EFFICIENT FAMILY OF EXPONENTIAL ESTIMATORS FOR THE POPULATION MEAN

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Abstract

This paper deals with the estimation of the population mean with improved family of exponential estimators for the variable under study using some known population parameters of the auxiliary variable. The expressions for the bias and mean square error (MSE) of the estimators of the proposed family have been derived to the first degree of approximation. A comparison has been made with the exponential family of estimators of Singh et al. [10]. An improvement has been shown over the family of estimators of Singh et al. [10] through an empirical study.

Keywords: Exponential estimator, auxiliary variable, bias, mean square error, efficiency.

2000 AMS Classification: 62D05

1. Introduction

The use of auxiliary information increases the precision of the estimates of the parameters under consideration. When the variable under study, \( y \), is highly correlated with the auxiliary variable, \( x \), the ratio and product estimators are used for the improved estimation of parameters of the variable under study. To obtain the most efficient estimator, many authors have proposed ratio and product type estimators using some known parameters of the auxiliary variable. The first estimator utilizing the auxiliary information is well known ratio estimator due to Cochran [2] as

\[
(1.1) \quad t_R = \bar{y}
\]

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To the first order of approximation, the bias and mean square error, respectively, are

\[ B(t_R) = f \hat{Y} C_y^2 [1 - C], \]
\[ MSE(t_R) = f \hat{Y}^2 [C_y^2 + C_x^2 (1 - 2C)], \]

where \( f = \frac{n - n}{N n} \). Bahl and Tuteja [1] was the first to suggest an exponential ratio type estimator as

\[ t_1 = \bar{y} \exp(\frac{\bar{X} - \bar{x}}{\bar{X} + \bar{x}}) \]

with the bias and mean square error, respectively, up to the first order of approximation as

\[ B(t_1) = f C_x^2 \left( \frac{Y}{8} \right) (3 - 4C), \]
\[ MSE(t_1) = f \hat{Y}^2 [C_y^2 + \frac{C_x^2}{4} (1 - 4C)] \]

Upadhyaya et al. [13] proposed a modified exponential ratio type estimator as

\[ t_{Re}^{(a)} = \bar{y} \exp(\frac{\bar{X} - \bar{x}}{\bar{X} + (a - 1)\bar{x}}) \]

where \( a \) is some suitable constant.

To the first order of approximation, the bias and mean square error, respectively, are

\[ B(t_{Re}^{(a)}) = f \hat{Y} \left( \frac{C_y^2}{2a^2} \right) |2a(1 - C) - 1|, \]
\[ MSE(t_{Re}^{(a)}) = f \hat{Y}^2 [C_y^2 + \frac{C_x^2}{a^2} (1 - 2aC)] \]

where \( C = \rho C_y C_x \).

MSE in (1.4) is minimum for \( a = \frac{1}{4} \) and the minimum mean square error is equal to the MSE of the usual linear regression estimator.

There are several authors who used the known parameters of the auxiliary variable to find more precise estimate of the population parameters of the variable under study, including Sen [7], Sisodia and Dwivedi [11], Upadhyaya and Singh [12], Singh and Kakran [8], Singh and Tailor [9], Kadilar and Cingi [3, 4] and Khoshnevisan et al. [5] have suggested modified ratio estimators utilizing different known values of population parameters of the auxiliary variable.

In this paper, we have proposed an improved exponential family of ratio type estimators for the population mean of the variable under study using some known parameters of the auxiliary variable.

Let the population consists of \( N \) units and a sample of size \( n \) is drawn with the simple random sampling without replacement (SRSWOR). Let \( Y_i \) and \( X_i \) be the values of the study and auxiliary variables for the \( i \)th unit \((i = 1, 2, \ldots, N)\) of the population, respectively. Further, let \( \bar{y} \) and \( \bar{x} \) be the sample means of the study and auxiliary variables, respectively.

