**q −Bernoulli Matrices and Their Some Properties**

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**ABSTRACT**

In this study, we define \(q\)-Bernoulli matrix \(qB\) and \(q\)-Bernoulli polynomial matrix \((x,q)B\) by using \(q\)-Bernoulli numbers and polynomials respectively. We obtain some properties of \(qB\) and \((x,q)B\). We obtain factorizations \(q\)-Bernoulli polynomial matrix and shifted \(q\)-Bernoulli matrix using special matrices.

**Keywords:** \(q\)-Bernoulli numbers, \(q\)-Bernoulli matrix, \(q\)-Vandermonde matrix.

1. **INTRODUCTION**

Bernoulli numbers are defined by Jacob Bernoulli ([1]). Nörlund ([2]) and Carlitz ([3]) obtained some properties of Bernoulli numbers and polynomials. Carlitz ([4, 5]) defined \(q\)-Bernoulli numbers and polynomials. Hegazi ([10]) studied \(q\)-Bernoulli numbers and polynomials.

Let \(n\) be a positive integer and \(q \in (0,1)\). The quantum integer or Gauss number \([n]_q\) is defined by

\[
[n]_q = \frac{q^n - 1}{q - 1} = 1 + q + q^2 + \cdots + q^{n-1}.
\]

The \(q\)-analogue of \(n!\) is defined as follows

\[
[n]_q! = \begin{cases} 1 & \text{if } n = 0, \\ [n]_q [n-1]_q \cdots [1]_q & \text{if } n = 1, 2, \ldots. \end{cases}
\]

Gaussian or \(q\)-binomial coefficients are defined for integers \(n \geq k \geq 1\) as

\[
\binom{n}{k}_q = \frac{[n]_q!}{[n-k]_q! [k]_q!}
\]

with \(\binom{n}{0}_q = 1\) and \(\binom{n}{k}_q = 0\) for \(n < k\) ([6]). Some properties of \(q\)-binomial coefficients are

\[
\binom{n}{k}_q = \binom{n}{n-k}_q \quad \text{and} \quad \binom{n}{k}_q \binom{k}{j}_q = \binom{n}{j}_q \binom{n-j}{k-j}_q.
\]

The \(q\)-analogue of \((x-a)^n\) denoted \((x-a)_q^n\) is

\[
(x-a)_q^n = \begin{cases} 1 & \text{if } n = 0, \\ (x-a)(x-qa) \cdots (x-q^{n-1}a) & \text{if } n = 1, 2, \ldots. \end{cases}
\]

for \(x\) variable. Using definition of \(q\)-binomial coefficients it can be obtained
\[(x + a)^n = \sum_{k=0}^{n} \binom{n}{k} q^k a^{n-k} \]  \quad (1.3)

is called Gauss’s binomial formula.

### 2. BERNOULLI NUMBERS AND POLYNOMIALS

Firstly, we mention that Bernoulli numbers. Then using these numbers, a matrix can be delivered. This matrix is called Bernoulli matrix. Extending this matrix some matrices are obtained.

In [7], the Bernoulli numbers are defined initial condition by

\[B_0 = 1\] and

\[B_n = -\frac{1}{n+1} \sum_{k=0}^{n} \binom{n+1}{k} B_k \quad n = 1, 2, 3, \ldots \]  \quad (2.1)

The exponential generating function of Bernoulli numbers is

\[\frac{t}{e^t - 1} = \sum_{n=0}^{\infty} B_n \frac{t^n}{n!}. \]  \quad (2.2)

Let \(n\) be a nonnegative integer, the Bernoulli polynomials \(B_n(x)\) are defined by

\[B_n(x) = \sum_{n=0}^{\infty} \binom{n}{k} B_k x^{n-k}. \]  \quad (2.3)

Zhang defined Bernoulli matrices by using Bernoulli numbers and polynomials. Also the author obtained factorization and some properties of Bernoulli matrices [8].

