspaces of (R^2, d) where $A=\{(x,y): x = \frac{1}{n}, 0 \le y \le \frac{1}{n}, n\ is\ odd\ n \in N\}$

$B = \{(x,y): x = \frac{1}{n}\ 0 \le y \le \frac{1}{n},\ n\ is\ even\ and\ n \in N\}$

$C = \{(x,0): \frac{1}{n_0} \le x \le \frac{1}{n_0+1}, n_0 \in N\}$.



Then $\langle A, B\rangle$ is ε- chainable for ε $>\frac{1}{2}$ but not ε-chainable for ε $\le \frac{1}{2}$. Then $\langle A, B\rangle$ is locallychainable at infinite number of points lying on the segments x = $\frac{1}{n_0}$ and x = $\frac{1}{n_0+1}$.

Various Resultson Locally Chainable Sets

Theorem 1

Let A,B ⊂ X, such that $\langle A, B\rangle$ is locally chainable and C and D are open sets in X such that C⊂ A, D ⊂ B. then $\langle C, D\rangle$ is locally chainable.

Proof : Let (a,b) ∈ (C×D) ⇒(a,b) ∈ (A×B). C, D are open subsets of X, hence $U_{\varepsilon_1}(a)$⊂ C ⊂ A and $U_{\varepsilon_1}(b)$⊂ D⊂ B for some $\varepsilon_1 > 0$. $\langle A, B\rangle$is locally chainable at (a,b) then $\langle U_\delta(a) \cap A^0, U_\delta(b) \cap B^0\rangle$ is ε_1-chainable and $U_\delta(a) \cap A^0 \subset U_{\varepsilon_1}(a) \subset$ C and $U_\delta(b) \cap B^0 \subset U_{\varepsilon_1}(b) \subset$ D for some δ > 0.

Let ε $>\varepsilon_1$ (without loss of generality). As C ⊂ A, D ⊂ B,

$U_\delta(a) \cap C = U_\delta(a) \cap C^0 \subset U_\delta(a) \cap A^0$ and $U_\delta(b) \cap D = U_\delta(b) \cap D^0 \subset U_\delta(b) \cap B^0$ also $U_\delta(a) \cap A^0 \subset U_\delta(a) \cap C = U_\delta(a) \cap C^0$

$U_\delta(b) \cap B^0 \subset U_\delta(b) \cap D = U_\delta(b) \cap D^0$

$\Rightarrow U_\delta(a) \cap A^0 = U_\delta(a) \cap C^0$ and $U_\delta(b) \cap B^0 = U_\delta(b) \cap D^0$

$\Rightarrow \langle U_\delta(a) \cap C^0, U_\delta(b) \cap D^0\rangle$ is ε_1-chainable and hence ε-chainable.

Also $U_\delta(a) \cap C^0 \subset U_{\varepsilon_1}(a) \subset U_\varepsilon(a)$ and $U_\delta(b) \cap D^0 \subset U_{\varepsilon_1}(b) \subset U_\varepsilon(b)$

$\Rightarrow \langle C, D\rangle$ is locally chainable at (a , b).

Remarks : (1) Let A,B ⊂ X , such that $\langle A, B\rangle$ is locally chainable, then $\langle A^0, B^0\rangle$is locally chainable.

(2) Let $\langle A, B\rangle$ be locally chainable sets and f : X $\longrightarrow$ X be an isomorphism then $\langle f(A), f(B)\rangle$ is locally chainable.

(3) Let A,B,C,D be the open sets of X such that $\langle A, B\rangle$ and $\langle C, D\rangle$ are locally chainable, then $\langle A\cap C\,,\ B\cap D\rangle$ is locally chainable.

Theorem 2

Let A,B,C and D be open subsets of X such that $\langle A, B\rangle$ and $\langle C, D\rangle$ are locally chainable at (a,b) then $\langle A\cup C\,,\ B\cup D\rangle$ is locally chainable.

Proof :$\langle A, B\rangle$ is locally chainable at (a,b) ⇒ for some $\delta_1 > 0$,

$$\langle U_{\delta_1}(a)\cap A\ ,U_{\delta_1}(b)\cap B\rangle$$

is ε-chainable where $U_{\delta_1}(a)\cap A \subset U_\varepsilon(a)$ and $U_{\delta_1}(b)\cap B \subset U_\varepsilon(b)$.

$\langle C, D\rangle$is locally chainable at (a,b) ⇒ for some $\delta_2 > 0$. $\langle U_{\delta_2}(a)\cap C\ ,U_{\delta_2}(b)\cap D\rangle$is ε-chainable, where

$U_{\delta_2}(a)\cap C \subset U_\varepsilon(a)andU_{\delta_2}(b)\cap D \subset U_\varepsilon(b)$.