To obtain the bias and mean square error (MSE) of the estimators, let \( \bar{y} = \hat{Y} (1 + e_0) \) and \( \bar{x} = \hat{X} (1 + e_1) \) such that \( E(e_i) = 0 \), \( i = 0, 1 \) and \( E(e_0^2) = f C_y^2 \), \( E(e_1^2) = f C_x^2 \), and \( E(e_0 e_1) = f C_{yx} = f \rho C_y C_x \), where \( C_y^2 = \frac{S_y^2}{n} \), and \( C_x^2 = \frac{S_x^2}{n} \).
2. The Suggested Exponential Family of Estimators

Singh et al. [10] defined an exponential family of estimators for the population mean in the simple random sampling as

\[ t = \bar{y} \exp \left( \frac{(a\bar{x} + b) - (a\bar{x} + b)}{(a\bar{x} + b) + (a\bar{x} + b)} \right), \]

where \( a(\neq 0) \) and \( b \) are either real numbers or the functions of the known parameters of the auxiliary variable, \( x \), such as coefficient of variation \( (C_x) \), coefficient of kurtosis \( (\beta_2(x)) \) and correlation coefficient \( (\rho) \).

The bias and MSE of this family of estimators, to the first degree of approximation are, respectively, as follows:

\[ B(t) = f \bar{Y} (2\theta^2 C_x^2 - \theta \rho C_y C_x), \]

\[ MSE(t) = f \bar{Y}^2 (C_y^2 + \theta^2 C_x^2 - 2\theta \rho C_y C_x), \]

where \( \theta = \frac{a\bar{X}}{2(a\bar{X} + \bar{b})} \).

The ratio estimators, given in Table 1, are the members of the \( t \)-family of estimators in (2.1) and the mean square error for these estimators is

\[ MSE(t_i) = f \bar{Y}^2 (C_y^2 + \theta_i^2 C_x^2 - 2\theta_i \rho C_y C_x), \quad i = 2, 3, \ldots, 10 \]

where \( \theta_2 = \frac{\bar{X}}{2(\bar{X} + \beta_2(x))}, \theta_3 = \frac{\bar{X}}{2(\bar{X} + \rho)}, \theta_4 = \frac{\bar{X}}{2(\bar{X} + \beta_2(x) + \beta_2(x))}, \theta_5 = \frac{\beta_2(x) \bar{X}}{2(\bar{X} + \beta_2(x))}, \theta_6 = \frac{\beta_2(x) \bar{X}}{2(\bar{X} + \beta_2(x) + \beta_2(x))}, \theta_7 = \frac{\beta_2(x) \bar{X}}{2(\bar{X} + \rho)}, \theta_8 = \frac{\beta_2(x) \bar{X}}{2(\bar{X} + \beta_2(x) + \beta_2(x))}, \theta_9 = \frac{\beta_2(x) \bar{X}}{2(\bar{X} + \rho)}, \theta_{10} = \frac{\beta_2(x) \bar{X}}{2(\bar{X} + \beta_2(x) + \beta_2(x))}. \]

Many more estimators can be generated from the estimator, \( t \), in (2.1) just by putting different values of \( a \) and \( b \).

Motivated by Koyuncu and Kadilar [6], we propose a new family of exponential estimators as

\[ \xi = k \bar{y} \exp \left( \frac{(a\bar{x} + b) - (a\bar{x} + b)}{(a\bar{x} + b) + (a\bar{x} + b)} \right), \]

where \( k \) is suitably chosen constant to be determined later and \( a(\neq 0) \) and \( b \) are either real numbers or the functions of the known parameters of the auxiliary variable, such as the coefficient of variation \( (C_x) \), the coefficient of kurtosis \( (\beta_2(x)) \), and the correlation coefficient \( (\rho) \).

Now, expressing the estimator \( \xi \) in terms of \( e_i(i = 0, 1) \), we can write (2.5) as

\[ \xi = k \bar{Y} (1 + e_0) \exp \left\{ -\theta e_1 (1 + \theta e_1)^{-1} \right\}. \]

Expanding (2.6) on right hand side to the first degree of approximation and subtracting \( \bar{Y} \) from both sides, we get

\[ \xi - \bar{Y} = k \bar{Y} (1 + e_0 - \theta e_1 + 2\theta^2 e_1^2 - \theta e_0 e_1) - \bar{Y}. \]

Taking the expectation on both sides of (2.7), we get the bias of the estimator \( \xi \) as

\[ B(\xi) = k \bar{Y} f (2\theta^2 C_x^2 - \theta \rho C_y C_x) + \bar{Y} (k - 1). \]

