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Research Article

The (j, m)-core inverse in rings with involution

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Abstract

Let R be a unital ring with involution. The (j, m)-core inverse of a complex matrix was extended to an element in R. New necessary and sufficient conditions such that an element in R to be (j, m)-core invertible are given. Moreover, several additive and product properties of two (j, m)-core invertible elements are investigated and a order related to the (j, m)-core inverse is introduced.

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1. Introduction

Throughout this paper, R denotes a unital ring with involution, i.e., a ring with unity 1, and a mapping $a \mapsto a^*$ that satisfies $(a^*)^* = a$, $(ab)^* = b^*a^*$ and $(a+b)^* = a^* + b^*$, for all $a, b \in R$. Let $a, x \in R$. If $axa = a$, $xax = x$, $(ax)^* = ax$ and $(xa)^* = xa$ hold, then x is called a *Moore-Penrose inverse* of a. If such an element x exists, then it is unique and denoted by $a^\dagger$. The set of all Moore-Penrose invertible elements in R will be denoted by $R^\dagger$. If the equation $axa = a$ and $(ax)^* = ax$ hold, then x is called a $\{1, 3\}$-inverse of a.

An element $a \in R$ is said to be *Drazin invertible* if there exists $x \in R$ such that $ax = xa$, $xax = x$ and $a^k = a^{k+1}x$ for some nonnegative integer k. The element x is unique if it exists and denoted by a^D [3]. The smallest positive integer k in the definition of the Drazin inverse is called the *index* of a, denoted by $\mathrm{ind}(a)$. If $\mathrm{ind}(a) \leq 1$, then a is group invertible and the group inverse of a is denoted by $a^\#$. Thus, $a^\#$ satisfies $a^\# a a^\# = a^\#$, $a^\# a = a a^\#$ and $a a^\# a = a$. The sets of all Drazin invertible and all group invertible elements in R will be denote by R^D and $R^\#$, respectively.

For an element a in a ring R, we denote $aR = \{ax \mid x \in R\}$ and $Ra = \{xa \mid x \in R\}$. The notion of the core inverse of a complex matrix was introduced by Baksalary and Trenkler [1]. In [8], Rakić et al. generalized the core inverse of a complex matrix to the case of an element in R. More precisely, let $a, x \in R$. If $axa = a$, $xR = aR$ and $Rx = Ra^*$, then x is called a *core inverse* of a. If such an element x exists, then it is unique and denoted by $a^{\textcircled{\#}}$. The set of all core invertible elements in R will be denoted by $R^{\textcircled{\#}}$. There are some generalizations of the core inverse, for example, the B-T inverse in [2] and the

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DMP-inverse in [5]. Moreover, the B-T inverse of a is $a^{\diamond} = (a^2a^{\dagger})^{\dagger}$ by [2, Definition 1] and the DMP-inverse of a is $a^{D,\dagger} = a^Daa^{\dagger}$ by [5, Theorem 2.2].

Let $\mathbb{N}$ denote the set of all positive integers and $\mathbb{C}^{n\times n}$ denote the set of all $n\times n$ complex matrices over the complex filed $\mathbb{C}$. A matrix $A\in\mathbb{C}^{n\times n}$ is called an *EP* (*range-Hermitian*) matrix if $\mathcal{R}(A)=\mathcal{R}(A^*)$ [9], where $\mathcal{R}(A)$ is the range (or column space) of A. An element $a\in R$ is said to be an *EP* element if $a\in R^{\dagger}\cap R^{\#}$ and $a^{\dagger}=a^{\#}$ (see [4]). The set of all EP elements in R will be denoted by R^{EP}.

The (j,m)-core inverse was introduced in [13] for a complex matrix. Let $A\in\mathbb{C}^{n\times n}$ and $j,m\in\mathbb{N}$. A matrix $X\in\mathbb{C}^{n\times n}$ is called a *(j,m)-core inverse* of A, if it satisfies $X=A^DAX$ and $A^mX=A^m(A^j)^{\dagger}$. If such X exists, then it is unique and denoted by $A^{\ominus}_{j,m}$.

We introduce and characterize the (j,m)-core inverse of an element in a ring with involution, as extension of corresponding inverse of a square complex matrix. Some additive and product properties of two (j,m)-core invertible elements are presented. Also, we define a order related to the (j,m)-core inverse.

2. The (j,m)-core inverse in rings

Let us start this section with some useful lemmas. The next lemma was proved for complex matrices in [13], but for elements in rings can be proved in a similar way, thus we omit the proof.

