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Bi-Periodic Balancing Quaternions

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Abstract. In this paper, we first define the bi-periodic balancing quaternions. We give the generating function and Binet formula for this quaternion. Then, we obtain some identities and properties including this quaternion.

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1. Introduction

Quaternion is a number system which extends the complex numbers and defined by Hamilton as

$$q = a_0e_0 + a_1e_1 + a_2e_2 + a_3e_3$$

where e_1, e_2, e_3 are standart basis in $\mathbb{R}^3$, $e_0 = 1$ and a_0, a_1, a_2, a_3 are real numbers [4].

There are so many researches on integer sequences such as Fibonacci, Lucas and Jacobsthal sequences [7–9, 11]. By the view of the definition of quaternions, the quaternions of some integer sequences was defined. Horadam [5] gave the definition of Fibonacci and Iyer [6] gave the definition of Lucas quaternions, respectively

$$Q_n = e_0F_n + e_1F_{n+1} + e_2F_{n+2} + e_3F_{n+3}$$

and

$$T_n = e_0L_n + e_1L_{n+1} + e_2L_{n+2} + e_3L_{n+3}$$

where F_n is the n-th Fibonacci number and L_n is the n-th Lucas number.

Also, Halıcı [3] gave the generating functions and Binet formulas for Fibonacci and Lucas quaternions. Finally, Tan et al. [10] introduced the bi-periodic Fibonacci quaternions which is a new generalization of the Fibonacci quaternions as

$$Q_n = \sum_{l=0}^{3} q_{n+l}e_l, \quad n \geq 0$$

where q_n is the n-th bi-periodic Fibonacci number which was defined by Edson and Yayenie [2].

In the second section, we will give the definition of bi-periodic balancing quaternion. Also, we will obtain the Binet formula, the generating function and some identities of this quaternion. Before these, we will recall the balancing numbers and bi-periodic balancing numbers.

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Balancing numbers $\{d_n\}_{n=0}^{\infty}$ was defined by Behera and Panda [1] with initial conditions $d_0 = 0, d_1 = 1$ and the recurrence relation

$$d_n = 6d_{n-1} - d_{n-2}, \quad n \geq 2.$$

Definition 1.1. For any two non-zero real numbers c and d, the bi-periodic balancing numbers $\{b_n\}_{n=0}^{\infty}$ is defined for $n \geq 2$ with initial conditions $b_0 = 0$, $b_1 = 1$ and the recurrence relation

$$b_n = \begin{cases} 6cb_{n-1} - b_{n-2}, & \textit{if n is even} \\ 6db_{n-1} - b_{n-2}, & \textit{if n is odd.} \end{cases}$$

When $c = d = 1$, we have the classic balancing numbers. If we set $c = d = k$, for any positive number, we get the k-balancing numbers. The first five elements of the bi-periodic balancing numbers are

$$b_0 = 0, b_1 = 1, b_2 = 6c, b_3 = 36cd - 1, b_4 = 216c^2d - 12c.$$

The quadratic equation for the bi-periodic balancing numbers is defined as

$$x^2 - 6cdx + cd = 0$$

with roots

$$\alpha = 3cd + \sqrt{9c^2d^2 - cd} \quad \textit{and} \quad \beta = 3cd - \sqrt{9c^2d^2 - cd}. \tag{1.1}$$

Lemma 1.2. *We can express the terms of the bi-periodic balancing numbers $\{b_n\}_{n=0}^{\infty}$ by using the Binet formula;*

$$b_m = \left(\frac{c^{1-\xi(m)}}{(cd)^{\lfloor \frac{m}{2} \rfloor}}\right)\frac{\alpha^m - \beta^m}{\alpha - \beta} \tag{1.2}$$

where $\lfloor c \rfloor$ is the floor function of c and $\xi(m) = m - 2\left\lfloor \frac{m}{2} \right\rfloor$ is the parity function.

