A GENERAL FRAMEWORK FOR COMPACTNESS IN $L$-TOPOLOGICAL SPACES

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Abstract
A general framework for the concepts of compactness, countable compactness, and the Lindelöf property are introduced in $L$-topological spaces by means of several kinds of open $L$-sets and their inequalities when $L$ is a complete DeMorgan algebra. The method used in this paper shows that these results are valid for any kind of open $L$-sets and thus we do not need to repeat it for each kind separately.

Keywords: $L$-topological space, Compactness, Countable compactness, Lindelöf property.

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1. Introduction
The concept of compactness of an $I$-topological space was first introduced by Chang [6] in terms of open covers. Chang’s compactness has been greatly extended to the variable-basis case by Rodabaugh [12], and it can be regarded as a successful definition of compactness in poslat topology from the categorical point of view (see [12, 18]). Moreover, Gantner et al. introduced $\alpha$-compactness [8], Lowen introduced fuzzy compactness, strong fuzzy compactness and ultra-fuzzy compactness [17, 16], Chadwick [5] generalized Lowen’s compactness, Liu introduced $Q$-compactness [15], Li introduced strong $Q$-compactness [13] which is equivalent to the strong fuzzy compactness in [16], Wang and Zhao introduced $N$-compactness [29, 31], and Shi introduced $S^*$-compactness [24].

Recently, Shi presented a new definition of fuzzy compactness in $L$-topological spaces [20, 25] by means of open $L$-sets and their inequality where $L$ is a complete DeMorgan algebra. The new definition does not depend on the structure of $L$. When $L$ is completely distributive, it is equivalent to the notion of fuzzy compactness in [14, 17, 28].

In this paper, following the lines of [20, 24, 25], we will introduce a general framework of compactness in $L$-topological spaces by means of $m$-open $L$-sets and their inequality,

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where $m$ means the kind of openness of the $L$-sets. We also introduce countable $m$-compactness and the $m$-Lindelöf property in $L$-topology.

## 2. Preliminaries

Throughout this paper $(L, \leq, \land, \lor, t)$ is a complete De Morgan algebra, $X$ a nonempty set. The smallest element and the largest element in $L$ are denoted by $0$ and $1$, respectively. By $L_0$ and $L_1$ we mean $L\{0\}$ and $L\{1\}$, respectively. $L^X$ is the set of all $L$-fuzzy sets (or $L$-sets, for short) on $X$. The smallest element and the largest element in $L^X$ are denoted by $\chi_\emptyset$ and $\chi_X$, respectively. We often do not distinguish a crisp subset $A$ of $X$ and its characteristic function $\chi_A$.

A complete lattice $L$ is a complete Heyting algebra if it satisfies the following infinite distributive law: For all $a \in L$ and all $B \subseteq L$, $a \land \lor B = \lor\{a \land b \mid b \in B\}$.

An element $a$ in $L$ is called a prime element if $a \geq b \land c$ implies $a \geq b$ or $a \geq c$. An element $a$ in $L$ is called co-prime if $a'$ is prime [9]. The set of non-unit prime elements in $L$ is denoted by $P(L)$. The set of non-zero co-prime elements in $L$ is denoted by $M(L)$.

The binary relation $\prec$ in $L$ is defined as follows: for $a, b \in L$, $a \prec b$ if and only if for every subset $D \subseteq L$, the relation $b \leq \sup D$ always implies the existence of $d \in D$ with $a \leq d$ [7]. In a completely distributive De Morgan algebra $L$, each element $b$ is a sup of $\{a \in L \mid a \prec b\}$. A set $\{a \in L \mid a \prec b\}$ is called the greatest minimal family of $b$ in the sense of $[14, 28]$, denoted by $\beta(b)$, and $\beta^*(b) = \beta(b) \cap M(L)$. Moreover, for $b \in L$, we define $\alpha(b) = \{a \in L \mid a' \prec b\}$ and $\alpha^*(b) = \alpha(b) \cap P(L)$.

