Lyapunov-Type Inequalities for Two Classes of Difference Systems with Dirichlet Boundary Conditions

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ABSTRACT

In this paper, we establish Lyapunov-type inequalities for two classes of difference systems which improve all existing ones in the literature. Applying our inequalities, we obtain a lower bound for the eigenvalues of corresponding systems.

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1. INTRODUCTION

In 1983, Cheng [4] obtained the following inequality

\[ \Im(h - a) \sum_{\tau = a}^{b-2} f_1(\tau) \geq 4, \]  \hspace{1cm} (1.1)

where \( f_1(n) \geq 0 \) for all \( n \in \mathbb{Z} \) and

\[ \Im(z) = \begin{cases} \frac{z^2 - 1}{z}, & \text{if } z - 1 \text{ is even} \\ \frac{z}{z}, & \text{if } z - 1 \text{ is odd} \end{cases} \]  \hspace{1cm} (1.2)

if the second-order difference equation

\[ -\Delta^2 u_1(n) = f_1(n)u_1(n + 1) \]  \hspace{1cm} (1.3)

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has a real solution \( u_1(n) \) satisfying Dirichlet boundary conditions
\[
u_1(a) = 0 = u_1(b), u_1(n) \neq 0, n \in \mathbb{Z}[a,b], \tag{1.4}\]
a, b \in \mathbb{Z} with \( a \leq b - 2 \), and \( \mathbb{Z}[a,b] = (a, a + 1, a + 2, ..., b - 1, b) \), \( f_1 \) is a real-valued function defined on \( \mathbb{Z} \). The inequality (1.1) is a discrete analogue of the following so-called Lyapunov inequality
\[
(b - a) \int_a^b |f_1(s)| \, ds > 4 \tag{1.5}
\]
if Hill’s equation
\[
-u_1''(t) = f_1(t)u_1(t), \tag{1.6}
\]
where \( f_1 \in C([a,b], \mathbb{R}) \), has a real solution \( u_1(t) \) such that Dirichlet boundary conditions
\[
u_1(a) = 0 = u_1(b), u_1(t) \neq 0, t \in (a,b), \tag{1.7}
\]
where \( a, b \in \mathbb{R} \) with \( a < b \) [7].

In 2012, Zhang and Tang [15] obtained the following Lyapunov-type inequality for the 2k-th order difference equations
\[
-\Delta^{2k}u_1(n) = (-1)^{k-1}f_1(n)u_1(n + 1) \tag{1.8}
\]
with the boundary conditions
\[
\Delta^{2i}u_1(a) = 0 = \Delta^{2i}u_1(b), i = 0,1,..., k - 1; u_1(n) \neq 0, n \in \mathbb{Z}[a,b], \tag{1.9}
\]
where \( k \in \mathbb{N} \), \( n \in \mathbb{Z} \) and \( f_1(n) \) is a real-valued function defined on \( \mathbb{Z} \).

**Theorem A.** If (1.8) has a solution \( u_1(n) \) satisfying the boundary conditions (1.9), then the following inequality
\[
\sum_{\tau = a}^{b-1} |f_1(\tau)| (\tau - a + 1)(b - \tau - 1) \geq \frac{2^{3(k-1)}}{(b - a)^{2k-3}} \tag{1.10}
\]
holds.

It is easy to see that the inequality (1.10) is rewritten as
\[
\sum_{\tau = a}^{b-2} |f_1(\tau)| (\tau - a + 1)(b - \tau - 1) \geq \frac{2^{3(k-1)}}{(b - a)^{2k-3}}. \tag{1.11}
\]

Now, throughout the paper for the sake of brevity, we denote
\[
\zeta_i(n) = \sum_{\tau = a}^{n} r_{\tau}^{1/\{1 - p_i\}}(\tau) \quad \text{and} \quad \eta_i(n) = \sum_{\tau = a}^{b-2} r_{\tau}^{1/\{1 - p_i\}}(\tau) \tag{1.12}
\]
for \( i = 1,2,\ldots, m \).

In 2012, Zhang and Tang [14] obtained Lyapunov-type inequalities for the following systems
\[
\begin{cases}
-\Delta(r_1(n)\Delta u_1(n)|^{p_1-2}\Delta u_1(n)) = f_1(n)|u_1(n + 1)|^{\alpha_1}u_2(n + 1)\cdot \cdot \cdot |u_m(n + 1)|^{\alpha_m} \\
-\Delta(r_2(n)\Delta u_2(n)|^{p_2-2}\Delta u_2(n)) = f_2(n)|u_1(n + 1)|^{\beta_1}u_2(n + 1)\cdot \cdot \cdot |u_m(n + 1)|^{\beta_m} \tag{1.13}
\end{cases}
\]
and
\[
\begin{cases}
-\Delta(r_1(n)\Delta u_1(n)|^{p_1-2}\Delta u_1(n)) = f_1(n)|u_1(n + 1)|^{\alpha_1}u_2(n + 1)\cdot \cdot \cdot |u_m(n + 1)|^{\alpha_m} \\
-\Delta(r_2(n)\Delta u_2(n)|^{p_2-2}\Delta u_2(n)) = f_2(n)|u_1(n + 1)|^{\beta_1}u_2(n + 1)\cdot \cdot \cdot |u_m(n + 1)|^{\beta_m} \\
-\Delta(r_m(n)\Delta u_m(n)|^{p_m-2}\Delta u_m(n)) = f_m(n)|u_1(n + 1)|^{\alpha_1}u_2(n + 1)\cdot \cdot \cdot |u_m(n + 1)|^{\alpha_m} \cdot \cdot \cdot |u_m(n + 1)|^{\alpha_m}. \tag{1.14}
\end{cases}
\]