Squaring both sides of (2.7), we have

\[ (\xi - \bar{Y})^2 = k^2 \bar{Y}^2 (1 + e_0 - \theta e_1 + 2\theta^2 e_1^2 - \theta e_0 e_1)^2 + \]

\[ \bar{Y}^2 - 2k \bar{Y}^2 (1 + e_0 - \theta e_1 + 2\theta^2 e_1^2 - \theta e_0 e_1). \]

Taking expectation of both sides, we get the MSE of the estimator, \( \xi \), to the first degree of approximation as

\[ MSE(\xi) = \bar{Y}^2 \left\{ k^2 f C_x^2 + (5k^2 - 4k) f \theta^2 C_x^2 - 2(2k^2 - k) f \theta \rho C_y C_x + (k - 1)^2 \right\}. \]
The minimum $MSE(\xi)$ is obtained for the optimal value of $k$ which is $k_{opt} = \frac{A}{2}$, where $A = f(2\theta^2C_x^2 - \theta \rho C_y C_x) + 1$ and $B = f(C_y^2 + 5\theta^2C_x^2 - 4\theta \rho C_y C_x) + 1$. Therefore, the minimum $MSE$ of the estimator, $\xi$, is

\[ MSe_{\min}(\xi) = \bar{Y}^2 \left[ 1 - \frac{A^2}{B} \right]. \]

For the ratio estimators, given in Table 2, the expression for the $MSE$ is obtained by

\[ MSE(\xi_i) = \bar{Y}^2 \left\{ k_{opt}^2 f C_y^2 + (5k_{opt}^2 - 4k_{opt})f \theta^2 C_x^2 \right\} - 2(2k_{opt} - k_{opt})f \theta_1 \rho C_y C_x + (k_{opt} - 1)^2 \right\}; \quad i = 1, 2, 3, ..., 10. \]

Many more estimators can be generated from the estimator, $\xi$, in (2.5) just by putting different values of $a$ and $b$.

**Table 1. Some Members of the $t$-family of Estimators**

<table>
<thead>
<tr>
<th>Estimators</th>
<th>$a$</th>
<th>$b$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$t_0 = \bar{Y}$ (The Sample Mean)</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>$t_1 = \bar{Y} \exp \left( \frac{X - \bar{Y}}{\sigma \bar{Y}} \right)$ (Bahl and Tuteja [1])</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>$t_2 = \bar{Y} \exp \left( \frac{X - \bar{Y}}{\sigma \bar{Y}^2} \cdot \frac{X - \bar{Y}}{\sigma^2 \bar{Y}^2} \cdot (x) \right)$</td>
<td>1</td>
<td>$\beta_2(x)$</td>
</tr>
<tr>
<td>$t_3 = \bar{Y} \exp \left( \frac{X - \bar{Y}}{\sigma \bar{Y}^2} \cdot \frac{X - \bar{Y}}{\sigma^2 \bar{Y}^2} \cdot C_x \right)$</td>
<td>1</td>
<td>$C_x$</td>
</tr>
<tr>
<td>$t_4 = \bar{Y} \exp \left( \frac{X - \bar{Y}}{\sigma \bar{Y}^2} \cdot \frac{X - \bar{Y}}{\sigma^2 \bar{Y}^2} \cdot \rho \right)$</td>
<td>1</td>
<td>$\rho$</td>
</tr>
<tr>
<td>$t_5 = \bar{Y} \exp \left( \frac{\beta_2(x)}{(X - \bar{Y})} \cdot \frac{\beta_2(x)}{\beta_2(x)(X - \bar{Y}) + 2C_x} \cdot \beta_2(x) \cdot C_x \right)$</td>
<td>$\beta_2(x)$</td>
<td>$C_x$</td>
</tr>
<tr>
<td>$t_6 = \bar{Y} \exp \left( \frac{C_x(X - \bar{Y})}{\beta_2(x)(X - \bar{Y}) + 2C_x} \cdot \beta_2(x) \cdot C_x \right)$</td>
<td>$C_x$</td>
<td>$\beta_2(x)$</td>
</tr>
<tr>
<td>$t_7 = \bar{Y} \exp \left( \frac{C_x(X - \bar{Y})}{\rho(X - \bar{Y} + 2C_x)} \cdot \rho \cdot C_x \right)$</td>
<td>$C_x$</td>
<td>$\rho$</td>
</tr>
<tr>
<td>$t_8 = \bar{Y} \exp \left( \frac{\rho(X - \bar{Y})}{\rho(X - \bar{Y} + 2C_x)} \cdot \rho \cdot C_x \right)$</td>
<td>$\rho$</td>
<td>$C_x$</td>
</tr>
<tr>
<td>$t_9 = \bar{Y} \exp \left( \frac{\beta_2(x)}{(X - \bar{Y})} \cdot \frac{\beta_2(x)}{\beta_2(x)(X - \bar{Y}) + 2C_x} \cdot \beta_2(x) \right)$</td>
<td>$\beta_2(x)$</td>
<td>$\rho$</td>
</tr>
<tr>
<td>$t_{10} = \bar{Y} \exp \left( \frac{\rho(X - \bar{Y})}{\rho(X - \bar{Y} + 2\beta_2(x))} \cdot \rho \cdot \beta_2(x) \right)$</td>
<td>$\rho$</td>
<td>$\beta_2(x)$</td>
</tr>
</tbody>
</table>