**Definition 1.** [8] Let \(B_n\) be \(n^{th}\) Bernoulli number and \(B_n(x)\) be Bernoulli polynomial, \((n + 1) \times (n + 1)\) type Bernoulli matrix \(\mathbb{B} = [b_{ij}]\) and Bernoulli polynomial matrix \(\mathbb{B}(x) = [b_{ij}(x)]\) defined respectively as follows

\[b_{ij} = \binom{i+j}{i} B_{i-j} \quad \text{if } i \geq j, \quad \text{and otherwise,} \]  \quad (2.4)

and

\[b_{ij}(x) = \binom{i+j}{i} B_{i-j}(x) \quad \text{if } i \geq j, \quad \text{and otherwise.} \]  \quad (2.5)

It is know that the constant terms of \(B_n(x)\) Bernoulli polynomials are \(B_n\) Bernoulli numbers. Therefore we obtain Bernoulli \(\mathbb{B}\) matrix by using the constant term of \(\mathbb{B}(x)\) Bernoulli polynomial matrix [8].

Now we give definitions of \(q\) – Bernoulli numbers and \(q\) – Bernoulli polynomials.

**Definition 2.** [10] Let \(n\) be a nonnegative integer and \(B_n\) be \(n^{th}\) Bernoulli numbers. The \(q\) – Bernoulli numbers \(b_n(q)\) are defined by

\[b_n(q) = B_n \frac{[n]_q!}{n!}. \]  \quad (2.6)

The \(q\) – Bernoulli polynomials \(B_n(x, q)\) are defined by

\[B_n(x, q) = \sum_{k=0}^{n} \binom{n}{k}_q b_k(q) x^{n-k}. \]  \quad (2.7)

**Theorem 1.** [10] For \(q\) – commuting variables \(x\) and \(y\) such that \(xy = qxy\) we have

\[B_n(x + y, q) = \sum_{k=0}^{n} \binom{n}{k}_q b_k(q) y^{n-k} B_k(x, q). \]  \quad (2.8)

Similar considerations apply this theorem, it can easy to check that

\[B_n(x + y, q) = \sum_{k=0}^{n} \binom{n}{k}_q x^{n-k} B_k(y, q). \]  \quad (2.9)

### 3. \(q\) – BERNOULLI MATRICES

Zhang [8] defined generalized Bernoulli matrix by using Bernoulli numbers and polynomials. Then the author obtained factorization and some properties of the Bernoulli matrices.

Ernst [9] studied matrix form of \(q\) – Bernoulli polynomials and obtained recurrence formula using this matrix form. The author studied relation between \(q\) – Cauchy-Vandermonde matrix and the \(q\) – Bernoulli matrix. Then the author obtained \(q\) – analogue of the Bernoulli theorem by using the Jackson-Hahn-Cigler \(q\) – Bernoulli polynomials.

In this section, we define \(q\) – Bernoulli matrices by using \(q\) – Bernoulli numbers and \(q\) – Bernoulli polynomials. Then we obtain inverse of \(q\) – Bernoulli matrix and some theorems related to the generalized \(q\) – Bernoulli matrix.

**Definition 3.** Let \(b_n(q)\) be \(n^{th}\) \(q\) – Bernoulli number. The \(q\) – Bernoulli matrix \(\mathbb{B}(q) = [b_{ij}(q)]\) is defined by

\[b_{ij}(q) = \binom{i+j}{i} b_{i-j}(q) \quad \text{if } i \geq j, \quad \text{and otherwise,} \]  \quad (3.1)

where \(0 \leq i, j \leq n\).

5 \times 5 \(q\) – Bernoulli matrix is
\[\mathcal{B}(q) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ \frac{1}{2!} & 1 & 0 & 0 \\ \frac{3!}{2!} & \frac{3!}{2!} & 1 & 0 \\ \frac{4!}{3!} & \frac{4!}{3!} & \frac{4!}{3!} & 1 \end{pmatrix}\]

Following theorem is a generalization of Theorem 2.4 in [8].

**Theorem 2.** Let \(\mathcal{D}(q) = [d_{ij}(q)]\) be \((n + 1) \times (n + 1)\) matrix, is defined by

\[d_{ij}(q) = \binom{i}{j} q^{|i-j|} (i-j+1)\]  
if \(i \geq j\),

\[0\]  
otherwise. \hspace{1cm} (3.2)

Then \(\mathcal{D}(q)\) is the inverse of \(q\)–Bernoulli matrix.