For $a \in L$ and $A \subseteq L^X$, we use the following notations from [26].

\[
A[a] = \{x \in X \mid A(x) \geq a\}, \quad A^{(a)} = \{x \in X \mid A(x) \leq a\}, \quad A(a) = \{x \in X \mid a \in \beta(A(x))\}.
\]

An $L$-topological space (or $L$-space, for short) is a pair $(X, \tau)$, where $\tau$ is a subfamily of $L^X$ which contains $\chi_0$; $\chi_X$ and is closed for any suprema and finite infima. $\tau$ is called an $L$-topology on $X$. Members of $\tau$ are called open $L$-sets and their complements are called closed $L$-sets.

### 2.1. Definition. [14, 28] An $L$-space $(X, \tau)$ is called weakly induced if $\forall a \in L$, $A \subseteq L^X$, it follows that $A^{(a)} \subseteq \tau$, where $\tau$ denotes the topology formed by all the crisp sets in $\tau$.

### 2.2. Definition. [14, 28] For a topological space $(X, \tau)$, let $\omega_L(\tau)$ denote the family of all lower semi-continuous maps from $(X, \tau)$ to $L$, i.e., $\omega_L(\tau) = \{A \subseteq L^X \mid A^{(a)} \in \tau, \ a \in L\}$. Then $\omega_L(\tau)$ is an $L$-topology on $X$; in this case, $(X, \omega_L(\tau))$ is said to be topologically generated by $(X, \tau)$. A topologically generated $L$-space is also called an induced $L$-space.

### 2.3. Definition. [21] Let $(X, \tau)$ be an $L$-space, $a \in L_0$ and $G \subseteq L^X$. A family $\subseteq \subseteq L^X$ is called a $\beta_a$-cover of $G$ if for any $x \in X$, it follows that $a \in \beta(G(x) \lor \bigvee_{A \in L^X} A(x))$. $\subseteq$ is called a strong $\beta_a$-cover of $G$ if $a \in \beta(\bigwedge_{x \in X} (G(x) \lor \bigvee_{A \in L^X} A(x)))$.

### 2.4. Definition. [21] Let $(X, \tau)$ be an $L$-space, $a \in L_0$ and $G \subseteq L^X$. A family $\subseteq \subseteq L^X$ is called a $\beta_a$-cover of $G$ if for any $x \in X$, it follows that $G(x) \lor \bigvee_{A \in L^X} A(x) \geq a$.

It is obvious that a strong $\beta_a$-cover of $G$ is a $\beta_a$-cover of $G$, and a $\beta_a$-cover of $G$ is a $\beta_a$-cover of $G$. For $a \in L$ and a crisp subset $D \subseteq X$, we define $a \land D$ and $a \lor D$ as follows:

\[
(a \land D)(x) = \begin{cases} a, & x \in D; \\ 0, & x \not\in D. \end{cases} \quad (a \lor D)(x) = \begin{cases} 1, & x \in D; \\ 0, & x \not\in D. \end{cases}
\]
2.5. Theorem.\textsuperscript{[26]} For an L-set $A \subseteq L^X$, the following facts are true:

(1) $A = \bigvee_{a \in L}(a \wedge A(a)) = \bigvee_{a \in L}(a \wedge A^{(a)})$;

(2) $A = \bigwedge_{a \in L}(a \vee A^{(a)}) = \bigwedge_{a \in L}(a \vee A^{[a]})$.

□

2.6. Theorem.\textsuperscript{[26]} Let $(X, \omega_L(\tau))$ be the L-space topologically generated by $(X, \tau)$ and $A \subseteq L^X$. Then the following facts hold:

(1) $\text{cl}(A) = \bigvee_{a \in L}(a \wedge (A^{(a)})^\ominus) = \bigvee_{a \in L}(a \wedge (A^{[a]})^\ominus)$;

(2) $\text{cl}(A)^\ominus \subseteq (A^{(a)})^\ominus \subseteq \text{cl}(A)^\ominus$;

(3) $\text{cl}(A) = \bigwedge_{a \in L}(a \vee (A^{(a)})^\ominus) = \bigwedge_{a \in L}(a \vee (A^{[a]})^\ominus)$;

(4) $\text{cl}(A)^\ominus \subseteq (A^{(a)})^\ominus \subseteq \text{cl}(A)^\ominus$;