### 3. Efficiency Comparisons

The $t$-family of estimators is more efficient than the Bahl and Tuteja [1] estimator, $t_1$, if

\[ MSE(t_i) < MSE(t_1) \quad i = 2, ..., 10, \quad \text{i.e.,} \quad (\theta_1^2 - \frac{1}{4})C_x^2 + (2\theta_1 - 1)\rho C_y C_x > 0. \]

When the condition (3.1) is satisfied, we may say that $t$-family is more efficient than the estimator $t_1$.

The proposed $\xi$- family of estimators is more efficient than the estimator $t_1$ if

\[ MSE_{\min}(\xi_i) < MSE(t_1) \quad i = 1, 2, ..., 10, \quad \text{i.e.,} \quad (1 - \frac{A^2}{B}) < f(C_y^2 + C_x^2 - \rho C_y C_x). \]

When the condition (3.2) is satisfied, we may infer that $\xi$-family is more efficient than the estimator $t_1$. 
Table 2. Some Members of the $\xi$-family of Estimators

<table>
<thead>
<tr>
<th>Estimators</th>
<th>$a$</th>
<th>$b$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\xi_1 = k \bar{y} \exp \left[ \frac{X - \bar{x}}{\bar{y}^2 + 2\beta_2(x)} \right]$</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>$\xi_2 = k \bar{y} \exp \left[ \frac{X - \bar{x}}{\bar{y}^2 + 2\beta_2(x)} \right]$</td>
<td>1</td>
<td>$\beta_2(x)$</td>
</tr>
<tr>
<td>$\xi_3 = k \bar{y} \exp \left[ \frac{X - \bar{x}}{\bar{y}^2 + 2C_x} \right]$</td>
<td>1</td>
<td>$C_x$</td>
</tr>
<tr>
<td>$\xi_4 = k \bar{y} \exp \left[ \frac{X - \bar{x}}{\bar{y}^2 + 2\rho} \right]$</td>
<td>1</td>
<td>$\rho$</td>
</tr>
<tr>
<td>$\xi_5 = k \bar{y} \exp \left[ \frac{\beta_2(x)(X - \bar{x})}{\beta_2(x)(X - \bar{x}) + 2C_x} \right]$</td>
<td>$\beta_2(x)$</td>
<td>$C_x$</td>
</tr>
<tr>
<td>$\xi_6 = k \bar{y} \exp \left[ \frac{C_x(X - \bar{x})}{C_x(X - \bar{x}) + 2\beta_2(x)} \right]$</td>
<td>$C_x$</td>
<td>$\beta_2(x)$</td>
</tr>
<tr>
<td>$\xi_7 = k \bar{y} \exp \left[ \frac{\rho(X - \bar{x})}{\rho(X - \bar{x}) + 2\rho} \right]$</td>
<td>$C_x$</td>
<td>$\rho$</td>
</tr>
<tr>
<td>$\xi_8 = k \bar{y} \exp \left[ \frac{\rho(X - \bar{x})}{\rho(X - \bar{x}) + 2\rho} \right]$</td>
<td>$\rho$</td>
<td>$C_x$</td>
</tr>
<tr>
<td>$\xi_9 = k \bar{y} \exp \left[ \frac{\beta_2(x)(X - \bar{x})}{\beta_2(x)(X - \bar{x}) + 2\rho} \right]$</td>
<td>$\beta_2(x)$</td>
<td>$\rho$</td>
</tr>
<tr>
<td>$\xi_{10} = k \bar{y} \exp \left[ \frac{\rho(X - \bar{x})}{\rho(X - \bar{x}) + 2\rho} \right]$</td>
<td>$\rho$</td>
<td>$\beta_2(x)$</td>
</tr>
</tbody>
</table>