Letδ = min(δ_1, δ_2)

Let x $\epsilon U_\delta(a)\cap(A\cup C)\Rightarrow$ x $\epsilon U_{\delta_1}(a)\cap A$ or x $\epsilon U_{\delta_2}(a)\cap C$. Then x is ε-chainable to some point of $U_{\delta_1}(b)\cap B$or to some point of$U_{\delta_2}(b)\cap D$. As both these sets are contained in$U_\varepsilon(b)$. It follows that x is ε-chainable to b.

$\Rightarrow\langle U_\delta(a)\cap(A\cup C), U_\delta(b)\cap(B\cup D)\rangle$ is ε-chainable.

Moreover

$U_\delta(a)\cap(A\cup C)\subset(\,U_{\delta_1}(a)\cap A\,)\ \cup(\,U_{\delta_2}(a)\cap C\,)\subset U_\varepsilon(a)$

$U_\delta(b)\cap(B\cup D)\subset(\,U_{\delta_1}(b)\cap B\,)\ \cup\ (\,U_{\delta_2}(b)\cap D\,)\subset U_\varepsilon(b)$.

Theorem 3

Let A,B ⊂ X such that $\langle A, B\rangle$ is locally chainable at (a,b) and let C,D be subsets of A and B respectively such that a and b are interior points of C and D, then $\langle C, D\rangle$ is locally chainable at (a, b).

Proof : Now a $\epsilon\, C^o$, b $\epsilon\, D^o$, hence $U_{\varepsilon_1}(a)\subset C \subset A$ and $U_{\varepsilon_1}(b)\subset D \subset B$ for some $\varepsilon_1 > 0$. Local chainability of $\langle A, B\rangle$at (a, b)⇒$\langle U_\delta(a)\cap A^o\,, U_\delta(b)\cap B^o\rangle$ is ε_1-chainable for some δ>0.

And $U_\delta(a)\cap C^o \subset U_\delta(a)\cap A^o \subset U_{\varepsilon_1}$(a)

$U_\delta(b)\cap D^o \subset U_\delta(b)\cap B^o \subset U_{\varepsilon_1}$(b)

x $\epsilon\, U_\delta(a)\cap C^o \Rightarrow$ d(x, a) <ε_1.

a $\epsilon\, U_\delta(a)\cap A^o$ is ε_1-chainable to some point of $U_\delta(b)\cap B^o$and hence to b. Thus x isε_1-chainable to b $\epsilon\, U_\delta(b)\cap D^0$. Likewise every member of $U_\delta(b)\cap D^o$ is ε_1-chainable to a $\epsilon\, U_\delta(a)\cap C^o$. Hence $\langle U_\delta(a)\cap C^o\,, U_\delta(b)\cap D^o\rangle$is ε-chainable,

on choosing ε >ε_1 (without loss of generality).

Also $U_\delta(a)\cap C^o \subset U_\varepsilon(a)$ and $U_\delta(b)\cap D^o \subset U_\varepsilon(b)$

It follows$\langle C, D\rangle$ is locally chainable at (a, b).

Corollary

Let $\langle A, B\rangle$ be locally chainable at (a, b) and a $\epsilon\, C^o$ and b $\epsilon\, D^o$ then $\langle A\cap C, B\cap D\rangle$ is locally chainable at (a, b).

Remark

Let A, B ⊂ X such that $\langle \bar{A}, \bar{B}\rangle$ is locally chainable at (a, b) and a and b be interior points of A and B respectively then$\langle A, B\rangle$ is locally chainable at (a, b).

Theorem 4

Let A, B ⊂ X and a ϵA , b ϵ B. If every neighborhood $U_\varepsilon(a)$ of a and $U_\varepsilon(b)$ of b contains subsets C and D of

A and B respectively such that $\langle C, D\rangle$ is ε-chainable and a ϵ C^o and b ϵ D^o then $\langle A, B\rangle$ is locally chainable at (a, b).

Proof : We have $U_{\varepsilon_1}(a) \subset C^o \subset A^o$ and b ϵ $U_{\varepsilon_1}(b) \subset D^0 \subset B^0$ for some $\varepsilon_1 > \varepsilon$.

Let x $\epsilon U_{\varepsilon_1}(a) \cap A^o = U_{\varepsilon_1}(a)$. Then x ϵC, and hence is ε-chainable to some y ϵ D.

As d(y, b) <ε, x is ε-chainable to b $\epsilon U_{\varepsilon_1}(b) \cap B^o$

$\Rightarrow \langle U_{\varepsilon_1}(a) \cap A^o, U_{\varepsilon_1}(b) \cap B^o\rangle$ is ε-chainable, hence $\langle A, B\rangle$ is locally chainable at (a, b).

