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ON ONE WEIGHTED INEQUALITIES FOR CONVOLUTION TYPE OPERATOR

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Abstract

In this paper we prove the boundedness of certain convolution operator in a weighted Lebesgue space with kernel satisfying the generalized Hörmander's condition. The sufficient conditions for the pair of general weights ensuring the validity of two-weight inequalities of a strong type and of a weak type for convolution operator with kernel satisfying the generalized Hörmander's condition are found.

Keywords: Weighted Lebesgue space, Singular integral, Kernel, Generalized Hörmander's condition, Boundedness.

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1. Introduction.

Let $\mathbb{R}^n$ be n-dimensional Euclidean spaces of points $x = (x_1, \dots, x_n)$, where $n \in \mathbb{N}$ and $\mathbb{R}^n_0 = \mathbb{R}^n \setminus \{0\}$. Suppose that ω is a non-negative, Lebesgue measurable and real function defined on $\mathbb{R}^n$, i.e., ω is a weight function defined on $\mathbb{R}^n$. By $L_{p,\,\omega}(\mathbb{R}^n)$ we denote the weighted Lebesgue space of measurable functions f on $\mathbb{R}^n$ such that

$$\|f\|_{L_{p,\,\omega}(\mathbb{R}^n)} = \|f\|_{p,\,\omega} = \left(\int\limits_{\mathbb{R}^n} |f(x)|^p \,\omega(x)\,dx\right)^{1/p} < \infty,\ \ 1 \le p < \infty.$$

In the case $p = \infty$, the norm on the space $L_{\infty,\omega}(\mathbb{R}^n)$ is defined as

$$\|f\|_{L_{\infty,\omega}(\mathbb{R}^n)} = \|f\|_\infty = ess \sup_{x\in\mathbb{R}^n} |f(x)|.$$

For $\omega = 1$ we obtain the nonweighted L_p spaces, i.e., $\|f\|_{L_{p,\,1}(\mathbb{R}^n)} = \|f\|_{L_p(\mathbb{R}^n)} = \|f\|_p$.

Our aim in this paper is to show the boundedness of certain convolution operator in a weighted Lebesgue space with kernel satisfying the generalized Hörmander's condition.

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The sufficient conditions for the pair of general weights ensuring the validity of two-weight inequalities of a strong type and of a weak type for convolution operator with kernel satisfying the generalized Hörmander's condition are found. In particular, is given a class $B(u,v)$ of weight pair which is generalized earlier obtained results (see below). Also, in this paper we give a weight pairs which satisfy the condition of obtain results.

Now we give a chronological development of earlier results. Let $K:\mathbb{R}_0^n\to\mathbb{R}$, $K\in L_1^{loc}(\mathbb{R}_0^n)$ is a function satisfy following conditions:

1) $K(tx)\equiv K(tx_1,\ldots,tx_n)=t^{-n}K(x)$ for all $t>0$ and $x\in\mathbb{R}_0^n$;

2) $\int\limits_{|\xi|=1} K(\xi)\,d\sigma(\xi)=0$;

3) $\int\limits_0^1 \frac{w(t)}{t}\,dt<\infty$, where $w(t)=\sup\limits_{|\xi-\eta|\le t}|K(\xi)-K(\eta)|$ for $|\xi|=|\eta|=1$.

We consider the following singular integral

$$Af(x)=\lim_{\varepsilon\to+0}\int\limits_{|x-y|>\varepsilon} K(x-y)\,f(y)\,dy=p.v.\int\limits_{\mathbb{R}^n} K(x-y)\,f(y)\,dy, \tag{1.1}$$

where $f\in C_0^\infty(\mathbb{R}^n)$ and last integral is understood in the sense of principal value.

The following Calderon-Zygmund Theorem is valid.

1.1. Theorem. [3, 4] *Let $1<p<\infty$ and A be a singular integral operator with kernel K satisfying conditions 1)-3). Then singular integral Af is exist for almost every (a.e.) $x\in R^n$ and the inequality*

$$\|Tf\|_p\le C\|f\|_p$$

holds, where a constant $C>0$ is independent of $f\in L_p(\mathbb{R}^n)$.

Further development of this theory is closely related the boundedness of Calderon-Zygmund singular integral operator in the weighted Lebesgue space with power weights. Namely, in the paper [13] Stein proved the following Theorem.

1.2. Theorem. [13] *Let $1<p<\infty$, $-n<\alpha<n(p-1)$ and A be singular integral operator* (1.1) *with kernel K satisfying conditions 1)-3). Then singular integral Af is exist for a.e. $x\in R^n$ and the inequality*

$$\|Tf\|_{p,\,|x|^\alpha}\le C\|f\|_{p,\,|x|^\alpha}$$

holds, where a constant $C>0$ is independent on $f\in L_{p,\,|x|^\alpha}(\mathbb{R}^n)$.