Lemma 1.3. *The bi-periodic balancing numbers $\{b_n\}_{n=0}^{\infty}$ satisfies the following properties:*

$$b_{2n} = (36cd - 2)b_{2n-2} - b_{2n-4},$$

$$b_{2n+1} = (36cd - 2)b_{2n-1} - b_{2n-3}.$$

Proof. The proof can be seen by the recurrence relation of bi-periodic balancing numbers. □

2. Main Results

Definition 2.1. For $n \geq 0$, the bi-periodic balancing quaternions B_n are defined as

$$B_n = \sum_{l=0}^{3} b_{n+l}e_l \tag{2.1}$$

where b_n is the n-th bi-periodic balancing number.

Theorem 2.2. *The generating function for the bi-periodic balancing quaternion B_n is*

$$G(t) = \frac{B_0 + (B_1 - 6dB_0)\,t + 6(c - d)S\,(t)}{1 - 6dt + t^2}$$

where

$$S\,(t) = tg(t)e_0 + (g(t) - t)e_1 + \left(\frac{g(t)}{t} - 1\right)e_2 + \left(\frac{g(t) - t - (36cd - 1)t^3}{t^2}\right)e_3,$$

$$g(t) = \sum_{n=1}^{\infty} b_{2n-1}t^{2n-1} = \frac{t + t^3}{1 - (36cd - 2)t^2 + t^4}.$$

Proof. The formal power series representation of the generating function for B_n is

$$G(t) = B_0 + B_1 t + B_2 t^2 + \cdots + B_r t^r + \cdots = \sum_{n=0}^{\infty} B_n t^n.$$

By multiplying this series by $6dt$ and t^2 respectively, we can get the following series;

$$6dtG(t) = 6dB_0 t + 6dB_1 t^2 + 6dB_2 t^3 + \cdots + 6dB_r t^{r+1} + \cdots$$

and

$$t^2 G(t) = B_0 t^2 + B_1 t^3 + B_2 t^4 + \cdots + B_r t^{r+2} + \cdots .$$

Since $b_{2k+1} = 6db_{2k} - b_{2k-1}$ and $b_{2k} = 6cb_{2k-1} - b_{2k-2}$, we can write

$$\begin{aligned}(1 - 6dt + t^2)G(t) &= B_0 + (B_1 - 6dB_0)\,t + \sum_{n=2}^{\infty}(B_n - 6dB_{n-1} + B_{n-2})t^n \\ &= B_0 + (B_1 - 6dB_0)\,t + 6(c-d)t\left(\sum_{n=1}^{\infty} b_{2n-1}t^{2n-1}\right)e_0 \\ &\quad + 6(c-d)\left(\sum_{n=2}^{\infty} b_{2n-1}t^{2n-1}\right)e_1 + 6(c-d)\left(\sum_{n=1}^{\infty} b_{2n+1}t^{2n}\right)e_2 \\ &\quad + 6(c-d)\left(\sum_{n=2}^{\infty} b_{2n+1}t^{2n-1}\right)e_3.\end{aligned}$$

Now we define $g(t)$ as

$$g(t) = \sum_{n=1}^{\infty} b_{2n-1}x^{2n-1}. \tag{2.2}$$

By writing (2.2) in the above expansion, we get

$$\begin{aligned}(1 - 6dt + t^2)G(t) &= B_0 + (B_1 - 6dB_0)\,t + 6(c-d)tg(t)e_0 \\ &\quad + 6(c-d)\,(g(t) - t)\,e_1 + 6(c-d)\left(\frac{g(t)}{t} - 1\right)e_2 \\ &\quad + 6(c-d)\left(\frac{g(t) - t - (36cd-1)t^3}{t^2}\right)e_3.\end{aligned} \tag{2.3}$$

Now, we define $S(t)$ series as

$$S(t) = tg(t)e_0 + (g(t) - t)e_1 + \left(\frac{g(t)}{t} - 1\right)e_2 + \left(\frac{g(t) - t - (36cd-1)t^3}{t^2}\right)e_3.$$