The proposed family $\xi$ of estimators is more efficient than $t$-family of estimators if

$$MSE_{\min}(\xi_i) < MSE(t_i) \quad i = 1, \ldots, 10,$$

i.e., 

$$1 - \frac{A^2}{B} < f(C_y^2 + \theta_1^2C_x^2 - 2\theta_1\rho C_y C_x).$$

The proposed family $\xi$ of estimators is more efficient than Upadhyaya et al. [13] estimator if

$$1 - \frac{A^2}{B} < f [C_y^2 + \frac{C_x^2}{a^2}(1 - 2a\rho)]$$

for suitable positive value of $a$.

4. The Empirical Study

We have used the data in Koyuncu and Kadilar [6], given in Table 3, to compare the efficiencies between the $t$-family and the proposed $\xi$-family of estimators for the population mean under the simple random sampling.

Table 3. Data Statistics

<table>
<thead>
<tr>
<th>$N$</th>
<th>$n$</th>
<th>$Y$</th>
</tr>
</thead>
<tbody>
<tr>
<td>923</td>
<td>180</td>
<td>436.4345</td>
</tr>
<tr>
<td>$X$</td>
<td>$C_y = 1.7183$</td>
<td>$C_x = 1.8645$</td>
</tr>
<tr>
<td>$\rho$</td>
<td>$\beta_1(x) = 3.9365$</td>
<td>$\beta_2(x) = 18.7208$</td>
</tr>
</tbody>
</table>

The MSE values of the $t$ and $\xi$ estimators have been obtained using (2.4) and (2.11), respectively, and these values are presented in Table 4.
Table 4. Mean Square Error of the $t$ and $\xi$ Families

<table>
<thead>
<tr>
<th>Estimators</th>
<th>MSE</th>
<th>Estimators</th>
<th>MSE</th>
</tr>
</thead>
<tbody>
<tr>
<td>$t_1$</td>
<td>655.1199</td>
<td>$\xi_1$</td>
<td>651.4819</td>
</tr>
<tr>
<td>$t_2$</td>
<td>656.9777</td>
<td>$\xi_2$</td>
<td>653.3295</td>
</tr>
<tr>
<td>$t_3$</td>
<td>655.3104</td>
<td>$\xi_3$</td>
<td>651.6914</td>
</tr>
<tr>
<td>$t_4$</td>
<td>655.2152</td>
<td>$\xi_4$</td>
<td>651.5962</td>
</tr>
<tr>
<td>$t_5$</td>
<td>655.1390</td>
<td>$\xi_5$</td>
<td>651.5009</td>
</tr>
<tr>
<td>$t_6$</td>
<td>656.1295</td>
<td>$\xi_6$</td>
<td>652.4724</td>
</tr>
<tr>
<td>$t_7$</td>
<td>655.1771</td>
<td>$\xi_7$</td>
<td>651.5199</td>
</tr>
<tr>
<td>$t_8$</td>
<td>655.3295</td>
<td>$\xi_8$</td>
<td>651.6724</td>
</tr>
<tr>
<td>$t_9$</td>
<td>655.1390</td>
<td>$\xi_9$</td>
<td>651.5009</td>
</tr>
<tr>
<td>$t_{10}$</td>
<td>657.0628</td>
<td>$\xi_{10}$</td>
<td>653.4247</td>
</tr>
</tbody>
</table>

5. Conclusion

From Table 4, we observe that $MSE_{\text{min}}(\xi_i) < MSE(t_i)$ for $i = 1, 2, 3, \ldots, 10$ and therefore $\xi_i$ as well as every member of $\xi$-family is more efficient than the Bahl and Tuteja [1] estimator. Every member of $\xi$-family is more efficient than the corresponding as well as every member of $t$-family given by Singh et al. [10]. Thus, from the results of the empirical study, we may conclude that the proposed $\xi$-family of exponential estimators of the population mean is more efficient than $t$-family of exponential estimators. Hence, the use of suggested estimators should be preferred in practice.

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References