**Proof.** Let \(\mathcal{B}(q)\) be \(q\)–Bernoulli matrix and \(\mathcal{D}(q)\) defined as in (3.2).

\[(\mathcal{B}(q) \mathcal{D}(q))_{ij} = \sum_{k=0}^{n} b_{ik}(q) d_{kj}(q)\]

\[= \sum_{k=j}^{i} \binom{i}{j} \binom{j}{k} b_{i-k}(q) \binom{k}{j} q^{k-j+1} \]

\[= \sum_{k=j}^{i} \binom{i}{j} \binom{j}{k} q^{k-j+1} b_{i-k}(q)\]

\[= \binom{i}{j} \sum_{k=j}^{i} (-1)^{i-k} q^{k-j+1} b_{i-k}(q)\]

\[= \binom{i}{j} \sum_{t=0}^{i-j} (-1)^{i-j} q^{k-j+1} \binom{t}{q} q^{t+1} b_{i-j-t}(q)\]

\[= \binom{i}{j} \sum_{t=0}^{i-j} (-1)^{i-j} q^{k-j+1} \binom{t}{q} q^{t+1} B_{i-j-t} (i-j-t)!\]

\[= \binom{i}{j} \sum_{t=0}^{i-j} (-1)^{i-j} q^{k-j+1} \binom{t}{q} q^{t+1} B_{i-j-t} (i-j-t)!\]

\[= \binom{i}{j} \sum_{t=0}^{i-j} (-1)^{i-j} q^{k-j+1} \binom{t}{q} q^{t+1} B_{i-j-t} (i-j-t)!\]

Using the orthogonality relation for Bernoulli numbers

\[\sum_{k=0}^{n} \binom{n}{k} \frac{1}{k+1} B_{n-k} = \delta_{n,0}\]  
(3.3)

(see [8]), we obtain

\[(\mathcal{B}(q) \mathcal{D}(q))_{ij} = \binom{i}{j} q^{i-j} (i-j)!\delta_{i-j,0} = \delta_{i,j}.

**Definition 4.** Let \(B_n(x,q)\) be \(n\)th \(q\)–Bernoulli polynomial. The \(q\)–Bernoulli polynomial matrix \(\mathcal{B}(x,q) = [b_{ij}(x,q)]\) is defined as follows

\[b_{ij}(x,q) = \binom{i}{j} B_{i-j}(x,q)\]  
if \(i \geq j\),

\[0\]  
otherwise. \hspace{1cm} (3.3)

**4. \(q\)–BERNOULLI AND \(q\)–PASCAL MATRICES**

Ernst [9] defined \((n + 1) \times (n + 1)\) generalized \(q\)–Pascal matrix \(\mathcal{P}(x,q) = [p_{ij}(q)]\) by

\[p_{ij}(q) = \binom{i}{j} \frac{x^{i-j}}{q!}\]  
if \(i \geq j\),

\[0\]  
otherwise. \hspace{1cm} (4.1)

The inverse of generalized \(q\)–Pascal matrix \(\mathcal{P}^{-1}(x,q) = [p_{ij}(q)]\) is

\[p_{ij}(q) = \binom{i}{j} q^{i-j} (-x)^{i-j}\]  
if \(i \geq j\),

\[0\]  
otherwise. \hspace{1cm} (4.2)

Now using the Zhang’s methods in [8] we can generalize the factorization \(q\)–Bernoulli matrices.

**Theorem 3.** Let \(\mathcal{B}(x,q)\) be \(q\)–Bernoulli polynomial matrix and \(\mathcal{P}(x,q)\) be generalized \(q\)–Pascal matrix, then

\[\mathcal{B}(x+y,q) = \mathcal{P}(x,q) \mathcal{B}(y,q) = \mathcal{P}(x,q) \mathcal{B}(y,q)\]  
(4.3)

and specially

\[\mathcal{B}(x,q) = \mathcal{P}(x,q) \mathcal{B}(y,q)\]. \hspace{1cm} (4.4)