(5) $\text{int}(A) = \bigvee_{a \in L}(a \wedge (A^{(a)})^\ominus) = \bigvee_{a \in L}(a \wedge (A^{[a]})^\ominus)$;

(6) $\text{int}(A)^\ominus \subseteq (A^{(a)})^\ominus \subseteq \text{int}(A)^\ominus$;

(7) $\text{int}(A) = \bigwedge_{a \in L}(a \vee (A^{(a)})^\ominus) = \bigwedge_{a \in L}(a \vee (A^{[a]})^\ominus)$;

(8) $\text{int}(A)^\ominus \subseteq (A^{(a)})^\ominus \subseteq \text{int}(A)^\ominus$;

where $(A^{(a)})^\ominus$ and $(A^{[a]})^\ominus$ denote respectively the closure and the interior of $A(a)$ in $(X, \tau)$ and so on, $\text{cl}(A)$ and $\text{int}(A)$ denote respectively the closure and the interior of $A$ in $(X, \omega_L(\tau))$.

□

2.7. Definition.\textsuperscript{[21]} Let $(X, \mathcal{T})$ be an L-space, $a \in L_1$ and $G \subseteq L^X$. A family $\mathcal{A} \subseteq L^X$ is said to be:

(1) An a-shading of $G$ if for any $x \in X$, $(G^x \vee \bigvee_{A \in \mathcal{A}} A(x)) \notin a$.

(2) A strong a-shading of $G$ if $\bigvee_{x \in X} (G(x) \vee \bigvee_{A \in \mathcal{A}} A(x)) \notin a$.

(3) An a-remote family of $G$ if for any $x \in X$, $(G(x) \wedge \bigwedge_{B \in \mathcal{B}} A(x)) \notin a$.

(4) A strong a-remote family of $G$ if $\bigwedge_{x \in X} (G(x) \wedge \bigwedge_{B \in \mathcal{B}} A(x)) \notin a$.

2.8. Definition.\textsuperscript{[21]} Let $a \in L_0$ and $G \subseteq L^X$. A subfamily $\mathcal{U}$ of $L^X$ is said to have a weak a-nonempty intersection in $G$ if $\bigvee_{x \in X} (G(x) \wedge \bigwedge_{A \in \mathcal{U}} A(x)) \geq a$. $\mathcal{U}$ is said to have the finite (countable) weak a-intersection property in $G$ if every finite (countable) subfamily $\mathcal{P}$ of $\mathcal{U}$ has a weak a-nonempty intersection in $G$.

2.9. Definition.\textsuperscript{[21]} Let $a \in L_0$ and $G \subseteq L^X$. A subfamily $\mathcal{U}$ of $L^X$ is said to be a weak $a$-filter relative to $G$ if any finite intersection of members in $\mathcal{U}$ is weak $a$-nonempty in $G$. A subfamily $\mathcal{B}$ of $L^X$ is said to be a weak $a$-filterbase relative to $G$ if

$$\{A \in L^X \mid \text{there exists } B \in \mathcal{B} \text{ such that } B \subseteq A\}$$

is a weak $a$-filter relative to $G$.

For a subfamily $\Phi \subseteq L^X$, $2^{[#]}$ denotes the set of all finite subfamilies of $\Phi$ and $2^{[\Phi]}$ the set of all countable subfamilies of $\Phi$.

2.10. Definition. Let $L$ be an L-set of an L-space $(X, \mathcal{T})$. $G$ is called a semiopen L-set $\textsuperscript{[2]}$ (resp. a preopen L-set $\textsuperscript{[27]}$, an open L-set $\textsuperscript{[4]}$, a $\beta$-open L-set $\textsuperscript{[3]}$, a $\gamma$-open L-set $\textsuperscript{[11]}$) if $G \subseteq \text{cl}(\text{int}(G))$ (resp. $G \subseteq \text{int}(\text{cl}(G))$), $G \subseteq \text{int}(\text{cl}(G))$, $G \subseteq \text{cl}(\text{int}(\text{cl}(G)))$, $G \subseteq \text{cl}(\text{int}(\text{cl}(G)))$ and $G \subseteq \text{cl}(\text{int}(\text{cl}(G)))$.