For the sake of convenience, we give the following hypotheses:

\( (H_1) \) \( r_i(n) \) and \( f_i(n) \) are real-valued functions and \( r_i(n) > 0, \forall n \in \mathbb{Z} \) and \( i = 1,2,\ldots, m \),

\( (H_2) \) \( 1 < p_1, p_2, \alpha_1, \beta_1 < \infty, \alpha_2, \beta_2 \geq 0 \) satisfy \( \frac{\alpha_1}{p_1} + \frac{\alpha_2}{p_2} = 1 \) and \( \frac{\beta_1}{p_1} + \frac{\beta_2}{p_2} = 1 \).
(Hₙ) 1 < pₖ < ∞, and αₖ ≥ 0 for k = 1, 2, ..., m satisfy \( \sum_{k=1}^{m} \frac{αₖ}{pₖ} = 1 \).

**Theorem B.** Let \( a, b \in \mathbb{Z} \) with \( a \leq b - 2 \). Suppose that hypotheses (H₂) with \( i = 1, 2 \) and (H₂) are satisfied. If the system (1.13) has a solution \( (u₁(n), u₂(n)) \) satisfying Dirichlet boundary conditions

\[
u_i(a) = 0 = u_i(b), \quad u_i(n) \neq 0, \quad n \in \mathbb{Z}[a, b], \quad i = 1, 2,
\]

then the following inequality

\[
\left( \sum_{t=a}^{b-2} \frac{(Lᵢ(t)‖ni(t)‖_{pᵢ-1}(t))^{pᵢ-1}}{ξᵢ^{pᵢ-1}(t) + ηᵢ^{pᵢ-1}(t)} fᵢ⁺(t) \right) \frac{αᵢ}{pᵢ} \left( \sum_{t=a}^{b-2} \frac{(Lᵢ(t)‖ni(t)‖_{pᵢ-1}(t))^{pᵢ-1}}{ξᵢ^{pᵢ-1}(t) + ηᵢ^{pᵢ-1}(t)} fᵢ⁺(t) \right) \geq 1
\]

holds, where \( fᵢ⁺(n) = \max\{0, fᵢ(n)\}, \quad i = 1, 2 \).

**Theorem C.** Let \( a, b \in \mathbb{Z} \) with \( a \leq b - 2 \). Suppose that hypotheses (H₁) and (H₂) are satisfied. If the system (1.14) has a solution \( (u₁(n), u₂(n), ..., uₘ(n)) \) satisfying Dirichlet boundary conditions

\[
u_i(a) = 0 = u_i(b), \quad u_i(n) \neq 0, \quad n \in \mathbb{Z}[a, b], \quad i = 1, 2, ..., m,
\]

then the following inequality

\[
\prod_{k=1}^{m} \prod_{l=1}^{m} \left( \sum_{t=a}^{b-2} \frac{(L_k(t)‖ni(t)‖_{p_k-1}(t))^{p_k-1}}{ξ_k^{p_k-1}(t) + η_k^{p_k-1}(t)} f_k⁺(t) \right) \frac{α_k}{p_k} \geq 1
\]

holds, where \( f_k⁺(n) = \max\{0, f_k(n)\}, \quad i = 1, 2, ..., m \).

**Remark 1.1.** It is clear that the system (1.13) with (1.4), (H₂), and the condition \( α₂=0 \) or \( β₁=0 \), or the system (1.14) with (1.4) and (H₂) for \( m = 1 \) reduces to the following problem

\[
\Delta(\{rₙ\}Δ₁u₁(n))|^{p₁-2}Δ₁u₁(n) = f₁(n)|u₁(n+1)|^{p₁-2}u₁(n+1)
\]

\[
u₁(a) = 0 = u₁(b).
\]

Moreover, when \( α₁ = p₁ \) for \( i = 1, 2, ..., m \) and for \( k \neq i, αₖ = 0 \) for \( k = 1, 2, ..., m \), we obtain a single equation similar to the equation (1.19) from the system (1.14).

Aktas et al. [1], Aktas [2], Canmak and Tiryaki [5, 6], Tang and He [9], and Tiryaki et al. [11] established Lyapunov-type inequalities for the continuous cases of systems (1.13) and/or (1.14) and their special cases. For some of the most recent works on Lyapunov-type inequalities, the reader is referred to [4, 6, 8-10, 12]. Motivated by the above-mentioned papers, we establish Lyapunov-type inequalities for systems (1.13) and (1.14) which are better than that of Zhang and Tang [14].