Further Hörmander in the paper [9] replacing the condition 3) weaker condition proved the following Theorem.

1.3. Theorem. [9] *Let $1<p<\infty$ and A be singular integral operator* (1.1) *with kernel K satisfying conditions 1), 2) and*

$$\int\limits_{|x|>2|y|} |K(x-y)-K(x)|\,dx\le C_1, \tag{1.2}$$

where $C_1>0$ doesn't depend on $y\in\mathbb{R}_0^n$. Then singular integral Af is exist for a.e. $x\in R^n$ and the inequality

$$\|Af\|_p\le C\|f\|_p$$

holds, where a constant $C>0$ is independent of $f\in L_p(\mathbb{R}^n)$.

On the other hand, the convolution operators whose kernels do not satisfy Hörmander's condition (1.2) have been widely considered (for example, oscillatory and other singular integral) (see [5]).

Now we formulated the known results connected with generalized Hörmander's condition.

1.4. Definition. [7] A positive measurable and locally integrable function g is said to satisfy the reverse Hölder RH_∞ condition or $g \in RH_\infty(\mathbb{R}^n)$ if

$$0 < \sup_{x \in B} g(x) \le C \frac{1}{|B|} \int_B g(x)\, dx,$$

where B is an arbitrary ball centered at the origin and $C > 0$ is a constant independent of B.

Let $K \in L_2(\mathbb{R}^n)$ is a function satisfy the following conditions:

(a) $\left\| \widehat{K} \right\|_\infty \le C$;

(b) there exist functions $A_1, \dots, A_m$ and $\Phi = \{\varphi_1, \dots, \varphi_m\}$ such that $\varphi_i \in L_\infty(\mathbb{R}^n)$ and $|det\,[\varphi_j(y_i)]|^2 \in RH_\infty(\mathbb{R}^{nm})$, $y_i \in \mathbb{R}^n$, $i, j = 1, \dots, m$;

(c) for a fixed $\gamma > 0$ and for any $|x| > 2|y| > 0$ the inequality

$$\int_{|x|>2|y|} \left| K(x-y) - \sum_{i=1}^m A_i(x)\, \varphi_i(y) \right| dx \le C$$

is valid;

(d) $|K(x)| \le \dfrac{C}{|x|^n}$.

It is obvious that condition (c) is a generalization the condition (1.2) for $m = 1$, $A_1(x) = K(x)$ and $\varphi_1(x) \equiv 1$.

For $f \in C_0^\infty(\mathbb{R}^n)$ we define the convolution operator associated to the kernel K by

$$Tf(x) = \int_{\mathbb{R}^n} K(x-y)\, f(y)\, dy. \tag{1.3}$$

1.5. Theorem. [7] *Let $1 < p < \infty$ and T be a convolution operator with kernel K satisfying (a)-(c). Then the inequality*

$$\|Tf\|_p \le C \|f\|_p$$

holds, where a constant C depend only on p, n and the constant in the RH_∞-condition for the functions φ_j.

For $p = 1$ there exists a constant C such that

$$|\{x : \ |Tf(x)| > \lambda\}| \le \frac{C}{\lambda} \int_{\mathbb{R}^n} |f(x)|\, dx,$$

for every smooth function f with compact support and $\lambda > 0$.

Note that Theorem 1.3 is particular case of Theorem 1.5 for $m = 1$, $A_1(x) = K(x)$ and $\varphi_1(x) \equiv 1$.

2. Preliminaries

2.1. Remark. It is clear that from condition RH_∞ implies the well known reverse Hölder inequality

$$\left(\frac{1}{|B|}\int\limits_B [g(x)]^{1+\varepsilon}\,dx\right)^{\frac{1}{1+\varepsilon}} \le C\,\left(\frac{1}{|B|}\int\limits_B g(x)\,dx\right),$$

where $\varepsilon > 0$. It is well known that the reverse Hölder condition be characterized the condition $A_p(\mathbb{R}^n)$ (see [5]).

2.2. Example. Let $m = 2$, $K(x) = \dfrac{\sin x}{x}$, $x \in \mathbb{R}\setminus\{0\}$, $A_1(x) = \dfrac{e^{ix}}{2i\,x}$ $A_2(x) = -\dfrac{e^{ix}}{2i\,x}$, $\varphi_1(y) = e^{-iy}$ and $\varphi_2(y) = e^{iy}$. Then the conditions (a)-(d) hold (see [2]).

We will also need the following theorem.