The set of all semiopen L-sets (resp. preopen L-sets, open L-sets, $\beta$-open L-sets) in $(X, \mathcal{T})$ will be denoted by $\text{SO}(X, \mathcal{T})$ (resp. $\text{PO}(X, \mathcal{T})$, $\alpha\text{O}(X, \mathcal{T})$, $\beta\text{O}(X, \mathcal{T})$, $\gamma\text{O}(X, \mathcal{T})$). Generally, $\text{mO}(X, \mathcal{T})$ denotes the set of all $m$-open L-sets.

2.11. Lemma.\textsuperscript{[25]} Let $(X, \mathcal{T}_1)$ and $(Y, \mathcal{T}_2)$ be two L-spaces, where $L$ is a complete Heyting algebra, let $f : X \rightarrow Y$ be a mapping, $f^+_{L^Y} : L^X \rightarrow L^Y$ the extension of $f$. Then for any $P \subseteq L^Y$, we have that

$$\bigvee_{y \in Y} \left( f^+_{L^Y}(G)(y) \wedge \bigwedge_{B \in \mathcal{P}} B(y) \right) = \bigvee_{x \in X} \left( G(x) \wedge \bigwedge_{B \in \mathcal{P}} f^+_{L^Y}(B)(x) \right).$$

□
3. A notion of $m$-compactness

3.1. Definition. Let $(X, \mathcal{T})$ be an $L$-space. $G \in L^X$ is called (countably) $m$-compact if for every (countable) family $\mathcal{U} \subseteq L^X$ of $m$-open $L$-sets, it follows that

$$\bigwedge_{x \in X} \left( G'(x) \lor \bigvee_{A \in \mathcal{U}} A(x) \right) \leq \bigvee_{\varphi \in 2^{(U)}} \bigwedge_{x \in X} \left( G'(x) \lor \bigvee_{A \in \varphi} A(x) \right).$$

3.2. Definition. Let $(X, \mathcal{T})$ be an $L$-space. $G \in L^X$ is said to have the $m$-Lindelöf property (or to be an $m$-Lindelöf $L$-set) if for every family $\mathcal{U}$ of $m$-open $L$-sets, it follows that

$$\bigwedge_{x \in X} \left( G'(x) \lor \bigvee_{A \in \mathcal{U}} A(x) \right) \leq \bigvee_{\varphi \in 2^{(|U|)}} \bigwedge_{x \in X} \left( G'(x) \lor \bigvee_{A \in \varphi} A(x) \right).$$

3.3. Remark. $m$-compactness implies countable $m$-compactness and the $m$-Lindelöf property. Moreover, an $L$-set having the $m$-Lindelöf property is $m$-compact if and only if it is countably $m$-compact.

3.4. Theorem. Let $(X, \mathcal{T})$ be an $L$-space. Then $G \in L^X$ is (countably) $m$-compact if and only if for every (countable) family $\mathcal{B}$ of $m$-closed $L$-sets, it follows that

$$\bigvee_{x \in X} \left( G(x) \land \bigwedge_{B \in \mathcal{B}} B(x) \right) \geq \bigwedge_{\varphi \in 2^{(|\mathcal{B}|)}} \bigvee_{x \in X} \left( G(x) \land \bigwedge_{B \in \varphi} B(x) \right).$$

Proof. Straightforward. \hfill \Box

3.5. Theorem. Let $(X, \mathcal{T})$ be an $L$-space. Then $G \in L^X$ has the $m$-Lindelöf property if and only if for every family $\mathcal{B}$ of $m$-closed $L$-sets, it follows that

$$\bigvee_{x \in X} \left( G(x) \land \bigwedge_{B \in \mathcal{B}} B(x) \right) \geq \bigwedge_{\varphi \in 2^{(|\mathcal{B}|)}} \bigvee_{x \in X} \left( G(x) \land \bigwedge_{B \in \varphi} B(x) \right).$$