By writing $S(t)$ in equation (2.3), we get

$$(1 - 6dt + t^2)G(t) = B_0 + (B_1 - 6dB_0)\,t + 6(c-d)S(t).$$

Simplifying the above equation, we get the generating function as

$$G(t) = \frac{B_0 + (B_1 - 6dB_0)\,t + 6(c-d)S(t)}{1 - 6dt + t^2}.$$

By Lemma 1.3, we have $b_{2n-1} = (36cd-2)b_{2n-3} - b_{2n-5}$, so we get

$$(1 - (36cd-2)t^2 + t^4)g(t) = \sum_{n=1}^{\infty} b_{2n-1}t^{2n-1} - (36cd-2)\sum_{n=2}^{\infty} b_{2n-3}t^{2n-1} + \sum_{n=3}^{\infty} b_{2n-5}t^{2n-1}.$$

By using the above expansion, we have

$$(1-(36cd-2)t^2+t^4)g(t) = b_1t + b_3t^3 - (36cd-2)b_1t^3 + \sum_{n=3}^{\infty}(b_{2n-1} - (36cd-2)b_{2n-3} + b_{2n-5})t^{2n-1}$$
$$= t + t^3.$$

Therefore, we hold

$$g(t) = \sum_{n=1}^{\infty} b_{2n-1}x^{2n-1} = \frac{t+t^3}{1-(36cd-2)t^2+t^4}.$$ □

Theorem 2.3. *We can express the terms of the bi-periodic balancing quaternions $\{B_n\}_{n=0}^{\infty}$ by using the Binet formula;*

$$B_n = \begin{cases} \dfrac{1}{(cd)^{\lfloor \frac{n}{2} \rfloor}}\left(\dfrac{\alpha^*\alpha^n - \beta^*\beta^n}{\alpha-\beta}\right), & \text{if } n \text{ is even} \\ \dfrac{1}{(cd)^{\lfloor \frac{n}{2} \rfloor}}\left(\dfrac{\alpha^{**}\alpha^n - \beta^{**}\beta^n}{\alpha-\beta}\right), & \text{if } n \text{ is odd} \end{cases} \tag{2.4}$$

where α and β are the roots of quadratic equation of bi-periodic balancing numbers in (1.1), $\lfloor c \rfloor$ is the floor function of c and $\xi(m) = m - 2\left\lfloor \frac{m}{2} \right\rfloor$ is the parity function and

$$\alpha^* = \sum_{l=0}^{3} \frac{c^{\xi(l+1)}}{(cd)^{\lfloor \frac{l}{2} \rfloor}}\alpha^l e_l, \qquad \beta^* = \sum_{l=0}^{3} \frac{c^{\xi(l+1)}}{(cd)^{\lfloor \frac{l}{2} \rfloor}}\beta^l e_l,$$

$$\alpha^{**} = \sum_{l=0}^{3} \frac{c^{\xi(l)}}{(cd)^{\lfloor \frac{l+1}{2} \rfloor}}\alpha^l e_l, \qquad \beta^{**} = \sum_{l=0}^{3} \frac{c^{\xi(l)}}{(cd)^{\lfloor \frac{l+1}{2} \rfloor}}\beta^l e_l.$$