2.3. Theorem. [12] *Let $1 < q < p < \infty$ and $u(t)$ and $v(t)$ be positive functions on $(0,\infty)$. Suppose that $F : (0,\infty) \mapsto \mathbb{R}$ be a Lebesgue measurable function.*

1. For the validity of the inequality

$$\left(\int\limits_0^\infty u(t)\left|\int\limits_0^t F(\tau)\,d\tau\right|^q dt\right)^{1/q} \le C_1\,\left(\int\limits_0^\infty |F(t)|^p\,v(t)\,dt\right)^{1/p}$$

it is necessary and sufficient that

$$\int\limits_0^\infty\left[\left(\int\limits_t^\infty u(\tau)\,d\tau\right)\left(\int\limits_0^t v^{1-p'}(\tau)\,d\tau\right)^{q-1}\right]^{\frac{p}{p-q}} v^{1-p'}(t)\,dt < \infty,$$

where $C_1 > 0$ is independent of F.

2. For the validity of the inequality

$$\left(\int\limits_0^\infty u(t)\left|\int\limits_t^\infty F(\tau)\,d\tau\right|^q dt\right)^{1/q} \le C_2\,\left(\int\limits_0^\infty |F(t)|^p\,v(t)\,dt\right)^{1/p}$$

it is necessary and sufficient that

$$\int\limits_0^\infty\left[\left(\int\limits_0^t u(\tau)\,d\tau\right)\left(\int\limits_t^\infty v^{1-p'}(\tau)\,d\tau\right)^{q-1}\right]^{\frac{p}{p-q}} v^{1-p'}(t)\,dt < \infty,$$

where $C_2 > 0$ is independent of F.

For $q = 1$ the following Lemma is valid.

2.4. Lemma. [11] *Let $p > 1$ and $u(t)$ and $v(t)$ be positive functions on $(0,\infty)$.*

1. If a pair (u, v) satisfies the condition

$$\int\limits_0^\infty\left(\int\limits_t^\infty u(\tau)\,d\tau\right)^{p'} v^{1-p'}(t)\,dt < \infty,$$

then there exists a positive constant C_1 such that for an arbitrary function $F : (0,\infty) \mapsto \mathbb{R}$ the inequality

$$\int\limits_0^\infty u(t)\left|\int\limits_0^t F(\tau)\,d\tau\right| dt \le C_1\,\left(\int\limits_0^\infty |F(t)|^p\,v(t)\,dt\right)^{1/p}$$

holds.

2. *If a pair (u,v) satisfies the condition*

$$\int_0^\infty \left(\int_0^t u(\tau)\,d\tau \right)^{p'} v^{1-p'}(t)\,dt < \infty,$$

then there exists a positive constant C_2 such that for an arbitrary function $F:(0,\infty)\mapsto\mathbb{R}$ the inequality

$$\int_0^\infty u(t) \left| \int_t^\infty F(\tau)\,d\tau \right| dt \le C_2 \left(\int_0^\infty |F(t)|^p\, v(t)\,dt \right)^{1/p}$$

holds.

2.5. Theorem. [10] *Let $1 \le q < p < \infty$ and $u(x)$ and $v(x)$ be weight functions on $\mathbb{R}^n$. Then the condition*

$$A = \int_{\mathbb{R}^n} [u(x)]^{\frac{p}{p-q}}\, [v(x)]^{-\frac{q}{p-q}}\, dx < \infty \tag{2.1}$$

is necessary and sufficient for the validity of the inequality

$$\left(\int_{\mathbb{R}^n} |f(x)|^q\, u(x)\,dx \right)^{1/q} \le A^{\frac{1}{q}-\frac{1}{p}} \left(\int_{\mathbb{R}^n} |f(x)|^p\, v(x)\,dx \right)^{1/p}. \tag{2.2}$$

3. Main results

Let $\mathbb{Z} = \{0, \pm 1, \pm 2, \dots, \}$. By $B_{u,\,v}$ we denote the pair $(u,\,v)$ satisfy the condition

$$\left(\sum_{k\in\mathbb{Z}} \sup_{2^k<|x|\le 2^{k+1}} u(x) \int_{2^k<|x|\le 2^{k+1}} |f(x)|^q\,dx \right)^{1/q} \le C \left(\int_{\mathbb{R}^n} |f(x)|^p\, v(x)\,dx \right)^{1/p}, \tag{3.1}$$

where the constant C independent of $k\in\mathbb{Z}$.

3.1. Remark. Let $(u,\,v)\in B_{u,v}$. It is clear that

$$\left(\int_{\mathbb{R}^n} |f(x)|^q\, u(x)\,dx \right)^{1/q} = \left(\sum_{k\in\mathbb{Z}} \int_{2^k<|x|\le 2^{k+1}} |f(x)|^q\, u(x)\,dx \right)^{1/q} \le$$

$$\le \left(\sum_{k\in\mathbb{Z}} \sup_{2^k<|x|\le 2^{k+1}} u(x) \int_{2^k<|x|\le 2^{k+1}} |f(x)|^q\,dx \right)^{1/q} \le C \left(\int_{\mathbb{R}^n} |f(x)|^p\, v(x)\,dx \right)^{1/p}.$$

Therefore, the weight pair $(u,\,v)$ satisfies the condition (2.1).