Proof. Straightforward. \hfill \Box

3.6. Theorem. Let $(X, \mathcal{T})$ be an $L$-space and $G \in L^X$. Then the following conditions are equivalent:

1. $G$ is a (countably) $m$-compact.
2. For any $a \in L_1$, each (countable) $m$-open strong a-shading $\mathcal{U}$ of $G$ has a finite subfamily which is a strong a-shading of $G$.
3. For any $a \in L_0$, each (countable) $m$-closed strong a-remote family $\mathcal{P}$ of $G$ has a finite subfamily which is a strong a-remote family of $G$.
4. For any $a \in L_0$, each (countable) family of $m$-closed $L$-sets which has the finite weak $a$-intersection property in $G$ has a weak a-nonempty intersection in $G$.
5. For each $a \in L_0$, every $m$-closed (countable) weak $a$-filterbase relative to $G$ has a weak a-nonempty intersection in $G$. \hfill \Box

3.7. Theorem. Let $(X, \mathcal{T})$ be an $L$-space and $G \in L^X$. Then the following conditions are equivalent:

1. $G$ has the $m$-Lindelöf property.
2. For any $a \in L_1$, each $m$-open strong $a$-shading $\mathcal{U}$ of $G$ has a countable subfamily which is a strong $a$-shading of $G$.
3. For any $a \in L_0$, each $m$-closed strong $a$-remote family $\mathcal{P}$ of $G$ has a countable subfamily which is a strong $a$-remote family of $G$.
4. For any $a \in L_0$, each family of $m$-closed $L$-sets which has the countable weak $a$-intersection property in $G$ has a weak a-nonempty intersection in $G$. \hfill \Box
4. Properties of (countable) \( m \)-compactness

4.1. Theorem. Let \( L \) be a complete Heyting algebra. If both \( G \) and \( H \) are (countably) \( m \)-compact, then \( G \lor H \) is (countably) \( m \)-compact.

Proof. For any (countable) family \( \mathcal{B} \) of \( m \)-closed \( L \)-sets, we have by Theorem 3.4 that
\[
\bigvee_{x \in X} \left( (G \lor H)(x) \land \bigwedge_{B \in \mathcal{B}} B(x) \right)
= \left\{ \bigvee_{x \in X} \left( G(x) \land \bigwedge_{B \in \mathcal{B}} B(x) \right) \right\} \lor \left\{ \bigvee_{x \in X} \left( H(x) \land \bigwedge_{B \in \mathcal{B}} B(x) \right) \right\}
\geq \left\{ \bigwedge_{\varnothing \in 2^{\mathcal{B}}(\varnothing)} \bigvee_{x \in X} \left( G(x) \land \bigwedge_{B \in \varnothing} B(x) \right) \right\} \lor \left\{ \bigwedge_{\varnothing \in 2^{\mathcal{B}}(\varnothing)} \bigvee_{x \in X} \left( H(x) \land \bigwedge_{B \in \varnothing} B(x) \right) \right\}
= \bigwedge_{\varnothing \in 2^{\mathcal{B}}(\varnothing)} \bigvee_{x \in X} \left( (G \lor H)(x) \land \bigwedge_{B \in \varnothing} B(x) \right).
\]
This shows that \( G \lor H \) is (countably) \( m \)-compact. \( \Box \)

Analogously we have the following result.

4.2. Theorem. Let \( L \) be a complete Heyting algebra. If both \( G \) and \( H \) have the \( m \)-Lindelöf property, then \( G \lor H \) has the \( m \)-Lindelöf property. \( \Box \)

4.3. Theorem. If \( G \) is (countably) \( m \)-compact and \( H \) is \( m \)-closed, then \( G \land H \) is (countably) \( m \)-compact.

Proof. For any (countable) family \( \mathcal{B} \) of \( m \)-closed \( L \)-sets, we have by Theorem 3.4 that
\[
\bigvee_{x \in X} \left( (G \land H)(x) \land \bigwedge_{B \in \mathcal{B}} B(x) \right)
= \bigvee_{x \in X} \left( G(x) \land \bigwedge_{B \in \mathcal{B}} \bigwedge_{(B \cup \{H\})} B(x) \right)
\geq \bigwedge_{\varnothing \in 2^{\mathcal{B}}(\varnothing)} \bigvee_{x \in X} \left( G(x) \land \bigwedge_{B \in \varnothing} B(x) \right)
= \left\{ \bigwedge_{\varnothing \in 2^{\mathcal{B}}(\varnothing)} \bigvee_{x \in X} \left( G(x) \land \bigwedge_{B \in \varnothing} B(x) \right) \right\} \land \left\{ \bigwedge_{\varnothing \in 2^{\mathcal{B}}(\varnothing)} \bigvee_{x \in X} \left( G(x) \land H(x) \land \bigwedge_{B \in \varnothing} B(x) \right) \right\}
= \left\{ \bigwedge_{\varnothing \in 2^{\mathcal{B}}(\varnothing)} \bigvee_{x \in X} \left( G(x) \land H(x) \land \bigwedge_{B \in \varnothing} B(x) \right) \right\}
= \bigwedge_{\varnothing \in 2^{\mathcal{B}}(\varnothing)} \bigvee_{x \in X} \left( (G \land H)(x) \land \bigwedge_{B \in \varnothing} B(x) \right).
\]
This shows that \( G \land H \) is (countably) \( m \)-compact. \( \Box \)