### 2. MAIN RESULTS

One of the main results of this paper for the system (1.13) is as follows.

**Theorem 2.1.** Let \( a, b \in \mathbb{Z} \) with \( a \leq b - 2 \). Suppose that hypotheses (H₁) with \( i = 1, 2 \) and (H₂) are satisfied. If the system (1.13) has a solution \( (u₁(n), u₂(n)) \) satisfying Dirichlet boundary conditions (1.15), then the following inequality

\[
\left( \sum_{t=a}^{b-2} \frac{(L₁(t)‖ni(t)‖_{p₁-1}(t))^{p₁-1}}{ξ₁^{p₁-1}(t) + η₁^{p₁-1}(t)} f₁⁺(t) \right) \frac{α₁}{p₁} \left( \sum_{t=a}^{b-2} \frac{(L₁(t)‖ni(t)‖_{p₁-1}(t))^{p₁-1}}{ξ₁^{p₁-1}(t) + η₁^{p₁-1}(t)} f₁⁺(t) \right) \geq 1
\]

holds, where \( f₁⁺(n) = \max\{0, f₁(n)\} \) for \( i = 1, 2 \).
Proof. Let \( u_i(\alpha) = 0 = u_i(b) \) and \( u_i(n) \neq 0, n \in \mathbb{Z}[a, b], i = 1, 2 \) hold. Multiplying the first equation of system (1.13) by \( u_1(n+1) \) and the second equation of system (1.13) by \( u_2(n+1) \), summing from \( a \) to \( b-2 \) and taking into account that \( u_i(\alpha) = 0 = u_i(b) \) for \( i = 1, 2 \), we get

\[
\sum_{t=a}^{b-1} r_1(t) |\Delta u_1(t)|^{p_1} = \sum_{t=a}^{b-2} f_1(t) |u_1(t+1)|^{\alpha_1} |u_2(t+1)|^{\alpha_2} \leq \sum_{t=a}^{b-2} f_1^+(t) |u_1(t+1)|^{\alpha_1} |u_2(t+1)|^{\alpha_2}
\]  

(2.2)

and

\[
\sum_{t=a}^{b-1} r_2(t) |\Delta u_2(t)|^{p_1} = \sum_{t=a}^{b-2} f_2(t) |u_1(t+1)|^{\beta_1} |u_2(t+1)|^{\beta_2} \leq \sum_{t=a}^{b-2} f_2^+(t) |u_1(t+1)|^{\beta_1} |u_2(t+1)|^{\beta_2}
\]  

(2.3)

It follows from (1.12), (1.15), and Hölder’s inequality that

\[
|u_i(n+1)|^{p_i} = \left( \sum_{t=a}^{n} r_i(t) |\Delta u_i(t)|^{p_i} \right)^{1/p_i} \leq \left( \sum_{t=a}^{n} (\Delta u_i(t))^{p_i} \right)^{1/p_i}
\]  

(2.4)

and

\[
\left( \sum_{t=n+1}^{b-1} r_i(t) |\Delta u_i(t)|^{p_i} \right)^{1/p_i} \leq \sum_{t=a}^{b-1} r_i(t) |\Delta u_i(t)|^{p_i}
\]

(2.5)

for \( i = 1, 2 \) and \( a \leq n \leq b-1 \). Adding (2.4) and (2.5), we have

\[
|u_i(n+1)|^{p_i} \leq \left( \frac{\zeta_i(n) \eta_i(n)}{\zeta_i^{p_i-1}(n) + \eta_i^{p_i-1}(n)} \right)^{1/p_i} \sum_{t=a}^{b-1} r_i(t) |\Delta u_i(t)|^{p_i}
\]

(2.6)

for \( i = 1, 2 \) and \( a \leq n \leq b-1 \). If we take the \( \frac{\alpha_1}{p_1} \)-th and \( \frac{\beta_1}{p_1} \)-th powers of both sides of the inequality (2.6) with \( i = 1 \), we have

\[
|u_1(n+1)|^{\alpha_1} \leq \left( \frac{\zeta_1(n) \eta_1(n)}{\zeta_1^{p_1-1}(n) + \eta_1^{p_1-1}(n)} \right)^{\frac{\alpha_1}{p_1}} \left( \sum_{t=a}^{b-1} r_1(t) |\Delta u_1(t)|^{p_1} \right)^{\frac{\alpha_1}{p_1}}
\]  

(2.7)

and

\[
|u_1(n+1)|^{\beta_1} \leq \left( \frac{\zeta_1(n) \eta_1(n)}{\zeta_1^{p_1-1}(n) + \eta_1^{p_1-1}(n)} \right)^{\frac{\beta_1}{p_1}} \left( \sum_{t=a}^{b-1} r_1(t) |\Delta u_1(t)|^{p_1} \right)^{\frac{\beta_1}{p_1}}
\]  