**Proof.** Let \(\mathcal{P}(y,q)\) be generalized \(q\)–Pascal matrix and \(\mathcal{B}(x,q)\) be \(q\)–Bernoulli polynomial matrix. Then

\[(\mathcal{P}(y,q) \mathcal{B}(x,q))_{ij} = \sum_{k=0}^{n} p_{ik}(q) b_{kj}(x,q)\]

\[= \sum_{k=0}^{n} \binom{i}{k} q^{i-k} \binom{k}{q} B_{k-j}(x,q)\]

\[= \sum_{k=0}^{n} \binom{i}{k} q^{i-k} \binom{k}{q} B_{k-j}(x,q)\]

\[= \binom{i}{q} \sum_{t=0}^{i-j} (-1)^{i-j} q^{i-j} B_{t}(x,q).\]

Using (2.8), we have
\((\mathcal{P}(y, q) \mathcal{B}(x, q))_{ij} = \binom{i}{j} B_{i-j}(x+y, q) = (\mathcal{B}(x+y, q))_{ij}\)

and similarly it can be prove that
\[\mathcal{B}(x+y, q) = \mathcal{P}(x, q) \mathcal{B}(y, q).\]

Now, we show that
\[\mathcal{B}(x, q) = \mathcal{P}(x, q) \mathcal{B}(q).\]

\((\mathcal{P}(x, q) \mathcal{B}(q))_{ij} = \sum_{k=j}^{n} p_{jk}(q) b_{k-j}(q)
= \left( \sum_{k=j}^{i} \binom{i}{k} x^{i-k} \binom{k}{j} q b_{k-j}(q) \right)
= \binom{i}{j} \sum_{k=j}^{i} \binom{-j}{k} x^{i-k} b_{k-j}(q)
= \binom{i}{j} B_{i-j}(x, q)
= (\mathcal{B}(x, q))_{ij}\]

We give two examples of this theorem for \(3 \times 3\) and \(4 \times 4\) \(q\)-Bernoulli polynomial matrix and \(q\)-Pascal matrix.

\((\mathcal{P}(y, q) \mathcal{B}(x, q))_{ij}
= \begin{pmatrix} 1 & 0 & 0 \\ y & 1 & 0 \\ y^2 & 2y & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ x^{1/2} & 1 & 0 \\ x^2 + 2y & 2y & 1 \end{pmatrix}
= \mathcal{B}(x+y, q)\)

\((\mathcal{P}(x, q) \mathcal{B}(q))_{ij}
= \begin{pmatrix} 1 & 0 & 0 & 0 \\ x & 1 & 0 & 0 \\ x^2 & 2x & 1 & 0 \\ x^3 & 3x^2 & 2x & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ -\frac{1}{2} & 1 & 0 & 0 \\ \frac{1}{2}[x^2] & -\frac{1}{2}[x] & 1 & 0 \\ 0 & \frac{1}{2}[x^3] & -\frac{1}{2}[x^2] & 1 \end{pmatrix}
= \mathcal{B}(x, q)\)

**Corollary 1.** Let \(\mathcal{B}(x, q)\) be \(q\)-Bernoulli polynomial matrix then \(\mathcal{B}^{-1}(x, q) = [c_{ij}(q)]\) is
\[c_{ij}(q) = \begin{cases} \binom{i}{j} \sum_{t=0}^{i-j} \frac{q(t)}{(i-j-t+1)!} & \text{if } i \geq j, \\ 0 & \text{otherwise}. \end{cases}\]

**Proof.** Let \(\mathcal{B}(q)\) be \(q\)-Bernoulli matrix and \(\mathcal{P}(x, q)\) be generalized \(q\)-Pascal matrix. Using factorization of \(\mathcal{B}(x, q)\) in (4.4)
\[\mathcal{B}^{-1}(x, q) = \mathcal{B}^{-1}(q) \mathcal{P}^{-1}(x, q) = \mathcal{D}(q) \mathcal{D}^{-1}(x, q)\]
and inverse of generalized \(q\)-Pascal matrix (4.2), we obtain
\[(\mathcal{D}(q) \mathcal{D}^{-1}(x, q))_{ij} = \sum_{k=0}^{n} d_{ik}(q) p_{kj}(q)
= \sum_{k=j}^{i} \binom{i}{k} \binom{[i-k]_q}{k} \frac{q((i-k)_q)}{(i-k+1)!} x^{i-k} q^{((i-k)_q)}
= \sum_{k=j}^{i} \binom{i}{k} \binom{-j}{k} \frac{q((-j)_q)}{(i-k+1)!} x^{-j} q^{((-j)_q)}
= \binom{i}{j} \sum_{k=j}^{i} \frac{q((-j)_q)}{(i-j-t+1)!} x^{-j} q^{((-j)_q)}
= c_{ij}(q)\]