3.2. Lemma. *Let $1\le q<p<\infty$ and $u(x)$ and $v(x)$ be weight functions on $\mathbb{R}^n$. Let there exists a constant M such that for any $k\in\mathbb{Z}$ the inequality*

$$\sup_{2^k<|x|\le 2^{k+1}} u(x) \le M \inf_{2^k<|x|\le 2^{k+1}} u(x)$$

holds. Then the conditions (2.2) *and* (3.1) *is equivalent.*

Proof. (2.2) ⇒ (3.1). We have

$$\left(\sum_{k\in\mathbb{Z}} \sup_{2^k<|x|\le 2^{k+1}} u(x) \int\limits_{2^k<|x|\le 2^{k+1}} |f(x)|^q\,dx\right)^{1/q}$$

$$= \left(\sum_{k\in\mathbb{Z}} \sup_{2^k<|x|\le 2^{k+1}} u(x) \int\limits_{2^k<|x|\le 2^{k+1}} |f(x)|^q\,u(x)[u(x)]^{-1}\,dx\right)^{1/q}$$

$$\le \left(\sum_{k\in\mathbb{Z}} \frac{\sup\limits_{2^k<|x|\le 2^{k+1}} u(x)}{\inf\limits_{2^k<|x|\le 2^{k+1}} u(x)} \int\limits_{2^k<|x|\le 2^{k+1}} |f(x)|^q\,u(x)\,dx\right)^{1/q}$$

$$\le M^{1/q}\left(\sum_{k\in\mathbb{Z}} \int\limits_{2^k<|x|\le 2^{k+1}} |f(x)|^q\,u(x)\,dx\right)^{1/q}$$

$$= M^{1/q}\,\|f\|_{q,\,u} \le M^{1/q}\,A^{\frac{1}{q}-\frac{1}{p}}\,\|f\|_{p,\,v}.$$

The fact (3.1) ⇒ (2.2) automatically implies from Remark 2. □

3.3. Lemma. *Let $1\le q<p<\infty$, $u(x)$ and $v(x)$ be weight functions on $\mathbb{R}^n$ and $v\in L_1(\mathbb{R}^n)$. Let there exists a constant M_1 such that for any $k\in\mathbb{Z}$ the inequality*

$$\sup_{2^k<|x|\le 2^{k+1}} u(x) \le M_1 \inf_{2^k<|x|\le 2^{k+1}} v(x)$$

holds. Then the inequality

$$\left(\sum_{k\in Z} \sup_{2^k<|x|\le 2^{k+1}} u(x) \int\limits_{2^k<|x|\le 2^{k+1}} |f(x)|^q\,dx\right)^{1/q} \le$$

$$\le M_1^{1/q}\left(\int\limits_{\mathbb{R}^n} v(x)\,dx\right)^{\frac{1}{q}-\frac{1}{p}} \|f\|_{p,\,v}$$

is valid.

Proof. Indeed, we have

$$\left(\sum_{k\in\mathbb{Z}} \sup_{2^k<|x|\le 2^{k+1}} u(x) \int\limits_{2^k<|x|\le 2^{k+1}} |f(x)|^q\,dx\right)^{1/q} \le$$

$$\le M_1^{1/q}\left(\sum_{k\in\mathbb{Z}} \inf_{2^k<|x|\le 2^{k+1}} v(x) \int\limits_{2^k<|x|\le 2^{k+1}} |f(x)|^q\,dx\right)^{1/q} =$$

$$= M_1^{1/q}\left(\sum_{k\in\mathbb{Z}} \int\limits_{2^k<|x|\le 2^{k+1}} |f(x)|^q \inf_{2^k<|x|\le 2^{k+1}} v(x)\,dx\right)^{1/q} \le$$

$$\le M_1^{1/q}\left(\sum_{k\in\mathbb{Z}}\int\limits_{2^k<|x|\le 2^{k+1}}|f(x)|^q\,v(x)\,dx\right)^{1/q}=M_1^{1/q}\left(\int\limits_{\mathbb{R}^n}|f(x)|^q\,v(x)\,dx\right)^{1/q}\le$$

$$\le M_1^{1/q}\left(\int\limits_{\mathbb{R}^n}v(x)\,dx\right)^{\frac{1}{q}-\frac{1}{p}}\|f\|_{p,\,v}.$$

□

The sufficient condition for pair of general weights guaranteeing the two-weight inequalities of a strong type (p,q) for convolution operator (1.3) are proved in the following Theorem.