(2.8)

respectively. Thus, from (2.2), we have

\[
|u_1(n+1)|^{\alpha_1} \leq \left( \frac{\zeta_1(n) \eta_1(n)}{\zeta_1^{p_1-1}(n) + \eta_1^{p_1-1}(n)} \right)^{\frac{\alpha_1}{p_1}} \left( \sum_{t=a}^{b-2} f_1^+(t) |u_1(t+1)|^{\alpha_1} |u_2(t+1)|^{\alpha_2} \right)^{\frac{\alpha_1}{p_1}}
\]

(2.9)

and
Similarly, if we take the \(\frac{\alpha_2}{p_2}\)-th and \(\frac{\beta_2}{p_2}\)-th powers of both sides of the inequality (2.6) with \(i = 2\), we have

\[
|u_2(n + 1)|^{\alpha_2} \leq \left( \frac{(\xi_2(n)\eta_2(n))^{p_2-1}}{\xi_2^{p_2-1}(n) + \eta_2^{p_2-1}(n)} \right)^{\frac{\alpha_2}{p_2}} \left( \sum_{t=a}^{b-2} f_2^+(t)|u_1(t + 1)|^{\beta_2}|u_2(t + 1)|^{\beta_2} \right) \tag{2.12}
\]

and

\[
|u_2(n + 1)|^{\beta_2} \leq \left( \frac{(\xi_2(n)\eta_2(n))^{p_2-1}}{\xi_2^{p_2-1}(n) + \eta_2^{p_2-1}(n)} \right)^{\frac{\beta_2}{p_2}} \left( \sum_{t=a}^{b-2} f_2^+(t)|u_1(t + 1)|^{\beta_2}|u_2(t + 1)|^{\beta_2} \right) \tag{2.13}
\]

respectively. Multiplying both sides of (2.13) by \(f_2^+(n)|u_1(n + 1)|^{\beta_1}\), summing from \(a\) to \(b - 2\), we have

\[
\left( \sum_{t=a}^{b-2} f_2^+(t)|u_1(t + 1)|^{\beta_1}|u_2(t + 1)|^{\beta_2} \right)^{\frac{1}{1 - \frac{\beta_1}{p_1}}} \leq \sum_{t=a}^{b-2} f_2^+(t)|u_1(t + 1)|^{\beta_1}|u_2(t + 1)|^{\beta_2} \tag{2.14}
\]

By using (2.12) in (2.11) and (2.10) in (2.14), we have

\[
\left( \sum_{t=a}^{b-2} f_1^+(t)|u_1(t + 1)|^{\alpha_1}|u_2(t + 1)|^{\alpha_2} \right)^{\frac{1}{1 - \frac{\alpha_1}{p_1}}} \leq M_1 \left( \sum_{t=a}^{b-2} f_1^+(t)|u_1(t + 1)|^{\beta_1}|u_2(t + 1)|^{\beta_2} \right) \tag{2.15}
\]

and

\[
\left( \sum_{t=a}^{b-2} f_2^+(t)|u_1(t + 1)|^{\beta_1}|u_2(t + 1)|^{\beta_2} \right)^{\frac{1}{1 - \frac{\beta_1}{p_1}}} \leq M_2 \left( \sum_{t=a}^{b-2} f_2^+(t)|u_1(t + 1)|^{\beta_1}|u_2(t + 1)|^{\beta_2} \right) \tag{2.16}
\]

where

\[
M_1 = \sum_{t=a}^{b-2} f_1^+(t) \left( \frac{(\xi_1(t)\eta_1(t))^{p_1-1}}{\xi_1^{p_1-1}(t) + \eta_1^{p_1-1}(t)} \right)^{\frac{\alpha_1}{p_1}} \left( \frac{(\xi_2(t)\eta_2(t))^{p_2-1}}{\xi_2^{p_2-1}(t) + \eta_2^{p_2-1}(t)} \right)^{\frac{\alpha_2}{p_2}} \tag{2.17}
\]

and

\[
M_2 = \sum_{t=a}^{b-2} f_2^+(t) \left( \frac{(\xi_1(t)\eta_1(t))^{p_1-1}}{\xi_1^{p_1-1}(t) + \eta_1^{p_1-1}(t)} \right)^{\frac{\beta_1}{p_1}} \left( \frac{(\xi_2(t)\eta_2(t))^{p_2-1}}{\xi_2^{p_2-1}(t) + \eta_2^{p_2-1}(t)} \right)^{\frac{\beta_2}{p_2}} \tag{2.18}
\]

respectively. If we take \(e_1\)-th and \(e_2\)-th powers of both sides of inequalities (2.15) and (2.16), and multiplying the resulting inequalities, we obtain

\[
\left( \sum_{t=a}^{b-2} f_1^+(t)|u_1(t + 1)|^{\alpha_1}|u_2(t + 1)|^{\alpha_2} \right)^{\frac{1}{1 - \frac{\alpha_1}{p_1}}} \left( \sum_{t=a}^{b-2} f_2^+(t)|u_1(t + 1)|^{\beta_1}|u_2(t + 1)|^{\beta_2} \right)^{\frac{1}{1 - \frac{\beta_1}{p_1}}} \leq M_1^{\frac{\alpha_1}{p_1}} \left( \sum_{t=a}^{b-2} f_1^+(t)|u_1(t + 1)|^{\alpha_1}|u_2(t + 1)|^{\alpha_2} \right) \tag{2.19}
\]