Proof. By using the definition of bi-periodic balancing quaternions (2.1) and Binet formula of bi-periodic balancing numbers (1.2) for n is even, we can write

$$\begin{aligned} B_n &= \sum_{l=0}^{3} b_{n+l}e_l \\ &= \sum_{l=0}^{3}\left(\frac{c^{1-\xi(n+l)}}{(cd)^{\lfloor \frac{n+l}{2} \rfloor}}\right)\left(\frac{\alpha^{n+l}-\beta^{n+l}}{\alpha-\beta}\right)e_l \\ &= \frac{c}{(cd)^{\lfloor \frac{n}{2} \rfloor}}\left(\frac{\alpha^n-\beta^n}{\alpha-\beta}\right)e_0 + \frac{1}{(cd)^{\lfloor \frac{n+1}{2} \rfloor}}\left(\frac{\alpha^{n+1}-\beta^{n+1}}{\alpha-\beta}\right)e_1 \\ &\quad + \frac{c}{(cd)^{\lfloor \frac{n+2}{2} \rfloor}}\left(\frac{\alpha^{n+2}-\beta^{n+2}}{\alpha-\beta}\right)e_2 + \frac{1}{(cd)^{\lfloor \frac{n+3}{2} \rfloor}}\left(\frac{\alpha^{n+3}-\beta^{n+3}}{\alpha-\beta}\right)e_3 \\ &= \frac{1}{(cd)^{\lfloor \frac{n}{2} \rfloor}}\left(\frac{\alpha^*\alpha^n-\beta^*\beta^n}{\alpha-\beta}\right), \end{aligned}$$

and for n is odd we get,

$$\begin{aligned}
B_n &= \sum_{l=0}^{3} b_{n+l} e_l \\
&= \sum_{l=0}^{3} \left(\frac{c^{1-\xi(n+l)}}{(cd)^{\lfloor \frac{n+l}{2} \rfloor}}\right)\left(\frac{\alpha^{n+l}-\beta^{n+l}}{\alpha-\beta}\right) e_l \\
&= \frac{1}{(cd)^{\lfloor \frac{n}{2} \rfloor}}\left(\frac{\alpha^{n}-\beta^{n}}{\alpha-\beta}\right) e_0 + \frac{c}{(cd)^{\lfloor \frac{n+1}{2} \rfloor}}\left(\frac{\alpha^{n+1}-\beta^{n+1}}{\alpha-\beta}\right) e_1 \\
&\quad + \frac{1}{(cd)^{\lfloor \frac{n+2}{2} \rfloor}}\left(\frac{\alpha^{n+2}-\beta^{n+2}}{\alpha-\beta}\right) e_2 + \frac{c}{(cd)^{\lfloor \frac{n+3}{2} \rfloor}}\left(\frac{\alpha^{n+3}-\beta^{n+3}}{\alpha-\beta}\right) e_3 \\
&= \frac{1}{(cd)^{\lfloor \frac{n}{2} \rfloor}}\left(\frac{\alpha^{**}\alpha^{n}-\beta^{**}\beta^{n}}{\alpha-\beta}\right).
\end{aligned}$$

□

Theorem 2.4. *For any two nonnegative integer n and r, with $r \leq n$, we have*

$$B_{2(n+r)}B_{2(n-r)} - B_{2n}^2 = \frac{\alpha^*\beta^*((cd)^{2r}-\alpha^{4r}) + \beta^*\alpha^*((cd)^{2r}-\beta^{4r})}{(\alpha-\beta)^2(cd)^{2r}}.$$