Lemma 2.1. *Let $a\in R$. If there exists $x\in R$ such that $ax^{k+1}=x^k$ and $xa^{k+1}=a^k$ for some $k\in\mathbb{N}$, then*

(1) $a^k = x^ka^{2k} = a^kx^ka^k = axa^k$;
(2) $x^k = a^kx^{2k} = x^ka^kx^k = xax^k$;
(3) $a^kx^k = a^{k+1}x^{k+1}$;
(4) $x^ka^k = x^{k+1}a^{k+1}$.

The following lemma was proved for complex matrices in [13, Lemma 2.5], but it is also valid in a ring. For the convenience of the readers, here we will give the proof.

Lemma 2.2. *Let $a\in R$. Then $a\in R^D$ if and only if there exists $x\in R$ such that $ax^{k+1}=x^k$ and $xa^{k+1}=a^k$ for some $k\in\mathbb{N}\cup\{0\}$. In this case, $a^D=x^{k+1}a^k$.*

Proof. Assume $a\in R^D$ with $\mathrm{ind}(a)=k$. If we let $x=a^D$, then it is easy to check that $ax^{k+1}=x^k$ and $xa^{k+1}=a^k$. Conversely, let $y=x^{k+1}a^k$, we shall prove that y is the Drazin inverse of a. Have in mind $ax^{k+1}=x^k$ and $xa^{k+1}=a^k$, we get

$$a(x^{k+1}a^k) = x^ka^k = x^{k+1}a^ka, \tag{2.1}$$

that is, $x^{k+1}a^k$ and a commute. Then, by (1) and (4) in Lemma 2.1, we have that

$$\begin{aligned}(x^{k+1}a^k)a(x^{k+1}a^k) &= x^{k+1}a^{k+1}x^{k+1}a^k = x^ka^k(x^{k+1}a^k)\\ &= x^kx^{k+1}a^ka^k = x^{k+1}x^ka^{2k} = x^{k+1}a^k.\end{aligned} \tag{2.2}$$

From (1) in Lemma 2.1, we obtain

$$(x^{k+1}a^k)a^{k+1} = x(x^ka^{2k})a = xa^ka = xa^{k+1} = a^k. \tag{2.3}$$

Thus, we deduce that $a^D=x^{k+1}a^k$, by the definition of the Drazin inverse and in view of (2.1), (2.2) and (2.3). □

Corollary 2.3. *Let $a\in R$. Then $a\in R^{\#}$ if and only if there exists $x\in R$ such that $ax^2=x$ and $xa^2=a$.*

Now, we introduce the definition of the (j,m)-core inverse for an element in a ring.

Definition 2.4. Let $a \in R^D$ and $a^j \in R^\dagger$ and $j, m \in \mathbb{N}$. An element $x \in R$ is called a (j, m)-core inverse of a, if it satisfies

$$x = a^D a x \ \text{ and } \ a^m x = a^m (a^j)^\dagger. \tag{2.4}$$

If a is (j,m)-core invertible, then the solution of (2.4) is unique and denoted by $a^{\ominus}_{j,m}$. In fact, if x satisfies (2.4), then $x = a^D a x = (a^D)^m a^m x = (a^D)^m a^m (a^j)^\dagger = a^D a (a^j)^\dagger$. It is easy to check that if $\text{ind}(a) \leq m$, then $x = a^D a (a^j)^\dagger$ is the unique solution of (2.4). In [13, Example 4.4], the authors have shown that if $m < \text{ind}(A)$, then the equations in (2.4) may be not consistent. That is, if we let $A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$, it is easy to get $\text{ind}(A) = 2$ and $A^D = 0$. Let $m = j = 1$ and suppose that X is the solution of system in (2.4), then $X = A^D A X = 0$, which gives $AA^\dagger = AX = 0$, thus $A = AA^\dagger A = 0$, this is a contradiction.

Theorem 2.5. *Let $a \in R^D$, $a^j \in R^\dagger$ and $j, m \in \mathbb{N}$. Then the followings are equivalent:*

(1) *a is (j, m)-core invertible;*
(2) *there exists $x \in R$ such that $x = a^D a x$ and $a^m (a^j)^\dagger = a^D a^{m+1} x$;*
(3) *there exists $x \in R$ such that $x = a^D a x$, $a^{m+1} (a^j)^\dagger = a^{m+1} x$ and $a^m (a^j)^\dagger = a^D a^{m+1} (a^j)^\dagger$.*

Furthermore, the above element x is unique and $x = a^{\ominus}_{j,m}$.