Next, we prove that
If (2.20) is not true, then
\[ 0 < \sum_{r=a}^{b-2} f_1^+(r)|u_1(r+1)|^{\alpha_1}|u_2(r+1)|^{\alpha_2}. \]  
(2.20)

From (2.2) and (2.21), we have
\[ 0 \leq \sum_{r=a}^{b-1} r_1(r)|\Delta u_1(r)|^{p_1} = \sum_{r=a}^{b-2} f_1(r)|u_1(r+1)|^{\alpha_1}|u_2(r+1)|^{\alpha_2} \leq \sum_{r=a}^{b-2} f_1^+(r)|u_1(r+1)|^{\alpha_1}|u_2(r+1)|^{\alpha_2} = 0. \]  
(2.22)

It follows from (H_i) with \( i = 1 \) that
\[ \Delta u_1(n) \equiv 0 \]  
(2.23)
for \( a \leq n \leq b - 1 \). Combining (2.6) for \( i = 1 \) with (2.23), we obtain that \( u_1(n) \equiv 0 \) for \( a \leq n \leq b \), which contradicts (1.15) with \( i = 1 \). Therefore, (2.20) holds. Similarly, we have
\[ 0 < \sum_{r=a}^{b-2} f_2^+(r)|u_1(r+1)|^{\beta_1}|u_2(r+1)|^{\beta_2}. \]  
(2.24)

Now, we choose \( e_1 \) and \( e_2 \) such that
\[ 0 < \sum_{r=a}^{b-2} f_1^+(r)|u_1(r+1)|^{\alpha_1}|u_2(r+1)|^{\alpha_2} \quad \text{and} \quad 0 < \sum_{r=a}^{b-2} f_2^+(r)|u_1(r+1)|^{\beta_1}|u_2(r+1)|^{\beta_2} \]  
(2.25)
cancel out in the inequality (2.19), i.e. solve the homogeneous linear system
\[ \begin{cases} (1 - \frac{\alpha_1}{p_1}) e_1 - \frac{\beta_1}{p_1} e_2 = 0 \\ \alpha_2 e_1 - (1 - \frac{\beta_2}{p_2}) e_2 = 0. \end{cases} \]  
(2.26)

We observe that by hypotheses \( \frac{\alpha_1}{p_1} + \frac{\alpha_2}{p_2} = 1 \) and \( \frac{\beta_1}{p_1} + \frac{\beta_2}{p_2} = 1 \), this system admits a nontrivial solution, indeed all equations are equivalent to \( (1 - \frac{\alpha_1}{p_1}) e_1 = \frac{\beta_1}{p_1} e_2 \) and \( \alpha_2 e_1 = (1 - \frac{\beta_2}{p_2}) e_2 \). Hence, we may take \( e_1 = \frac{\beta_1}{p_1} \) and \( e_2 = \frac{\alpha_2}{p_2} \), and we get the inequality (2.1) which completes the proof.  

The following result gives the new Lyapunov-type inequality for the system (1.14).

**Theorem 2.2.** Let \( a, b \in \mathbb{Z} \) with \( a \leq b - 2 \). Suppose that hypotheses (H_2) and (H_3) are satisfied. If the system (1.14) has a solution \( (u_1(n), u_2(n), \ldots, u_m(n)) \) satisfying Dirichlet boundary conditions (1.17), then the following inequality
\[ \sum_{i=1}^{m} \left( \sum_{r=a}^{b-2} f_i^+(r) \left( \frac{\xi_k(r)\eta_k(r)}{\xi_k^{p_k-1}(r) + \eta_k^{p_k-1}(r)} \right)^{\frac{1}{p_i}} \right)^{p_i} \geq 1 \]  
(2.27)
holds, where \( f_i^+(n) = \max(0, f_i(n)) \) for \( i = 1, 2, \ldots, m \).

**Proof.** Let \( u_i(a) = 0 = u_i(b) \) and \( u_i(n) \not\equiv 0, n \in \mathbb{Z}[a, b], i = 1, 2, \ldots, m \) hold. Multiplying the \( i \)-th equation of system (1.14) by \( u_i(n+1) \) and summing from \( a \) to \( b - 2 \) and taking into account that \( u_i(a) = 0 = u_i(b) \) for \( i = 1, 2, \ldots, m \), we get
\[ \sum_{r=a}^{b-1} r_1(r)|\Delta u_i(r)|^{p_i} = \sum_{r=a}^{b-2} f_i(r) \left( \frac{\xi_k(r)\eta_k(r)}{\xi_k^{p_k-1}(r) + \eta_k^{p_k-1}(r)} \right)^{\frac{1}{p_i}} \leq \sum_{r=a}^{b-2} f_i^+(r) \left( \frac{\xi_k(r)\eta_k(r)}{\xi_k^{p_k-1}(r) + \eta_k^{p_k-1}(r)} \right)^{\frac{1}{p_i}} \]  
(2.28)
for \( i = 1, 2, \ldots, m \). By using \( u_i(a) = 0, (1.12) \) and Hölder’s inequality, we get
\[ |u_i(n+1)|^{p_i} = \left| \sum_{r=a}^{b} \Delta u_i(r) \right|^{p_i} \leq \]
for $i = 1, 2, ..., m$ and $a \leq n \leq b - 1$. Similarly, by using $u_i(b) = 0$, (1.12) and Hölder’s inequality, we get