Proof. Using the Binet formula (2.4), we obtain

$$\begin{aligned}
B_{2(n+r)}B_{2(n-r)} - B_{2n}^2 &= \left[\frac{1}{(cd)^{n+r}}\left(\frac{\alpha^*\alpha^{2n+2r}-\beta^*\beta^{2n+2r}}{\alpha-\beta}\right)\right]\left[\frac{1}{(cd)^{n-r}}\left(\frac{\alpha^*\alpha^{2n-2r}-\beta^*\beta^{2n-2r}}{\alpha-\beta}\right)\right] \\
&\quad - \left[\frac{1}{(cd)^{2n}}\left(\frac{\alpha^*\alpha^{2n}-\beta^*\beta^{2n}}{\alpha-\beta}\right)\left(\frac{\alpha^*\alpha^{2n}-\beta^*\beta^{2n}}{\alpha-\beta}\right)\right] \\
&= \frac{1}{(cd)^{2n}}\frac{1}{(\alpha-\beta)^2}\left[\left(\alpha^*\alpha^{2n+2r}-\beta^*\beta^{2n+2r}\right)\left(\alpha^*\alpha^{2n-2r}-\beta^*\beta^{2n-2r}\right)\right. \\
&\quad \left. - \left(\alpha^*\alpha^{2n}-\beta^*\beta^{2n}\right)\left(\alpha^*\alpha^{2n}-\beta^*\beta^{2n}\right)\right] \\
&= \frac{1}{(\alpha\beta)^{2n}}\frac{1}{(\alpha-\beta)^2}\left[\left(\alpha^*\alpha^{2n}\right)^2 - \alpha^*\beta^*\alpha^{2n+2r}\beta^{2n-2r} - \alpha^*\beta^*\alpha^{2n-2r}\beta^{2n+2r}\right. \\
&\quad \left. + \left(\beta^*\beta^{2n}\right)^2 - \left(\alpha^*\alpha^{2n}\right)^2 + 2\alpha^*\alpha^{2n}\beta^*\beta^{2n} - \left(\beta^*\beta^{2n}\right)^2\right] \\
&= \frac{1}{(\alpha-\beta)^2}\left[\alpha^*\beta^*\left(1-\left(\frac{\alpha}{\beta}\right)^{2r}\right) + \beta^*\alpha^*\left(1-\left(\frac{\beta}{\alpha}\right)^{2r}\right)\right] \\
&= \frac{\alpha^*\beta^*((cd)^{2r}-\alpha^{4r}) + \beta^*\alpha^*((cd)^{2r}-\beta^{4r})}{(\alpha-\beta)^2(cd)^{2r}}.
\end{aligned}$$

□

Theorem 2.5. *For any nonnegative integer n, we have*

$$B_{2(n+1)}B_{2(n-1)} - B_{2n}^2 = \frac{\alpha^*\beta^*((cd)^{2}-\alpha^{4}) + \beta^*\alpha^*((cd)^{2}-\beta^{4})}{(\alpha-\beta)^2(cd)^{2}}.$$

Proof. In the previous theorem, if we take $r = 1$ we get the proof. □

Theorem 2.6 (Catalan's Identity)**.** *For nonnegative integer n and nonnegative even integer r, with $r \leq n$, we have*

$$B_{n+r}B_{n-r} - B_n^2 = \begin{cases} \dfrac{\alpha^*\beta^*((cd)^r - \alpha^{2r}) + \beta^*\alpha^*((cd)^r - \beta^{2r})}{(cd)^r\,(\alpha-\beta)^2}, & \text{if } n \text{ is even} \\ \dfrac{\alpha^{**}\beta^{**}((cd)^r - \alpha^{2r}) + \beta^{**}\alpha^{**}((cd)^r - \beta^{2r})}{(cd)^{r-1}\,(\alpha-\beta)^2}, & \text{if } n \text{ is odd.} \end{cases}$$