Proof. (1) $\Rightarrow$ (3). Suppose that a is (j, m)-core invertible. Then $a^{\ominus}_{j,m} = a^D a a^{\ominus}_{j,m}$ and $a^m a^{\ominus}_{j,m} = a^m (a^j)^\dagger$. The equality $a^{m+1} (a^j)^\dagger = a^{m+1} x$ is trivial and

$$\begin{aligned} a^m (a^j)^\dagger &= a^m a^{\ominus}_{j,m} = a^m (a^D a a^{\ominus}_{j,m}) = a^D a^{m+1} a^{\ominus}_{j,m} \\ &= a^D a^{m+1} (a^D a a^{\ominus}_{j,m}) = a^D a^{m+1} (a^D)^m a^m a^{\ominus}_{j,m} \\ &= a^D a^{m+1} (a^D)^m a^m (a^j)^\dagger = a^D a^2 a^D a^m (a^j)^\dagger \\ &= a^D a^{m+1} (a^j)^\dagger. \end{aligned}$$

That is, we have $a^m (a^j)^\dagger = a^D a^{m+1} (a^j)^\dagger$.

(3) $\Rightarrow$ (2). It is sufficient to prove $a^m (a^j)^\dagger = a^D a^{m+1} x$. We have $a^m (a^j)^\dagger = a^D a a^m (a^j)^\dagger = a^D a^{m+1} x$.

(2) $\Rightarrow$ (1). Since $a^m x = a^m (a^D a x) = a^D a^{m+1} x = a^m (a^j)^\dagger$, thus x is the (j, m)-core inverse of a by definition. ☐

If we take $j = 1$ and $m = \text{ind}(a)$, the (j, m)-core inverse of a is the DMP-inverse of a. That is, the (j, m)-core inverse of a is a generalization of the DMP-inverse of a. By Theorem 2.5, we have the following corollary.

Corollary 2.6. *Let $a \in R^D \cap R^\dagger$ with $\text{ind}(a) = k$. Then the following are equivalent:*

(1) *a is DMP-invertible;*
(2) *there exists $x \in R$ such that $x = a^D a x$ and $a^k a^\dagger = a^k x$;*
(3) *there exists $x \in R$ such that $x = a^D a x$ and $a^{k+1} a^\dagger = a^{k+1} x$.*

Furthermore, the above element x is unique and $x = a^{D,\dagger}$.

Proposition 2.7. *Let $a \in R^D$ with $\text{ind}(a) \leq m$. If there exists $x \in R$ such that $(a^k x^k)^* = a^k x^k$, $(x^k a^k)^* = x^k a^k$, $a x^{k+1} = x^k$ and $x a^{k+1} = a^k$ for some $k \in \mathbb{N}$, then a is (k, m)-core invertible and $a^{\ominus}_{k,m} = x^k$.*

Proof. By Lemma 2.1 and Lemma 2.2, we have $a^k x^k a^k = a^k$, $x^k a^k x^k = x^k$, $a^k = x^k a^{2k}$ and $a^D = x^{k+1} a^k$. Equalities $(a^k x^k)^* = a^k x^k$ and $(x^k a^k)^* = x^k a^k$ imply that x^k is the Moore-Penrose inverse of a^k. Thus, a is (k, m)-core invertible by $\text{ind}(a) \leq m$. From $a^D = x^{k+1} a^k$, we can obtain $(a^D)^l = x^{l-1} a^D$ for arbitrary $l \in \mathbb{N}$ by induction. Thus

$$a^{\ominus}_{k,m} = a^D a (a^k)^\dagger = (a^D)^k a^k x^k = x^{k-1} a^D a^k x^k = x^k (x^k a^{2k}) x^k = x^k a^k x^k = x^k$$

That is $a^{\ominus}_{k,m} = x^k$. □

Example 2.8. The (j,m)-core inverse is different from the DMP-inverse, B-T inverse and core inverse. Let $a = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \in \mathbb{C}^{3\times 3}$ and $j \geq 2$. Then it is easy to check that a is not core invertible by $\mathrm{ind}(a) = 2$, $a^{D,\dagger} = \begin{bmatrix} 1 & 2 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ by $a^{D,\dagger} = a^D a a^\dagger$ and $a^\diamond = \begin{bmatrix} 1/5 & 0 & 0 \\ 2/5 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ by $a^\diamond = (a^2 a^\dagger)^\dagger$, but $a^{\ominus}_{j,m} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$.