$$|u_i(n + 1)|^{p_i} \leq \eta_i^{p_i-1}(n) \sum_{r=a+1}^{b-1} r_i(\tau)|\Delta u_i(\tau)|^{p_i}$$

(2.30)

for $i = 1, 2, ..., m$ and $a \leq n \leq b - 1$. Adding (2.29) and (2.30), we have

$$|u_i(n + 1)|^{p_i} \leq \left(\frac{\zeta_i(n)}{\zeta_i(n)}\right)^{p_i-1}(n) + \eta_i^{p_i-1}(n) \sum_{r=a}^{b-1} r_i(\tau)|\Delta u_i(\tau)|^{p_i}$$

(2.31)

for $i = 1, 2, ..., m$ and $a \leq n \leq b - 1$. If we take the $\frac{a_i}{p_i}$-th power of both sides of the inequality (2.31), we obtain

$$|u_i(n + 1)|^{a_i} \leq \left(\frac{\zeta_i(n)}{\zeta_i(n)}\right)^{a_i/p_i} \left(\sum_{r=a}^{b-1} r_i(\tau)|\Delta u_i(\tau)|^{p_i}\right)^{a_i/p_i}$$

(2.32)

Multiplying both sides of (2.31) by $f_i^+(n) \prod_{k=1}^{m} |u_k(n + 1)|^{a_k}$, summing from $a$ to $b - 2$, we have

$$\sum_{r=a}^{b-2} f_i^+(\tau) \prod_{k=1}^{m} |u_k(\tau + 1)|^{a_k} \leq \sum_{r=a}^{b-2} \frac{\zeta_i(\tau)}{\zeta_i(\tau)} \eta_i^{p_i-1}(\tau) \left(\sum_{r=a}^{b-1} r_i(\tau)|\Delta u_i(\tau)|^{p_i}\right)^{a_i/p_i} \left(\sum_{r=a}^{b-1} r_i(\tau)|\Delta u_i(\tau)|^{p_i}\right) f_i^+(\tau) \prod_{k=1}^{m} |u_k(\tau + 1)|^{a_k}$$

(2.33)

for $i = 1, 2, ..., m$. By using (2.28) in (2.33), we have

$$\sum_{r=a}^{b-1} r_i(\tau)|\Delta u_i(\tau)|^{p_i} \leq \sum_{r=a}^{b-1} \frac{\zeta_i(\tau)}{\zeta_i(\tau)} \eta_i^{p_i-1}(\tau) \left(\sum_{r=a}^{b-1} r_i(\tau)|\Delta u_i(\tau)|^{p_i}\right)^{a_i/p_i} \left(\sum_{r=a}^{b-1} r_i(\tau)|\Delta u_i(\tau)|^{p_i}\right) f_i^+(\tau) \prod_{k=1}^{m} |u_k(\tau + 1)|^{a_k}$$

(2.34)

for $i = 1, 2, ..., m$. Therefore, by using (2.31) in (2.34), we have

$$\left(\sum_{r=a}^{b-1} r_i(\tau)|\Delta u_i(\tau)|^{p_i}\right)^{1/a_i} \leq \prod_{k=1}^{m} \left(\sum_{r=a}^{b-1} r_k(\tau)|\Delta u_k(\tau)|^{p_k}\right)^{a_k/a_k} \prod_{k=1}^{m} \left(\frac{\zeta_k(\tau)}{\zeta_k(\tau)} \eta_k^{p_k-1}(\tau) \right)^{a_k/p_k}$$

(2.35)

for $i = 1, 2, ..., m$. If we take the $\epsilon_i$-th power of both side of the inequalities (2.35) for $i = 1, 2, ..., m$, and multiplying the resulting inequalities, we obtain

$$\prod_{i=1}^{m} \left(\sum_{r=a}^{b-1} r_i(\tau)|\Delta u_i(\tau)|^{p_i}\right)^{1/a_i} \leq \prod_{i=1}^{m} \left(\sum_{r=a}^{b-1} r_k(\tau)|\Delta u_k(\tau)|^{p_k}\right)^{a_k/a_k} \prod_{k=1}^{m} \left(\frac{\zeta_k(\tau)}{\zeta_k(\tau)} \eta_k^{p_k-1}(\tau) \right)^{a_k/p_k}$$