3.4. Theorem. *Let $1<q<p<\infty$ and the kernel of convolution operator (1.3) satisfies the conditions (a)-(d). Let ω and ω_1 be weight functions on $\mathbb{R}^n$. Suppose that the weight pair (ω_1,ω) satisfy the following conditions:*

$$1)\ \int\limits_{\mathbb{R}^n}\left[\left(\int\limits_{|y|>|x|}\frac{\omega_1(y)}{|y|^{nq}}\,dy\right)\left(\int\limits_{|y|<|x|}\omega^{1-p'}(y)\,dy\right)^{q-1}\right]^{\frac{p}{p-q}}\omega^{1-p'}(x)\,dx<\infty;$$

$$2)\ \int\limits_{\mathbb{R}^n}\left[\left(\int\limits_{|y|<|x|}\omega_1(y)\,dy\right)\left(\int\limits_{|y|>|x|}\frac{\omega^{1-p'}(y)}{|y|^{np'}}\,dx\right)^{q-1}\right]^{\frac{p}{p-q}}\frac{\omega^{1-p'}(x)}{|x|^{np'}}\,dx<\infty;$$

3) *there exists a constantd >0 such that for any $f\in L_{p,\,\omega}(R^n)$ the inequality*

$$\left(\sum_{k\in\mathbb{Z}}\sup_{2^{k-1}<|x|\le 2^{k+2}}\omega_1(x)\int\limits_{2^{k-1}<|x|\le 2^{k+2}}|f(x)|^q\,dx\right)^{1/q}\le$$

$$d\left(\int\limits_{\mathbb{R}^n}|f(x)|^p\,\omega(x)\,dx\right)^{1/p}$$

holds. Then

(3.2) $\quad \|Tf\|_{L_{q,\,\omega_1}(\mathbb{R}^n)}\le C\|f\|_{L_{p,\,\omega}(\mathbb{R}^n)},$

where the constant $C>0$ is independent of f.

Proof. Estimate the left-hand side of inequality (3.2). We have

$$\left(\int\limits_{\mathbb{R}^n}|Tf(x)|^q\,\omega_1(x)\,dx\right)^{1/q}=\left(\sum_{k\in\mathbb{Z}}\int\limits_{2^k<|x|\le 2^{k+1}}|Tf(x)|^q\,\omega_1(x)\,dx\right)^{1/q}=$$

$$=\left(\sum_{k\in\mathbb{Z}}\int\limits_{2^k<|x|\le 2^{k+1}}\left|T\left(f\cdot\chi_{\{|y|\le 2^{k-1}\}}\right)(x)+T\left(f\cdot\chi_{\{2^{k-1}<|y|\le 2^{k+2}\}}\right)(x)+\right.\right.$$

$$\left.\left.+T\left(f\cdot\chi_{\{|y|>2^{k+2}\}}\right)(x)\right|^q\omega_1(x)\,dx\right)^{1/q}\le$$

$$\leq 4^{1/q'} \left(\sum_{k\in\mathbb{Z}} \int\limits_{2^k<|x|\leq 2^{k+1}} \left| T\left(f\cdot \chi_{\left\{|y|\leq 2^{k-1}\right\}} \right)(x) \right|^q \omega_1(x)\,dx \right)^{1/q} +$$

$$+4^{1/q'} \left(\sum_{k\in\mathbb{Z}} \int\limits_{2^k<|x|\leq 2^{k+1}} \left| T\left(f\cdot \chi_{\left\{2^{k-1}<|y|\leq 2^{k+2}\right\}} \right)(x) \right|^q \omega_1(x)\,dx \right)^{1/q} +$$

$$+4^{1/q'} \left(\sum_{k\in\mathbb{Z}} \int\limits_{2^k<|x|\leq 2^{k+1}} \left| T\left(f\cdot \chi_{\left\{|y|>2^{k+2}\right\}} \right)(x) \right|^q \omega_1(x)\,dx \right)^{1/q} =$$

$$= 4^{1/q'} \left(A_1 + A_2 + A_3 \right).$$

Now we estimate A_1. If $2^k < |x| \leq 2^{k+1}$ and $|y| \leq 2^{k-1}$, then $|y| \leq 2^{k-1} \leq \frac{|x|}{2} \leq |x|$ and $|x-y| \geq |x| - |y| \geq |x| - \frac{|x|}{2} = \frac{|x|}{2}$. We have

$$A_1 = \left(\sum_{k\in\mathbb{Z}} \int\limits_{2^k<|x|\leq 2^{k+1}} \left| \int\limits_{R^n} K(x-y)\, f(y)\, \chi_{\left\{|z|\leq 2^{k-1}\right\}}(y)\,dy \right|^q \omega_1(x)\,dx \right)^{1/q} \leq$$

$$\leq C \left(\sum_{k\in\mathbb{Z}} \int\limits_{2^k<|x|\leq 2^{k+1}} \left(\int\limits_{|y|\leq 2^{k-1}} \frac{|f(y)|}{|x-y|^n}\,dy \right)^q \omega_1(x)\,dx \right)^{1/q} \leq$$

$$\leq C_1 \left(\sum_{k\in\mathbb{Z}} \int\limits_{2^k<|x|\leq 2^{k+1}} \frac{\omega_1(x)}{|x|^{nq}} \left(\int\limits_{|y|\leq |x|} |f(y)|\,dy \right)^q dx \right)^{1/q} =$$