Lemma 2.9 ([14, Theorem 3.1])**.** *Let $a, x \in R$. Then a is core invertible with $a^{\text{\#}} = x$ if and only if $(ax)^* = ax$, $xa^2 = a$ and $ax^2 = x$.*

By Remark 4.7 in [13], if $\mathrm{ind}(a) \leq m$, it is not difficult to see that $a^{\ominus}_{j,m} = a^{\ominus}_{j,m+1}$. That is to say, the (j,m)-core inverse of a coincides with the $(j,m+1)$-core inverse of a. Thus, for notational convenience in the sequel, we only discuss the $\mathrm{ind}(a) = m$ case. For $j \in \mathbb{N}$, we shall assume that $R^{\ominus}_{j,m} = \{a \in R \mid a \text{ is } (j,m)\text{-core invertible with and } \mathrm{ind}(a) = m\}$.

Theorem 2.10. *Let $a \in R^{\ominus}_{j,m}$ with $\mathrm{ind}(a) \leq j$ and $x \in R$. Then the following are equivalent:*

(1) $a^{\ominus}_{j,m} = x$;
(2) $a^j x a^j = a^j$, $(a^j x)^* = a^j x$ *and* $a^j x^2 = x$;
(3) $a^j x a^j = a^j$, $(a^j x)^* = a^j x$, $x a^j x = x$ *and* $x a^j = a^D a$;
(4) *x is the core inverse of a^j (or equivalently $a^j x^2 = x$, $(a^j x)^* = a^j x$ and $x(a^j)^2 = a^j$).*

Proof. (1) ⇒ (2)-(4). Let $x = a^D a (a^j)^\dagger$. First notice that $a^j x = a^j (a^j)^\dagger$ is Hermitian, $a^j x a^j = a^j (a^j)^\dagger a^j = a^j$ and

$$a^j x^2 = (a^j x)x = a^j (a^j)^\dagger a^j (a^D)^j (a^j)^\dagger = a^j (a^D)^j (a^j)^\dagger = a^D a (a^j)^\dagger = x.$$

Further, $x a^j = (a^D)^j a^j (a^j)^\dagger a^j = (a^D)^j a^j = a^D a$ implies $x a^j x = a^D a x = x$ and $x(a^j)^2 = (x a^j) a^j = a^D a a^j = a^j$. Hence, x is the core inverse of a^j by Lemma 2.9.

(4) ⇒ (2). The equalities $a^j x^2 = x$, $(a^j x)^* = a^j x$ and $x(a^j)^2 = a^j$ yield $a^j x a^j = a^j x^2 (a^j)^2 = x(a^j)^2 = a^j$.

(2) ⇒ (1). Suppose that there exists $x \in R$ such that $a^j x a^j = a^j$, $(a^j x)^* = a^j x$ and $a^j x^2 = x$. Then $a^j (a^j)^\dagger = a^j x a^j (a^j)^\dagger = (a^j (a^j)^\dagger a^j x)^* = a^j x$ gives

$$a^m (a^j)^\dagger = a^m a^D a (a^j)^\dagger = a^m (a^D)^j a^j (a^j)^\dagger = a^m (a^D)^j a^j x = a^m x.$$

and

$$x = a^j x^2 = a^D a a^j x^2 = a^D a x,$$

i.e. x is the (j,m)-core inverse of a.

(3) ⇒ (1). If there exists $x \in R$ such that $a^j x a^j = a^j$, $(a^j x)^* = a^j x$, $x a^j x = x$ and $x a^j = a^D a$, then we obtain that x is the (j,m)-core inverse of a by $a^j (a^j)^{(1,3)} = a^j (a^j)^\dagger$ for arbitrary $\{1,3\}$-inverse $(a^j)^{(1,3)}$ of a^j and $x = x a^j x = a^D a x = (a^D)^j a^j x = (a^D)^j a^j (a^j)^\dagger = a^D a (a^j)^\dagger$. □

Recall that, for $e = e^2 \in R$, we can represent any $a \in R$ as a matrix form

$$a = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}_{e \times e},$$

where $a_{11} = eae$, $a_{12} = ea(1-e)$, $a_{21} = (1-e)ae$ and $a_{22} = (1-e)a(1-e)$.