4.4. Theorem. If \( G \) has the \( m \)-Lindelöf property and \( H \) is \( m \)-closed, then \( G \land H \) has the \( m \)-Lindelöf property.

Proof. Similar to Theorem 4.3. \( \Box \)
4.5. Definition. Let $(X, T_1)$ and $(Y, T_2)$ be two $L$-spaces. A map $f : (X, T_1) \to (Y, T_2)$ is called $m$-irresolute if $f^+_L(G)$ is $m$-open for each $m$-open $L$-set $G$.

4.6. Theorem. Let $L$ be a complete Heyting algebra and let $f : (X, T_1) \to (Y, T_2)$ be an $m$-irresolute map. If $G$ is an $m$-compact (or, countably $m$-compact, $m$-Lindelöf) $L$-set in $(X, T_1)$, then so is $f^+_L(G)$ in $(Y, T_2)$.

Proof. Suppose that $P$ is a family of $m$-closed $L$-sets, then
\[
\bigvee_{y \in Y} \left( f^+_L(G)(y) \land \bigwedge_{B \in P} B(y) \right) = \bigvee_{x \in X} \left( G(x) \land \bigwedge_{B \in P} f^+_L(B)(x) \right)
\geq \bigwedge_{\alpha \in 2^y} \bigvee_{x \in X} \left( G(x) \land \bigwedge_{B \in P} f^+_L(B)(x) \right)
= \bigwedge_{\alpha \in 2^y} \bigvee_{y \in Y} \left( f^+_L(G)(y) \land \bigwedge_{B \in P} B(y) \right).
\]
Therefore $f^+_L(G)$ is $m$-compact. \hfill \Box

4.7. Theorem. Let $L$ be a complete Heyting algebra and let $f : (X, T_1) \to (Y, T_2)$ be an $m$-continuous map. If $G$ is an $m$-compact (a countably $m$-compact, $m$-Lindelöf) $L$-set in $(X, T_1)$, then $f^+_L(G)$ is a compact (countably compact, Lindelöf) $L$-set in $(Y, T_2)$.

Proof. Straightforward. \hfill \Box

4.8. Definition. Let $(X, T_1)$ and $(Y, T_2)$ be two $L$-spaces. A map $f : (X, T_1) \to (Y, T_2)$ is called strongly $m$-irresolute if $f^+_L(G)$ is open in $(X, T_1)$ for every $m$-open $L$-set $G$ in $(Y, T_2)$.

It is obvious that a strongly $m$-irresolute map is $m$-irresolute and $m$-continuous. Analogously we have the following result.

4.9. Theorem. Let $L$ be a complete Heyting algebra and $f : (X, T_1) \to (Y, T_2)$ a strongly $m$-irresolute map. If $G$ is a compact (countably compact, Lindelöf) $L$-set in $(X, T_1)$, then $f^+_L(G)$ is an $m$-compact (a countably $m$-compact, $m$-Lindelöf) $L$-set in $(Y, T_2)$.