Theorem 5

Let $\langle A, B\rangle$ be locally chainable . Then for each ε >0 , there exists open sets C and D , C $\subset U_\varepsilon(A)$, D $\subset U_\varepsilon(B)$ such that $\langle C, D\rangle$ is ε- chainable.

Proof : Let a ϵ A ,b ϵ B be arbitrary. Then for every ε >0 ,

$\langle U_\delta(a) \cap A^o, U_\delta(b) \cap B^o\rangle$ε- chainable for some $\delta > 0$ and $U_\delta(a) \cap A^o \subset U_\varepsilon(a)$, $U_\delta(b) \cap B^o \subset U_\varepsilon(b)$

Let C $= \bigcup_{\substack{\delta>0 \\ a\epsilon A}} U_\delta(a) \cap A^o$, D $= \bigcup_{\substack{\delta>0 \\ a\epsilon A}} U_\delta(b) \cap B^o$

Then C and D are open sets and $\langle C, D\rangle$ is ε- chainable. Moreover C $\subset U_\varepsilon(A)$, D $\subset U_\varepsilon(B)$.

Remarks : The sets C and D being open sets in A and B respectively show that $\langle C, D\rangle$ is locally chainable.

Theorem 6

Let A, B be open subsets of X such that $\langle A, B\rangle$ is locally chainable then -

(i) C(A) = C(B)

(ii) A, B, C(A) = C(B) are self – chainable.

(iii) $\langle A, B\rangle$is strongly chainable.

(iv) C(A) = C(B) is open.

Proof : (i) Let x ϵ C(A) ⇒for each ε > 0, x $\epsilon C_\varepsilon(A) \Rightarrow$ for each ε > 0, there exists a ϵ A such that x is ε-chainable to a. Now $\langle A, B\rangle$ is locally chainable at (a, b) for some b ϵ B.

$\Rightarrow \langle U_\delta(a) \cap A, U_\delta(b) \cap B\rangle$ is ε-chainable and $U_\delta(a) \cap A \subset U_\varepsilon(a)$ and $U_\delta(b) \cap B \subset U_\varepsilon(b)$. A is ε-chainable to $b_1 \epsilon$ $U_\delta(b) \cap B \subset U_\varepsilon(b) \Rightarrow$ a is ε-chainable to b or x is ε-chainableto b ∀ ε > 0 ⇒ xϵ C(B) ⇒ C(A) ⊂ C(B). Interchanging the roles of C(A) and C(B) we get C(A) = C(B).

(ii) As A ⊂C (A) every two points x and y in A are chainable to some point b ϵ B for each ε > 0 (refer (i)) so, x and y are ε-chainable. Consequently A is self - chainable. Similarly B is self – chainable. The definition of C (A) and the fact that A is self – chainable establishes that C (A) is self – chainable and so is C(B).

(iii) A ⊂C (A) ⇒aϵ A is ε-chainable to every b ϵ B for every ε > 0 (refer proof (i)) $\Rightarrow \langle A, B\rangle$ is strongly chainable.

(iv) Let x ϵ C (A). Then x is ε-chainable to b ∀ ε > 0 and ∀ b ϵ B (refer proof (i)). As B is open $U_{\varepsilon_1}(b) \subset$ B for some positive $\varepsilon_1 < \varepsilon$.

Let y $\epsilon U_{\varepsilon_1}(x) \Rightarrow$ d(x, y) < $\varepsilon_1 < \varepsilon$

or y is ε-chainable to b ∀ ε > 0.

⇒y ϵ C(B) = C(A)

⇒C(A) is open.

Theorem 7

Let A and B be open subsets of X such that $\langle A, B\rangle$ is strongly chainable, then$\langle A, B\rangle$is locally chainable.

Proof : Let (a, b) ϵ (A×B) and ε > 0 then $\langle U_\varepsilon(a) \cap A, U_\varepsilon(b) \cap B\rangle$ is ε-chainable and hence $\langle A, B\rangle$ is locally chainable.

Note : The above theorem is the converse of (iii) part of the previous theorem.

Theorem 8

If A and B are self-chainable open subsets of X such that C(A) = C(B), then $\langle A, B\rangle$ is locally chainable.

Proof : Let a ϵ A , b ϵ B and ε > 0 be arbitrary. Then $U_\delta(a)\subset$A ,$U_\delta(b)\subset$ B for some positive δ < ε.

Let x $U_\delta(b) \cap A^o$. Then C(A) = C(B) ⇒ x is ε-chainable to b_1 for some $b_1 \epsilon$ B. Since B is self-chainable it follows that x is ε-chainable to b ϵ $U_\delta(b) \cap B^o$. On interchanging the roles of $U_\delta(a) \cap A^o$ and $U_\delta(b) \cap B^o$ it follows that $\langle A, B\rangle$ is locally chainable.

CONFLICT OF INTEREST

No conflict of interest was declared by the authors.

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