(2.36)

and hence

$$\prod_{i=1}^{m} \left(\sum_{r=a}^{b-1} r_i(\tau)|\Delta u_i(\tau)|^{p_i}\right)^{1/a_i} \leq \prod_{i=1}^{m} \left(\sum_{r=a}^{b-1} r_k(\tau)|\Delta u_k(\tau)|^{p_k}\right)^{a_k/a_k} \prod_{k=1}^{m} \left(\frac{\zeta_k(\tau)}{\zeta_k(\tau)} \eta_k^{p_k-1}(\tau) \right)^{a_k/p_k}$$

(2.37)

It is easy to see that by using similar technique to the proof of Theorem 2.1, we obtain the following inequality

$$0 < \sum_{r=a}^{b-1} r_i(\tau)|\Delta u_i(\tau)|^{p_i}$$

(2.38)
for \( i = 1, 2, \ldots, m \). Now, we choose \( e_i \) such that \( 0 < \sum_{r=d}^{b-1} r f_i(t) |\Delta u_i(t)|^{p_i} \) for \( i = 1, 2, \ldots, m \) cancel out in the inequality (2.37), i.e. solve the homogeneous linear system

\[
\begin{aligned}
(p_1 - \alpha_1) e_1 - \alpha_1 e_2 & - \alpha_1 e_3 - \cdots - \alpha_1 e_m = 0 \\
-\alpha_2 e_1 + (p_2 - \alpha_2) e_2 - \alpha_2 e_3 - \cdots - \alpha_2 e_m &= 0 , \\
-\alpha_M e_1 - \alpha_M e_2 - \cdots - (p_m - \alpha_M) e_m &= 0 .
\end{aligned}
\]

(2.39)

We observe that by hypothesis \( \sum_{i=1}^{m} \frac{e_i}{p_i} = 1 \), this system admits a nontrivial solution, indeed all equations are equivalent to

\[
\alpha_i \left( \sum_{k=1}^{m} e_k \right) = e_i \left( \sum_{k=1}^{m} \frac{e_k}{p_k} \right)
\]

for \( i = 1, 2, \ldots, m \). Hence, we may take \( e_i = \frac{e_i}{p_i} \) for \( i = 1, 2, \ldots, m \), and we get the inequality (2.27) which completes the proof.

**Remark 2.1.** It is easy to see that if we use generalized Hölder’s inequality to the inequalities (2.1) and (2.27), then they reduce to the inequalities (1.16) and (1.18) obtained by Zhang and Tang [14], respectively. Thus, they are sharper than (1.16) and (1.18). Moreover, if we take \( r_1(n) = 1 \) and \( p_i = 2 \) in the problem (1.19)-(1.20), then Theorems 2.1, 2.2, B, and C are equivalent. In this case, from the inequalities (1.16), (1.18), (2.1), and (2.27), we get

\[
\sum_{r=d}^{b-1} f_i^+(r) (r - a + 1) (b - r - 1) \geq b - a .
\]

(2.40)

If we also take \( m = 2 \) in the system (1.14), and \( \beta_1 = \alpha_1 \) and \( \beta_2 = \alpha_2 \) in the system (1.13), then Theorems 2.1 and 2.2 are equivalent.

**Remark 2.2.** Note that since \( f_i^+(n) \leq \| f_i(n) \| \), the inequality (2.40) is better than the inequality (1.11) with \( k = 1 \). Moreover, by using

\[
M(n) = (n - a + 1) (b - n - 1) \leq \max_{a \leq n < b - 1} M(n) = M \left( \frac{a + b}{2} - 1 \right) = \left( \frac{b - a}{2} \right)^2
\]

in the inequality (2.40), we get

\[
\sum_{r=d}^{b-1} f_i^+(r) \geq \frac{4}{b - a} .
\]

(2.41)

Therefore, if we take \( f_i(n) \geq 0 \), then when \( b - a - 1 \) is odd, (2.41) is the same as (1.1). However, (2.41) is worse than (1.1) when \( b - a - 1 \) is even.

Now, we apply our Lyapunov-type inequalities to obtain a lower bound for the first eigencurve in the generalized spectra. Let \( a, b \in \mathbb{Z} \) with \( a \leq b - 2 \). We consider here the following difference system

\[
\begin{aligned}
-\Delta (|\Delta u_1(n)|^{p_1-2} \Delta u_1(n)) &= \lambda_1 a_1 q(n) |u_1(n + 1)|^{a_1} |u_1(n + 1)|^{a_1} \cdots |u_1(n + 1)|^{a_1} \\
-\Delta (|\Delta u_2(n)|^{p_2-2} \Delta u_2(n)) &= \lambda_2 a_2 q(n) |u_2(n + 1)|^{a_2} |u_2(n + 1)|^{a_2} \cdots |u_2(n + 1)|^{a_2} \\
-\Delta (|\Delta u_m(n)|^{p_m-2} \Delta u_m(n)) &= \lambda_m a_m q(n) |u_m(n + 1)|^{a_m} |u_m(n + 1)|^{a_m} \cdots |u_m(n + 1)|^{a_m} ,
\end{aligned}
\]

(2.42)

where \( q(n) > 0, \lambda_i \in \mathbb{R}, p_i \) and \( a_i \) are the same as those in the hypothesis \( (H_i) \), and \( u_i \) satisfies Dirichlet boundary conditions

\[
u_i(a) = 0 = u_i(b), u_i(n) \neq 0, n \in \mathbb{Z}[a + 1, b - 1], i = 1, 2, \ldots, m .
\]

(2.43)

We define the generalized spectrum \( S \) of a nonlinear difference system as the set of vector \( (\lambda_1, \lambda_2, \ldots, \lambda_m) \in \mathbb{R}^m \) such that the eigenvalue problem (2.42)-(2.43) admits a nontrivial solution.