$$= C_2 \left(\int\limits_{\mathbb{R}^n} \frac{\omega_1(x)}{|x|^{nq}} \left(\int\limits_{|y|\leq |x|} |f(y)|\,dy \right)^q dx \right)^{1/q} =$$

$$= C_2 \left(\int\limits_{\mathbb{R}^n} \frac{\omega_1(x)}{|x|^{nq}} \left[\int\limits_0^{|x|} s^{n-1} \left(\int\limits_{|\xi|=1} |f(s\xi)|\,d\xi \right) ds \right]^q dx \right)^{1/q} =$$

$$= C_2 \left(\int\limits_0^{\infty} t^{n(1-q)-1} \left(\int\limits_{|\eta|=1} \omega_1(t\eta)\,d\eta \right) \left[\int\limits_0^{t} s^{n-1} \left(\int\limits_{|\xi|=1} |f(s\xi)|\,d\xi \right) ds \right]^q dt \right)^{1/q}.$$

Taking

$$u(t) = t^{n(1-q)-1} \left(\int\limits_{|\eta|=1} \omega_1(t\eta)\,d\eta \right),\ F(t) = t^{n-1} \left(\int\limits_{|\xi|=1} |f(t\xi)|\,d\xi \right),$$

$v(t) = t^{-(n-1)(p-1)} \left(\int\limits_{|\xi|=1} \omega^{1-p'}(t\xi)\, d\xi \right)^{1-p}$ and using the Theorem 2.3 (part one), we get

$$A_1 \le C_3 \left(\int\limits_0^\infty t^{(n-1)p} \left(\int\limits_{|\xi|=1} |f(t\xi)|\, d\xi \right)^p \left(\int\limits_{|\xi|=1} \omega^{1-p'}(t\xi)\, d\xi \right)^{1-p} t^{-(n-1)(p-1)}\, dt \right)^{1/p} =$$

$$= C_3 \left(\int\limits_0^\infty t^{n-1} \left(\int\limits_{|\xi|=1} |f(t\xi)|\, d\xi \right)^p \left(\int\limits_{|\xi|=1} \omega^{1-p'}(t\xi)\, d\xi \right)^{1-p} dt \right)^{1/p}.$$

Applying the Hölder's inequality, we have

$$\left(\int\limits_0^\infty t^{n-1} \left(\int\limits_{|\xi|=1} |f(t\xi)|\, d\xi \right)^p \left(\int\limits_{|\xi|=1} \omega^{1-p'}(t\xi)\, d\xi \right)^{1-p} dt \right)^{1/p} =$$

$$= \left(\int\limits_0^\infty t^{n-1} \left(\int\limits_{|\xi|=1} \left[|f(t\xi)|\, \omega^{\frac{1}{p}}(t\xi) \right] \omega^{-\frac{1}{p}}(t\xi) d\xi \right)^p \left(\int\limits_{|\xi|=1} \omega^{1-p'}(t\xi)\, d\xi \right)^{1-p} dt \right)^{1/p} \le$$

$$\le \left(\int\limits_0^\infty t^{n-1} \left(\int\limits_{|\xi|=1} |f(t\xi)|^p\, \omega(t\xi)\, d\xi \right) \left(\int\limits_{|\xi|=1} \omega^{-\frac{p'}{p}}(t\xi)\, d\xi \right)^{p/p'} \left(\int\limits_{|\xi|=1} \omega^{1-p'}(t\xi)\, d\xi \right)^{1-p} dt \right)^{1/p}$$

$$= \left(\int\limits_0^\infty t^{n-1} \left(\int\limits_{|\xi|=1} |f(t\xi)|^p\, \omega(t\xi)\, d\xi \right) \left(\int\limits_{|\xi|=1} \omega^{1-p'}(t\xi)\, d\xi \right)^{p-1} \left(\int\limits_{|\xi|=1} \omega^{1-p'}(t\xi)\, d\xi \right)^{1-p} dt \right)^{1/p}$$

$$= \left(\int\limits_0^\infty t^{n-1} \left(\int\limits_{|\xi|=1} |f(t\xi)|^p\, \omega(t\xi)\, d\xi \right) dt \right)^{1/p} = \left(\int\limits_{\mathbb{R}^n} |f(x)|^p\, \omega(x)\, dx \right)^{1/p}.$$

Therefore $A_1 \le C_3 \left(\int\limits_{\mathbb{R}^n} |f(x)|^p\, \omega(x)\, dx \right)^{1/p}$ and by condition 1) of Theorem 3.4

$$\int\limits_0^\infty \left[\left(\int\limits_t^\infty u(\tau)\, d\tau \right) \left(\int\limits_0^t v^{1-p'}(\tau)\, d\tau \right)^{q-1} \right]^{\frac{p}{p-q}} v^{1-p'}(t)\, dt =$$

$$= \int\limits_{\mathbb{R}^n} \left[\left(\int\limits_{|y|>|x|} \frac{\omega_1(y)}{|y|^{nq}}\, dy \right) \left(\int\limits_{|y|<|x|} \omega^{1-p'}(y)\, dy \right)^{q-1} \right]^{\frac{p}{p-q}} \omega^{1-p'}(x)\, dx < \infty.$$