Proof. By using the Binet formula (2.4), for n is even we get,

$$\begin{aligned} B_{n+r}B_{n-r} - B_n^2 &= \left[\frac{1}{(cd)^{\lfloor\frac{n+r}{2}\rfloor}}\left(\frac{\alpha^*\alpha^{n+r} - \beta^*\beta^{n+r}}{\alpha-\beta}\right)\right]\left[\frac{1}{(cd)^{\lfloor\frac{n-r}{2}\rfloor}}\left(\frac{\alpha^*\alpha^{n-r} - \beta^*\beta^{n-r}}{\alpha-\beta}\right)\right] \\ &\quad - \left[\frac{1}{(cd)^{\lfloor\frac{n}{2}\rfloor}}\left(\frac{\alpha^*\alpha^n - \beta^*\beta^n}{\alpha-\beta}\right)\right]^2 \\ &= \frac{1}{(cd)^n}\frac{1}{(\alpha-\beta)^2}\left[(\alpha^*\alpha^{n+r} - \beta^*\beta^{n+r})(\alpha^*\alpha^{n-r} - \beta^*\beta^{n-r})\right] \\ &\quad - \frac{1}{(cd)^n}\frac{1}{(\alpha-\beta)^2}(\alpha^*\alpha^n - \beta^*\beta^n)^2 \\ &= \frac{1}{(cd)^n}\frac{1}{(\alpha-\beta)^2}\left[\alpha^*\beta^*(\alpha\beta)^n\left(1-\left(\frac{\alpha}{\beta}\right)^r\right) + \beta^*\alpha^*(\alpha\beta)^n\left(1-\left(\frac{\beta}{\alpha}\right)^r\right)\right] \\ &= \frac{\alpha^*\beta^*\left((cd)^r - \alpha^{2r}\right) + \beta^*\alpha^*\left((cd)^r - \beta^{2r}\right)}{(cd)^r\,(\alpha-\beta)^2}, \end{aligned}$$

and for n is odd, we get

$$\begin{aligned} B_{n+r}B_{n-r} - B_n^2 &= \left[\frac{1}{(cd)^{\lfloor\frac{n+r}{2}\rfloor}}\left(\frac{\alpha^{**}\alpha^{n+r} - \beta^{**}\beta^{n+r}}{\alpha-\beta}\right)\right]\left[\frac{1}{(cd)^{\lfloor\frac{n-r}{2}\rfloor}}\left(\frac{\alpha^{**}\alpha^{n-r} - \beta^{**}\beta^{n-r}}{\alpha-\beta}\right)\right] \\ &\quad - \left[\frac{1}{(cd)^{\lfloor\frac{n}{2}\rfloor}}\left(\frac{\alpha^{**}\alpha^n - \beta^{**}\beta^n}{\alpha-\beta}\right)\right]^2 \\ &= \frac{1}{(cd)^{n-1}}\frac{1}{(\alpha-\beta)^2}\left[(\alpha^{**}\alpha^{n+r} - \beta^{**}\beta^{n+r})(\alpha^{**}\alpha^{n-r} - \beta^{**}\beta^{n-r})\right] \\ &\quad - \frac{1}{(cd)^{n-1}}\frac{1}{(\alpha-\beta)^2}(\alpha^{**}\alpha^n - \beta^{**}\beta^n)^2 \\ &= \frac{1}{(cd)^{n-1}}\frac{1}{(\alpha-\beta)^2}\left[\alpha^{**}\beta^{**}(\alpha\beta)^n\left(1-\left(\frac{\alpha}{\beta}\right)^r\right) + \beta^{**}\alpha^{**}(\alpha\beta)^n\left(1-\left(\frac{\beta}{\alpha}\right)^r\right)\right] \\ &= \frac{\alpha^{**}\beta^{**}\left((cd)^r - \alpha^{2r}\right) + \beta^{**}\alpha^{**}\left((cd)^r - \beta^{2r}\right)}{(cd)^{r-1}\,(\alpha-\beta)^2}. \end{aligned}$$

□

Theorem 2.7. *For any integer $n \geq 0$, the following properties hold;*

$$\begin{aligned} B_n + \overline{B}_n &= 2b_n, \\ B_n^2 + B_n\overline{B}_n &= 2b_nB_n, \\ B_{n+2} &= 6B_{n+1} - B_n. \end{aligned}$$