Theorem 2.11. *Let $a \in R$ and $j \in \mathbb{N}$. Then $a \in R^{\ominus}_{j,m}$ if and only if $a \in R^D$*

$$a = \begin{bmatrix} a_1 & 0 \\ 0 & a_2 \end{bmatrix}_{e\times e} \quad \text{and} \quad a^m(a^j)^\dagger = \begin{bmatrix} q_1 & q_2 \\ 0 & 0 \end{bmatrix}_{e\times e},$$

where $e = aa^D$,a_1 is invertible in eRe. Moreover, the (j,m)-core inverse of a is given by

$$a^{\ominus}_{j,m} = \begin{bmatrix} a_1^{-m}q_1 & a_1^{-m}q_2 \\ 0 & 0 \end{bmatrix}_{e\times e}.$$

Proof. Suppose that $a \in R^{\ominus}_{j,m}$ and let $e = a^Da$. Then $e^2 = (a^Da)^2 = a^Da = e$, $ea(1-e) = aa^Da(1-a^Da) = 0$ and $(1-e)ae = 0$. Hence,

$$a = \begin{bmatrix} a_1 & 0 \\ 0 & a_2 \end{bmatrix}_{e\times e}.$$

Since $a_1a^D = a^Da^2a^D = a^Da = a^Da_1 = e$, so a_1 is invertible in eRe and $a_1^{-1} = a^D$. Thus,

$$a^D = a^Daa^Daa^D = \begin{bmatrix} a^D & 0 \\ 0 & 0 \end{bmatrix}_{e\times e} = \begin{bmatrix} a_1^{-1} & 0 \\ 0 & 0 \end{bmatrix}_{e\times e}.$$

Let

$$a^{\ominus}_{j,m} = \begin{bmatrix} x_1 & x_2 \\ x_3 & x_4 \end{bmatrix}_{e\times e} \quad \text{and} \quad a^m(a^j)^\dagger = \begin{bmatrix} q_1 & q_2 \\ q_3 & q_4 \end{bmatrix}_{e\times e}.$$

From $a^{\ominus}_{j,m} = a^Daa^{\ominus}_{j,m} = \begin{bmatrix} e & 0 \\ 0 & 0 \end{bmatrix}_{e\times e} a^{\ominus}_{j,m}$, we obtain $x_3 = x_4 = 0$. Since $a^m(a^j)^\dagger = a^Daa^m(a^j)^\dagger$, then $q_3 = q_4 = 0$.

Conversely, let

$$x = \begin{bmatrix} a_1^{-m}q_1 & a_1^{-m}q_2 \\ 0 & 0 \end{bmatrix}_{e\times e}$$

we get $x = a^Dax$ and $a^mx = a^m(a^j)^\dagger$. So, $a \in R^{\ominus}_{j,m}$ and $x = a^{\ominus}_{j,m}$. □

An element $a \in R$ is called $*$-*DMP* (Drazin-Moore-Penrose) of index k if k is the smallest natural number such that $(a^k)^\#$ and $(a^k)^\dagger$ exist and $(a^k)^\# = (a^k)^\dagger$ (see [7, Definition 6]). Before we answer when a (k,k)-core invertible element is an $*$-DMP element, some lemmas are necessary.

Lemma 2.12 ([12, Theorem 3.9])**.** *Let $a \in R$. Then the following are equivalent:*

(1) $a \in R^{\mathrm{EP}}$;
(2) $a \in R^\#$ *and* $aR \subseteq a^*R$;
(3) $a \in R^\#$ *and* $Ra \subseteq Ra^*$;
(4) $a \in R^\#$ *and* $a^*R \subseteq aR$;
(5) $a \in R^\#$ *and* $Ra^* \subseteq Ra$.

Lemma 2.13 ([7, Theorem 10])**.** *Let $a \in R$. Then a is $*$-DMP of index k if and only if a^D exists of index k and aa^D is Hermitian.*

For $a, b \in R$, the notations ${}^\circ a = \{x \in R \mid xa = 0\}$, $a^\circ = \{x \in R \mid ax = 0\}$ and $[a,b] = ab - ba$ will be used.

Lemma 2.14 ([10, Lemma 8])**.** *Let $a, b \in R$. Then:*

(1) $aR \subseteq bR$ *implies* ${}^\circ b \subseteq {}^\circ a$ *and the converse is valid whenever b is regular;*
(2) $Ra \subseteq Rb$ *implies* $b^\circ \subseteq a^\circ$ *and the converse is valid whenever b is regular.*

In the following theorem, we will give some necessary and sufficient conditions such that a (k,k)-core invertible element to be a $*$-DMP element.