Proof. Straightforward. \hfill \Box

5. Good extensions

5.1. Theorem. Let $(X, T)$ be an $L$-space and $G \in L^X$. Then the following conditions are equivalent:

1. $G$ is $m$-compact.
2. For any $a \in L_0$ ($a \in M(L)$), each $m$-closed strong $a$-remote family of $G$ has a finite subfamily which is an $a$-remote (a strong $a$-remote) family of $G$.
3. For any $a \in L_0$ ($a \in M(L)$) and any $m$-closed strong $a$-remote family $\mathcal{P}$ of $G$, there exists a finite subfamily $\mathcal{F}$ of $\mathcal{P}$ and $b \in \beta(a)$ ($b \in \beta^*(a)$) such that $\mathcal{F}$ is a (strong) $b$-remote family of $G$.
4. For any $a \in L_1$ ($a \in P(L)$), each $m$-open strong $a$-shading of $G$ has a finite subfamily which is an $a$-shading (a strong $a$-shading) of $G$.
5. For any $a \in L_1$ ($a \in P(L)$) and any $m$-open strong $a$-shading $\mathcal{U}$ of $G$, there exists a finite subfamily $\mathcal{V}$ of $\mathcal{U}$ and $b \in \beta(a)$ ($b \in \beta^*(a)$) such that $\mathcal{V}$ is a (strong) $b$-shading of $G$.
6. For any $a \in L_0$ ($a \in M(L)$), each $m$-open strong $\beta_\alpha$-cover of $G$ has a finite subfamily which is a (strong) $\beta_\alpha$-cover of $G$. 

(6) For any \( a \in L_0 \) (\( a \in M(L) \)) and any \( m \)-open strong \( \beta \)-cover \( \mathcal{U} \) of \( G \), there exists a finite subfamily \( \mathcal{V} \) of \( \mathcal{U} \) and \( b \in L \) (\( b \in M(L) \)) with \( a \in \beta(b) \) such that \( \mathcal{V} \) is a (strong) \( \beta \)-cover of \( G \).

(7) For any \( a \in L_0 \) (\( a \in M(L) \)) and any \( b \in \beta(a) \setminus \{0\} \), each \( m \)-open \( Q_{\alpha} \)-cover of \( G \) has a finite subfamily which is a \( Q_{\beta} \)-cover of \( G \).

(8) For any \( a \in L_0 \) (\( a \in M(L) \)) and any \( b \in \beta(a) \setminus \{0\} \) (\( b \in \beta^{*}(a) \)), each \( m \)-open \( Q_{\alpha} \)-cover of \( G \) has a finite subfamily which is a (strong) \( Q_{\beta} \)-cover of \( G \). \( \square \)

Analogously we also can present characterizations of countable \( m \)-compactness and the \( m \)-Lindelöf property.

If \( mO(X,T) \) denotes the set of \( m \)-open \( L \)-sets in \( (X,T) \), we will denote the corresponding set in \( (X,\tau) \) by \( MO(X,\tau) \). The following lemma can be proved separately using Theorem 2.6 for the special cases of \( mO(X,T) \) and \( MO(X,\tau) \).

5.2. Lemma. Let \( (X,\omega(L)) \) be generated topologically by \( (X,\tau) \). If \( A \) is an \( M \)-open set in \( (X,\tau) \), then \( \chi_{A} \) is an \( m \)-open \( L \)-set in \( (X,\omega_{L}(\tau)) \). If \( B \) is an \( m \)-open \( L \)-set in \( (X,\omega_{L}(\tau)) \), then \( B_{(a)} \) is an \( M \)-open set in \( (X,\tau) \) for every \( a \in L \). \( \square \)

The next two theorems show that \( m \)-compactness, countable \( m \)-compactness and the \( m \)-Lindelöf property are good extensions.

5.3. Theorem. Let \( (X,\omega_{L}(\tau)) \) be generated topologically by \( (X,\tau) \). Then \( (X,\omega_{L}(\tau)) \) is (countably) \( m \)-compact if and only if \( (X,\tau) \) is (countably) \( M \)-compact.