Boundary problem (2.42)-(2.43) is a generalization of the following \( p_1 \)-Laplacian difference equation

\[
-\Delta (|\Delta u_1(n)|^{p_1-2} \Delta u_1(n)) = \lambda_1 p_1 q(n) |u_1(n + 1)|^{p_1} \Delta u_1(n + 1)
\]

(2.44)

with Dirichlet boundary conditions

\[
u_1(a) = 0 = u_1(b), u_1(n) \neq 0, n \in \mathbb{Z}[a + 1, b - 1] .
\]

(2.45)

where \( p_1 > 1, \lambda_1 \in \mathbb{R} \), and \( q(n) > 0 \). When \( p_1 = 2 \), Atkinson [3, Theorems 4.3.1 and 4.3.5] investigated the existence of eigenvalues for (2.44)-(2.45), see also [13].

Let \( f_i(n) = \lambda_i a_i q(n) \) for \( i = 1, 2, \ldots, m \). Then we can apply Theorem 2.2 to boundary problem (2.42)-(2.43) and obtain a lower bound for the \( m \)-th component of any generalized eigenvalue \( (\lambda_1, \lambda_2, \ldots, \lambda_m) \) of the system (2.42).
Theorem 2.3. Let $a, b \in \mathbb{Z}$ with $a \leq b - 2$. Assume that $1 < p_i < \infty, a_i > 0$ satisfy $\sum_{i=1}^{m} \frac{a_i}{p_i} = 1$, and $q(n) > 0$ for all $n \in \mathbb{Z}$. Then there exists a function $h(\lambda_1, \lambda_2, \ldots, \lambda_{m-1})$ such that $|\lambda_m| \geq h(\lambda_1, \lambda_2, \ldots, \lambda_{m-1})$ for every generalized eigenvalue $(\lambda_1, \lambda_2, \ldots, \lambda_{m-1})$ of problem (2.42)-(2.43), where $h(\lambda_1, \lambda_2, \ldots, \lambda_{m-1})$ is given by

$$h(\lambda_1, \lambda_2, \ldots, \lambda_{m-1}) = \frac{1}{a_m} \left[ \prod_{k=1}^{m-1} (\lambda_k \alpha_k) \sum_{r=0}^{b-2} q(r) \prod_{k=1}^{m} \left( \frac{\zeta_k(t) \eta_k(t) \rho_{k-1}(t)}{\zeta_k \rho_{k-1}(t) + \eta \rho_{k-1}(t)} \right) \right].$$

Proof. For the eigenvalue $(\lambda_1, \lambda_2, \ldots, \lambda_{m-1})$, (2.42)-(2.43) has a nontrivial solution $(u_1(n), u_2(n), \ldots, u_m(n))$. That is the system (1.14) with $f_i(n) = \lambda_i a_i q(n)$ has a solution $(u_1(n), u_2(n), \ldots, u_m(n))$ satisfying (1.17), it follows from (2.27) that $f_i(n) = \lambda_i a_i q(n)$, for all $n \in \mathbb{Z}, i = 1, 2, \ldots, m$, and that

$$1 \leq \prod_{i=1}^{m} \left[ \sum_{r=0}^{b-2} f_i^2(t) \prod_{k=1}^{m} \left( \frac{\zeta_k(t) \eta_k(t) \rho_{k-1}(t)}{\zeta_k \rho_{k-1}(t) + \eta \rho_{k-1}(t)} \right) \right] \leq \prod_{i=1}^{m} \left[ \sum_{r=0}^{b-2} q(t) \prod_{k=1}^{m} \left( \frac{\zeta_k(t) \eta_k(t) \rho_{k-1}(t)}{\zeta_k \rho_{k-1}(t) + \eta \rho_{k-1}(t)} \right) \right].$$

Hence, we have

$$|\lambda_m| \geq \frac{1}{a_m} \left[ \prod_{k=1}^{m-1} (\lambda_k \alpha_k) \sum_{r=0}^{b-2} q(r) \prod_{k=1}^{m} \left( \frac{\zeta_k(t) \eta_k(t) \rho_{k-1}(t)}{\zeta_k \rho_{k-1}(t) + \eta \rho_{k-1}(t)} \right) \right].$$

This completes the proof. □

CONFLICT OF INTEREST

No conflict of interest was declared by the authors.

REFERENCES