Theorem 2.15. *Let $a \in R^{\ominus}_{k,k}$. Then the following are equivalent:*

(1) *a is $*$-DMP of index k;*
(2) *$a^kR \subseteq (a^k)^*R$;*
(3) *$Ra^k \subseteq R(a^k)^*$;*
(4) *$^\circ[(a^k)^*] \subseteq {}^\circ(a^k)$;*
(5) *$[(a^k)^*]^\circ \subseteq (a^k)^\circ$;*
(6) *$a^{\ominus}_{k,k}a^k$ is Hermitian;*
(7) *$[a^{\ominus}_{k,k}, a^k] = 0$;*
(8) *$a^{\ominus}_{k,k} = (a^D)^k$;*
(9) *$a^{\ominus}_{k,k} = (a^k)^\dagger$;*
(10) *$a^{\ominus}_{k,k}a^k(a^k)^* = (a^k)^*$;*
(11) *$a^{\ominus}_{k,k} = (a^k)^\dagger aa^D$;*
(12) *$[a^ka^{\ominus}_{k,k}, a^{\ominus}_{k,k}a^k] = 0$;*
(13) *$[a^{\ominus}_{k,k}, (a^D)^k] = 0$;*
(14) *there exists $x \in R$ such that $ax^{k+1} = x^k$, $xa^{k+1} = a^k$ and x^ka^k is Hermitian.*

Proof. (1) $\Rightarrow$ (2). Since a is $*$-DMP of index k, then a^k is EP, and thus $a^kR \subseteq (a^k)^*R$ by Lemma 2.12.

(2) $\Rightarrow$ (1). From $a^kR \subseteq (a^k)^*R$, we have $a^k = (a^k)^*r$ for some $r \in R$. Thus $a^k = (a^k)^*r = (a^ka^Da)^*r = (a^Da)^*(a^k)^*r = (a^Da)^*a^k$. Post-multiplying $a^k = (a^Da)^*a^k$ by $(a^D)^k$ yields that $aa^D = (a^Da)^*aa^D$, that is, a is $*$-DMP by Lemma 2.13 and a (k,k)-core invertible element is Drazin invertible.

The proof of (1) $\Leftrightarrow$ (3) can be proved in a similar way of (1) $\Leftrightarrow$ (2). The equivalences (2) $\Leftrightarrow$ (4) and (3) $\Leftrightarrow$ (5) follow by Lemma 2.14 and the regularity of a^k is regular (because $a \in R^{\ominus}_{k,k}$).

(1) $\Leftrightarrow$ (6). Since $a^{\ominus}_{k,k}a^k = a^Da(a^k)^\dagger a^k = (a^D)^ka^k(a^k)^\dagger a^k = (a^D)^ka^k = a^Da$, then a is $*$-DMP by Lemma 2.13. The opposite implication can be proved in a similar way.

(1) $\Leftrightarrow$ (7). It is easy to check that $[a^{\ominus}_{k,k}, a^k] = 0$ is equivalent to $aa^D = a^k(a^k)^\dagger$. Thus, the equivalence can be seen by Lemma 2.13.

(1) $\Rightarrow$ (8), (1) $\Rightarrow$ (9) and (1) $\Rightarrow$ (11) are follow by $(a^k)^\# = (a^D)^k$, for $\mathrm{ind}(a) = k$.

(8) $\Rightarrow$ (1). The hypothesis $a^{\ominus}_{k,k} = (a^D)^k$ implies $aa^D = a^k(a^D)^k = a^ka^{\ominus}_{k,m} = a^ka^Da(a^k)^\dagger = a^k(a^k)^\dagger$ is Hermitian. So, by Lemma 2.13, a is $*$-DMP.

(9) $\Rightarrow$ (6). Using $a^{\ominus}_{k,k} = (a^k)^\dagger$, we get $a^{\ominus}_{k,k}a^k = (a^k)^\dagger a^k$ is Hermitian.

(10) $\Leftrightarrow$ (6). Post-multiplying $a^{\ominus}_{k,k}a^k(a^k)^* = (a^k)^*$ by $[(a^k)^\dagger]^*$, we observe that $a^{\ominus}_{k,k}a^k = (a^k)^\dagger a^k$ is Hermitian. The converse is obvious.

(11) $\Rightarrow$ (6) follows because $a^{\ominus}_{k,k}a^k = (a^k)^\dagger aa^Da^k = (a^k)^\dagger a^k$ is Hermitian.

(7) $\Rightarrow$ (12) is evident.

(7) $\Rightarrow$ (13) is obvious by the commutativity of the Drazin inverse and $(a^D)^k = (a^k)^D$.