Proof. Necessity. Let \( A \) be an \( M \)-open cover (a countable \( M \)-open cover) of \( (X,\tau) \). Then \( \{\chi_{A} : A \in A\} \) is a family of \( m \)-open \( L \)-sets in \( (X,\omega_{L}(\tau)) \) with

\[
\bigwedge_{x \in X} \left( \bigvee_{A \in \mathcal{U}} \chi_{A}(x) \right) = 1.
\]

From the (countable) \( m \)-compactness of \( (X,\omega_{L}(\tau)) \) we know that

\[
1 \geq \bigvee_{\psi \in 2^{\mathcal{U}}} \bigwedge_{x \in X} \left( \bigvee_{A \in \psi} \chi_{A}(x) \right) \geq \bigwedge_{x \in X} \left( \bigvee_{A \in \mathcal{U}} \chi_{A}(x) \right) = 1.
\]

This implies that there exists \( \psi \in 2^{\mathcal{U}} \) such that \( \bigwedge_{x \in X} \left( \bigvee_{A \in \psi} \chi_{A}(x) \right) = 1 \). Hence \( \psi \) is a cover of \( (X,\tau) \). Therefore \( (X,\tau) \) is (countably) \( M \)-compact.

Sufficiency. Let \( \mathcal{U} \) be a (countable) family of \( m \)-open \( L \)-sets in \( (X,\omega_{L}(\tau)) \) and let \( \bigwedge_{x \in X} \left( \bigvee_{B \in \mathcal{U}} B(x) \right) = a \). If \( a = 0 \), then we obviously have

\[
\bigwedge_{x \in X} \left( \bigvee_{B \in \mathcal{U}} B(x) \right) \leq \bigvee_{\psi \in 2^{\mathcal{U}}} \bigwedge_{x \in X} \left( \bigvee_{A \in \psi} B(x) \right).
\]

Now we suppose that \( a \neq 0 \). In this case, for any \( b \in \beta(a) \setminus \{0\} \) we have

\[
b \in \beta \left( \bigwedge_{x \in X} \left( \bigvee_{B \in \mathcal{U}} B(x) \right) \right) \subseteq \bigcap_{x \in X} \beta \left( \bigvee_{B \in \mathcal{U}} B(x) \right) = \bigcap_{x \in X} \bigcup_{B \in \mathcal{U}} \beta(B(x)).
\]

By Lemma 5.2 this implies that \( \{B(b) \mid B \in \mathcal{U}\} \) is a \( M \)-open cover of \( (X,\tau) \). From the (countable) \( M \)-compactness of \( (X,\tau) \) we know that there exists \( \psi \in 2^{\mathcal{U}} \) such that \( \{B(b) \mid B \in \psi\} \) is a cover of \( (X,\tau) \). Hence \( b \leq \bigvee_{x \in X} \left( \bigwedge_{B \in \psi} B(x) \right) \). Furthermore we have

\[
b \leq \bigwedge_{x \in X} \left( \bigvee_{B \in \psi} B(x) \right) \leq \bigvee_{\psi \in 2^{\mathcal{U}}} \bigwedge_{x \in X} \left( \bigvee_{B \in \psi} B(x) \right).
\]
This implies that
\[ \bigwedge_{x \in X} \left( \bigvee_{B \in U} B(x) \right) = a = \bigvee \{ b : b \in \beta(a) \} \leq \bigvee_{\psi \in 2^U} \left( \bigvee_{x \in X} \left( \bigvee_{B \in \psi} B(x) \right) \right). \]

Therefore \((X, \omega_L(\tau))\) is (countably) \(m\)-compact. □

Analogously we have the following theorem.

5.4. Theorem. Let \((X, \omega_L(\tau))\) be generated topologically by \((X, \tau)\). Then \((X, \omega_L(\tau))\) has the \(m\)-Lindelöf property if and only if \((X, \tau)\) has the \(M\)-Lindelöf property. □

6. Conclusion and remarks

In this paper, we give a general framework for the concept of compactness in \(L\)-topological spaces. Instead of studying compactness for each type of open \(L\)-sets \(O(X, T)\) separately, we examine the compactness for open sets of type \(mO(X, T)\).

If \(mO(X, T) = SO(X, T)\), we get the study of Shi [23], when \(mO(X, T) = PO(X, T)\), we get the study of Shi [19]. In the case of \(mO(X, T) = \alpha O(X, T)\) we have the study of Shi [21]. This method can be applied for the cases of \(mO(X, T) = \beta O(X, T), mO(X, T) = \gamma O(X, T)\), and so on.

We conclude from this that there are no benefits from repeating the same study on other kinds of \(L\)-sets where we can get any kind of compactness by choosing a suitable type \(m\).

References