Proof. By using bi-periodic balancing number, quaternions and the definition of conjugate of quaternions we get,

$$
\begin{aligned}
B_n + \overline{B_n} &= \sum_{l=0}^{3} b_{n+l}e_l + \left(\overline{\sum_{l=0}^{3} b_{n+l}e_l}\right) \\
&= b_n e_0 + b_{n+1}e_1 + b_{n+2}e_2 + b_{n+3}e_3 + b_n\overline{e_0} + b_{n+1}\overline{e_1} + b_{n+2}\overline{e_2} + b_{n+3}\overline{e_3} \\
&= 2b_n + e_1\left(b_{n+1} - b_{n+1}\right) + e_2\left(b_{n+2} - b_{n+2}\right) + e_3\left(b_{n+3} - b_{n+3}\right) \\
&= 2b_n,
\end{aligned}
$$

$$
\begin{aligned}
B_n^2 + B_n\overline{B}_n &= \left(\sum_{l=0}^{3} b_{n+l}e_l\right)^2 + \sum_{l=0}^{3} b_{n+l}e_l\left(\overline{\sum_{l=0}^{3} b_{n+l}e_l}\right) \\
&= b_n^2e_0^2 + b_nb_{n+1}e_0e_1 + b_nb_{n+2}e_0e_2 + b_nb_{n+3}e_0e_3 \\
&\quad + b_{n+1}b_ne_1e_0 + b_{n+1}^2e_1^2 + b_{n+1}b_{n+2}e_1e_2 + b_{n+1}b_{n+3}e_1e_3 \\
&\quad + b_{n+2}b_ne_2e_0 + b_{n+2}b_{n+1}e_2e_1 + b_{n+2}^2e_2^2 + b_{n+2}b_{n+3}e_2e_3 \\
&\quad + b_{n+3}b_ne_3e_0 + b_{n+3}b_{n+1}e_3e_1 + b_{n+3}b_{n+2}e_3e_2 + b_{n+3}^2e_3^2 \\
&\quad + b_n^2e_0\overline{e_0} + b_nb_{n+1}e_0\overline{e_1} + b_nb_{n+2}e_0\overline{e_2} + b_nb_{n+3}e_0\overline{e_3} \\
&\quad + b_{n+1}b_ne_1\overline{e_0} + b_{n+1}^2e_1\overline{e_1} + b_{n+1}b_{n+2}e_1\overline{e_2} + b_{n+1}b_{n+3}e_1\overline{e_3} \\
&\quad + b_{n+2}b_ne_2\overline{e_0} + b_{n+2}b_{n+1}e_2\overline{e_1} + b_{n+2}^2e_2\overline{e_2} + b_{n+2}b_{n+3}e_2\overline{e_3} \\
&\quad + b_{n+3}b_ne_3\overline{e_0} + b_{n+3}b_{n+1}e_3\overline{e_1} + b_{n+3}b_{n+2}e_3\overline{e_2} + b_{n+3}^2e_3\overline{e_3} \\
&= 2b_nb_ne_0 + 2b_nb_{n+1}e_1 + 2b_nb_{n+2}e_2 + 2b_nb_{n+3}e_3 \\
&= 2b_nB_n
\end{aligned}
$$

and

$$
\begin{aligned}
6B_{n+1} - B_n &= 6\sum_{l=0}^{3} b_{(n+1)+l}e_l - \sum_{l=0}^{3} b_{n+l}e_l \\
&= 6b_{n+1}e_o + 6b_{n+2}e_1 + 6b_{n+3}e_2 + 6b_{n+4}e_3 - b_ne_o - b_{n+1}e_1 - b_{n+2}e_2 - b_{n+3}e_3 \\
&= (6b_{n+1}e_o - b_ne_o) + (6b_{n+2}e_1 - b_{n+1}e_1) + (6b_{n+3}e_2 - b_{n+2}e_2) + (6b_{n+4}e_3 - b_{n+3}e_3) \\
&= b_{n+2}e_o + b_{n+3}e_1 + b_{n+4}e_2 + b_{n+5}e_3 \\
&= \sum_{l=0}^{3} b_{(n+2)+l}e_l \\
&= B_{n+2}.
\end{aligned}
$$

□

Conflicts of Interest

The authors declare that there are no conflicts of interest regarding the publication of this article.

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