(12) $\Rightarrow$ (1). Applying $[a^ka^{\ominus}_{k,k}, a^{\ominus}_{k,k}a^k] = 0$, $a^ka^{\ominus}_{k,k}a^{\ominus}_{k,k}a^k = aa^D$ and $a^{\ominus}_{k,k}a^ka^ka^{\ominus}_{k,k} = a^k(a^k)^\dagger$, we note that $aa^D = a^k(a^k)^\dagger$ is Hermitian. Therefore, a is $*$-DMP by Lemma 2.13.

(13) $\Rightarrow$ (8). Because $a^{\ominus}_{k,k}(a^D)^k = (a^D)^ka^k(a^k)^\dagger a^k(a^D)^{2k} = a^Da(a^D)^{2k} = (a^D)^{2k}$ and $(a^D)^ka^{\ominus}_{k,k} = (a^D)^ka^Da(a^k)^\dagger = (a^D)^k(a^k)^\dagger$, the assumption $[a^{\ominus}_{k,k}, (a^D)^k] = 0$ gives $(a^D)^{2k} = (a^D)^k(a^k)^\dagger$. Thus, $(a^D)^k = a^k(a^D)^{2k} = a^k(a^D)^k(a^k)^\dagger = a^Da(a^k)^\dagger = a^{\ominus}_{k,k}$.

(1) $\Leftrightarrow$ (14). It is trivial by Lemma 2.2 and Lemma 2.13. Have in mind, $a^Da = x^{k+1}a^ka = x^{k+1}a^{k+1} = x^ka^k$. $\square$

There exists some partial orderings base on the core inverse, for example [1, 11], as the (j, m)-core inverse is a generalization of the core inverse, now we will introduce a ordering base on the the (j, m)-core inverse.

Lemma 2.16. *Let $a \in R^{\ominus}_{j,m}$ and $b \in R^{\ominus}_{l,n}$. If $ab = ba$ and $a^*b = ba^*$, then $ab^{\ominus}_{l,n} = b^{\ominus}_{l,n}a$, $a^{\ominus}_{j,m}b = ba^{\ominus}_{j,m}$ and $a^{\ominus}_{j,m}b^{\ominus}_{l,n} = b^{\ominus}_{l,n}a^{\ominus}_{j,m}$.*

Proof. Notice that $a^{\ominus}_{j,m} = a^D a(a^j)^\dagger$ and $b^{\ominus}_{l,n} = b^D b(b^l)^\dagger$. By $ab = ba$ and [3, Theorem 1], we have that a and a^D commute with b and b^D. Since $ab = ba$ and $ab^* = b^*a$, then a commute with b^l and $(b^l)^*$. Using [6, Lemma 1.1], we deduce that a commutes with $(b^l)^\dagger$, which implies that a commutes with $b^{\ominus}_{l,n}$. In the same way, we verify that $a^{\ominus}_{j,m}b = ba^{\ominus}_{j,m}$ and $a^{\ominus}_{j,m}b^* = b^*a^{\ominus}_{j,m}$, which imply $a^{\ominus}_{j,m}b^{\ominus}_{l,n} = b^{\ominus}_{l,n}a^{\ominus}_{j,m}$. □

Lemma 2.17. *Let $a, b \in R^D$ with $ab = ba$ and $p, q \in \mathbb{N}$. If $\mathrm{ind}(a) = p$ and $\mathrm{ind}(b) = q$, then $\mathrm{ind}(ab) \leq \max(p, q)$.*

Proof. Since $\mathrm{ind}(a) = p$ and $\mathrm{ind}(b) = q$, we have that $a^p = a^D a^{p+1}$ and $b^q = b^D b^{q+1}$. Suppose that $q \leq p$. By $ab = ba$ and [3, Theorem 1], we have that $(ab)^p = a^p b^p = a^D a^{p+1} b^D b^{p+1} = (ab)^D (ab)^{p+1}$, which gives that $\mathrm{ind}(ab) \leq p$. Similarly, if $p \leq q$, then $\mathrm{ind}(ab) \leq q$. Thus, $\mathrm{ind}(ab) \leq \max(p, q)$. □

By the definition of the (j, m)-core inverse, the condition $\mathrm{ind}(ab) \leq \max(p, q)$ in Lemma 2.17 is useful in the following theorem.