Now we estimate A_3. Note that if $2^k < |x| \le 2^{k+1}$ and $|y| > 2^{k+2}$, then $|x| \le \frac{|y|}{2}$ and $|x-y| \ge |y| - |x| \ge |y| - \frac{|y|}{2} = \frac{|y|}{2}$. We get

$$A_3 = \left(\sum_{k\in\mathbb{Z}} \int\limits_{2^k<|x|\le 2^{k+1}} \left| \int\limits_{R^n} K(x-y)\, f(y)\, \chi_{\{|z|>2^{k+2}\}}(y)\, dy \right|^q \omega_1(x)\, dx \right)^{1/q} \le$$

$$\le C \left(\sum_{k\in\mathbb{Z}} \int\limits_{2^k<|x|\le 2^{k+1}} \left(\int\limits_{|y|>2^{k+2}} \frac{|f(y)|}{|x-y|^n}\, dy \right)^q \omega_1(x)\, dx \right)^{1/q} \le$$

$$\leq C_1 \left(\sum_{k\in\mathbb{Z}} \int\limits_{2^k<|x|\leq 2^{k+1}} \omega_1(x) \left(\int\limits_{|y|\geq|x|} \frac{|f(y)|}{|y|^n}\, dy \right)^q dx \right)^{1/q} =$$

$$= C_2 \left(\int\limits_{\mathbb{R}^n} \omega_1(x) \left(\int\limits_{|y|\geq|x|} \frac{|f(y)|}{|y|^n}\, dy \right)^q dx \right)^{1/q} =$$

$$= C_2 \left(\int\limits_0^\infty t^{n-1} \left(\int\limits_{|\eta|=1} \omega_1(t\eta)\, d\eta \right) \left(\int\limits_t^\infty s^{-1} \left(\int\limits_{|\xi|=1} |f(s\xi)|\, d\xi \right) ds \right)^q dt \right)^{1/q}.$$

Further, using the Theorem 2.3 (part two) by condition 2) of Theorem 3.4 we get

$$A_3 \leq C_3 \left(\int\limits_{\mathbb{R}^n} |f(x)|^p\, \omega(x)\, dx \right)^{1/p}.$$

Finally, we estimate A_2. By Theorem 1.5 and by condition 3) of Theorem 3.4 we get

$$A_2 = \left(\sum_{k\in\mathbb{Z}} \int\limits_{2^k<|x|\leq 2^{k+1}} \left| T\left(f\cdot \chi_{\{2^{k-1}<|y|\leq 2^{k+2}\}} \right)(x) \right|^q \omega_1(x)\, dx \right)^{1/q} \leq$$

$$\leq \left(\sum_{k\in\mathbb{Z}} \sup_{2^k<|x|\leq 2^{k+1}} \omega_1(x) \int\limits_{\mathbb{R}^n} \left| T\left(f\cdot \chi_{\{2^{k-1}<|y|\leq 2^{k+2}\}} \right)(x) \right|^q \omega_1(x)\, dx \right)^{1/q} \leq$$

$$\leq C \left(\sum_{k\in\mathbb{Z}} \sup_{2^{k-1}<|x|\leq 2^{k+2}} \omega_1(x) \int\limits_{2^{k-1}<|x|\leq 2^{k+2}} |f(x)|^q\, \omega_1(x)\, dx \right)^{1/q} \leq$$

$$\leq C \left(\int\limits_{R^n} |f(x)|^p\, \omega(x)\, dx \right)^{1/p}.$$

This completes the proof of Theorem 3.4. □

3.5. Corollary. *Let* $1 < q < p < \infty$ *and the kernel of convolution operator* (1.3) *satisfies the conditions (a)-(d). Let* $\omega(t)$ *and* $\omega_1(t)$ *be positive increasing functions on* $(0, \infty)$ *satisfying condition 1) of Theorem 3.4. Then the inequality* (3.2) *holds.*

3.6. Corollary. *Let* $1 < q < p < \infty$ *and the kernel of convolution operator* (1.3) *satisfies the conditions (a)-(d). Let* $\omega(t)$ *and* $\omega_1(t)$ *be positive decreasing functions on* $(0, \infty)$ *satisfying condition 2) of Theorem 3.4. Then the inequality* (3.2) *holds.*