**5. Shifted \(q\)-BERNOULLI AND \(q\)-VANDERMONDE MATRICES**

In [8] Zhang defined shifted Bernoulli matrix, and obtained some relations between shifted Bernoulli matrix and Vandermonde matrix. In this section we define \(q\)-Vandermonde matrix and \(q\)-shifted Bernoulli matrix by using \(q\)-Bernoulli polynomials and give its relation with \(q\)-Vandermonde matrix.

**Definition 5.** Let \(B_n(x, q)\) be \(q\)-Bernoulli polynomial. The shifted \(q\)-Bernoulli matrix \(\mathcal{B}(y, q) = [b_{ij}(y, q)]\) is defined by
\[b_{ij}(y, q) = B_i(y+j, q)\]
where \(0 \leq i, j \leq n\).

**Definition 6.** ([11]) The \((n+1) \times (n+1)\) type \(q\)-Vandermonde matrix \(V(y, q) = [v_{ij}(y, q)]\) is defined by
\[v_{ij}(y, q) = (y+j)^i_q\]
\[ V(y, q) = \begin{pmatrix} 1 & 1 & 1 & 1 \\ y & y+1 & y+2 & y+3 \\ y^2 & (y+1)^2_q & (y+2)^2_q & (y+3)^2_q \\ y^3 & (y+1)^3_q & (y+2)^3_q & (y+3)^3_q \end{pmatrix} \]

In the following theorem, we obtain factorization shifted \( q - \) Bernoulli matrix by using \( q - \) Bernoulli matrix and \( q - \) Vandermonde matrix.

**Theorem 4.** Let \( V(y, q) \) be \( q - \) Vandermonde matrix and \( \tilde{\mathcal{B}}(q) \) be \( q - \) Bernoulli matrix. Then

\[
\tilde{\mathcal{B}}(y, q) = \tilde{\mathcal{B}}(q) V(y, q).
\]

**Proof.**

\[
(\tilde{\mathcal{B}}(q) V(y, q))_{ij} = \sum_{k=0}^{n} b_{ik}(q) v_{kj}(y, q)
\]

\[
= \sum_{k=0}^{n} \binom{i}{k} q b_{i-k}(y+j)^{k}.
\]

If we use the definition of \( q - \) Bernoulli polynomial, then

\[
(\tilde{\mathcal{B}}(q) V(y, q))_{ij} = B_i(y+j,q)
\]

we obtain

\[
\tilde{\mathcal{B}}(q) V(y, q) = \tilde{\mathcal{B}}(y, q).
\]

For \( q \to 1^- \), we can obtain Theorem 5.2 in [8].

The factorization of \( 3 \times 3 \) shifted \( q - \) Bernoulli matrix is as follows.

\[
\tilde{\mathcal{B}}(q) V(y, q)
\]

\[
= \begin{pmatrix} 1 & 0 & 0 \\ \frac{1}{2} & 1 & 0 \\ \frac{1}{2} & \frac{1}{2} & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 & 1 \\ y & y+1 & y+2 \\ y^2 & (y+1)^2_q & (y+2)^2_q \end{pmatrix}
\]

\[
= \begin{pmatrix} 1 & 0 & 0 \\ \frac{1}{2} & 1 & 0 \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix} \begin{pmatrix} 1 & 1 & 1 \\ y & y+1 & y+2 \\ y^2 & (y+1)^2_q & (y+2)^2_q \end{pmatrix}
\]

\[
= \frac{1}{2} ( y^2 - \frac{|2|}{2} y + \frac{|2|}{2} y^2 ) - 1
\]

**REFERENCES**


**CONFLICT OF INTEREST**

The authors declare that there is no conflict of interests regarding the publication of this paper.