3.7. Example. Let

$$\omega_1(t) = \begin{cases} t^{q-1} \ln^\beta \frac{1}{t} & \textit{for} \quad t < e^{-\frac{p}{p-q}} \\ e^{\frac{p(\lambda-q+1)}{p-q}} \left(\frac{p}{p-q}\right)^\beta t^\lambda & \textit{for} \quad t \geq e^{-\frac{p}{p-q}}, \end{cases}$$

$$\omega(t) = \begin{cases} t^{p-1} \ln^\gamma \frac{1}{t} & \textit{for} \quad t < e^{-\frac{p}{p-q}} \\ e^{\frac{p(\mu-p+1)}{p-q}} \left(\frac{p}{p-q}\right)^\gamma t^\mu & \textit{for} \quad t \geq e^{-\frac{p}{p-q}}, \end{cases}$$

where $p-1<\gamma<\dfrac{p(p-1)}{p-q}$, $\beta<\dfrac{q}{p}(\gamma+1)-q-1$, $\beta\neq-1$, $0\leq\lambda<\dfrac{q}{p}(\mu+1)-1$ and $\dfrac{q}{p}-1<\mu<p-1$. Then the pair (ω,ω_1) satisfies the condition of Theorem 3.4 for $n=1$.

The sufficient condition for pair of general weights guaranteeing the two-weight inequalities of a weak $(p,1)$ type for convolution operator (1.3) are formulate in the following Theorem.

3.8. Theorem. *Let* $1<p<\infty$ *and the kernel of convolution operator* (1.3) *satisfies the conditions (a)-(d). Let* ω *and* ω_1 *be positive functions on* R^n*. Suppose that the weight pair* (ω_1,ω) *satisfy the following conditions:*

$$1')\quad \int\limits_{R^n}\left(\int\limits_{|y|>|x|}\frac{\omega_1(y)}{|y|^n}\,dy\right)^{p'}\omega^{1-p'}(x)\,dx<\infty;$$

$$2')\quad \int\limits_{R^n}\left(\int\limits_{|y|<|x|}\omega_1(y)\,dy\right)^{p'}\frac{\omega^{1-p'}(x)}{|x|^{np'}}\,dx<\infty.$$

$3')$ *there exists a constant* $d_1>0$ *such that for any* $f\in L_{p,\,\omega}(R^n)$ *the inequality*

$$\sum_{k\in\mathbb{Z}}\sup_{2^{k-1}<|x|\leq 2^{k+2}}\omega_1(x)\int\limits_{2^{k-1}<|x|\leq 2^{k+2}}|f(x)|\,dx\leq d_1\left(\int\limits_{\mathbb{R}^n}|f(x)|^p\,\omega(x)\,dx\right)^{1/p}$$

holds. Then there exists a constant $C>0$ *such that for any* $f\in L_{p,\,\omega}(R^n)$ *and* $\lambda>0$ *the inequality*

$$\int\limits_{\{x\in R^n:\,|Tf(x)|>\lambda\}}\omega_1(x)\,dx\leq\frac{C}{\lambda}\left(\int\limits_{R^n}|f(x)|^p\,\omega(x)\,dx\right)^{1/p}\tag{3.3}$$

is valid.

3.9. Corollary. *Let* $1<q<p<\infty$ *and the kernel of convolution operator* (1.3) *satisfies the conditions (a)-(d). Let* $\omega_1(t)$ *be increasing and* $\omega(t)$ *be forall positive functions on* $(0,\infty)$ *satisfying condition 1') of Theorem 3.8. Then the inequality* (3.3) *holds.*

3.10. Corollary. *Let* $1<q<p<\infty$ *and the kernel of convolution operator* (1.3) *satisfies the conditions (a)-(d). Let* $\omega_1(t)$ *be decreasing and* $\omega(t)$ *be forall positive functions on* $(0,\infty)$ *satisfying condition 2') of Theorem 3.8. Then the inequality* (3.3) *holds.*

The Theorems 3.4 and 3.8 are pioneering results in the case $1\leq q<p<\infty$.

3.11. Example. Let

$$\omega(t)=\begin{cases}\frac{1}{t}\ln^{\beta}\frac{1}{t} & for\quad t<e^{2\beta}\\ e^{-2\beta(\lambda+1)}(-2\beta)^{\beta}t^{\lambda} & for\quad t\geq e^{2\beta},\end{cases}$$

$$\omega_1(t)=\begin{cases}\frac{1}{t}\ln^{\gamma}\frac{1}{t} & for\quad t<e^{2\beta}\\ e^{-2\beta(\mu+1)}(-2\beta)^{\gamma}t^{\mu} & for\quad t\geq e^{2\beta},\end{cases}$$

where $\mu>p(\lambda+1)-1$, $-1<\lambda<0$, $\beta<-1$ and $\gamma>p(\beta+2)+1$. Then the pair (ω,ω_1) satisfies the condition of Theorem 3.8.

3.12. Remark. Note that for $p=q$ of the special weights the Theorem 3.4 was proved in [1] (see also [2, 11]). Some others results for $p=q$ was proved in [8].

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