Theorem 2.18. *Let $a, b \in R^{\ominus}_{j,m}$ such that $ab = ba$ and $a^*b = ba^*$. Then $ab \in R^{\ominus}_{j,m}$ and $(ab)^{\ominus}_{j,m} = b^{\ominus}_{j,m}a^{\ominus}_{j,m} = a^{\ominus}_{j,m}b^{\ominus}_{j,m}$.*

Proof. The assumption $ab = ba$ gives that $ab \in R^D$ and $(ab)^D = b^D a^D = a^D b^D$. Also, we can easily prove that $(ab)^j \in R^\dagger$ and $[(ab)^j]^\dagger = (b^j)^\dagger (a^j)^\dagger = (a^j)^\dagger (b^j)^\dagger$. By Lemma 2.17, we have $\mathrm{ind}(ab) \leq \max\{\mathrm{ind}(a), \mathrm{ind}(b)\} = m$. Therefore, ab is (j, m)-core invertible and

$$(ab)^{\ominus}_{j,m} = (ab)^D ab[(ab)^j]^\dagger = b^D a^D ab(b^j)^\dagger (a^j)^\dagger = b^D b(b^j)^\dagger a^D a(a^j)^\dagger = b^{\ominus}_{j,m}a^{\ominus}_{j,m} = a^{\ominus}_{j,m}b^{\ominus}_{j,m}.$$

That is, $(ab)^{\ominus}_{j,m} = b^{\ominus}_{j,m}a^{\ominus}_{j,m} = a^{\ominus}_{j,m}b^{\ominus}_{j,m}$. □

It is well-known that for two Drazin invertible elements $a, b \in R^D$ with $ab = ba = 0$, then $(a + b)^D = a^D + b^D$.

Lemma 2.19. *Let $a, b \in R^D$ with $ab = ba = 0$ and $p, q \in \mathbb{N}$. If $\mathrm{ind}(a) = p$ and $\mathrm{ind}(b) = q$, then $\mathrm{ind}(a + b) \leq \max(p, q)$.*

Proof. Since $\mathrm{ind}(a) = p$ and $\mathrm{ind}(b) = q$, we have that $a^p = a^D a^{p+1}$ and $b^q = b^D b^{q+1}$. Suppose that $q \leq p$. By $ab = ba = 0$ and [3, Corollary 1], we have that $(a+b)^p = a^p + b^p = a^D a^{p+1} + b^D b^{p+1} = (a + b)^D (a + b)^{p+1}$, which gives that $\mathrm{ind}(a + b) \leq p$. Similarly, if $p \leq q$, then $\mathrm{ind}(a + b) \leq q$. Thus, $\mathrm{ind}(a + b) \leq \max(p, q)$. □

Theorem 2.20. *Let $a, b \in R^{\ominus}_{j,m}$ such that $ab = ba = 0 = a^*b = ba^*$. Then $a + b \in R^{\ominus}_{j,m}$ and $(a + b)^{\ominus}_{j,m} = a^{\ominus}_{j,m} + b^{\ominus}_{j,m}$.*

Proof. First, we have that $a+b \in R^D$ and $(a+b)^D = a^D + b^D$ by [3, Corollary 1]. Further, we can verify that $(a + b)^j \in R^\dagger$ and $[(a + b)^j]^\dagger = (a^j)^\dagger + (b^j)^\dagger$. By Lemma 2.19, we have $\mathrm{ind}(a + b) \leq \max\{\mathrm{ind}(a), \mathrm{ind}(b)\} = m$. So, $a + b$ is (j, m)-core invertible and

$$(a + b)^{\ominus}_{j,m} = (a^D + b^D)(a + b)[(a^j)^\dagger + (b^j)^\dagger] = a^D a(a^j)^\dagger + b^D b(b^j)^\dagger = a^{\ominus}_{j,m} + b^{\ominus}_{j,m}.$$

That is, $(a + b)^{\ominus}_{j,m} = a^{\ominus}_{j,m} + b^{\ominus}_{j,m}$. □

Let $R^{D,\dagger}$ denotes the set of all DMP invertible elements of R. Since the (j,m)-core inverse of a is a generalization of the DMP-inverse of a, thus by Theorem 2.18 and Theorem 2.20, we have the following two corollaries.

Corollary 2.21. *Let $a, b \in R^{D,\dagger}$ such that $ab = ba$ and $a^*b = ba^*$. Then ab is DMP invertible and $(ab)^{D,\dagger} = b^{D,\dagger}a^{D,\dagger} = a^{D,\dagger}b^{D,\dagger}$.*

Corollary 2.22. *Let $a, b \in R^{D,\dagger}$ such that such that $ab = ba = 0 = a^*b = ba^*$. Then $a+b$ is DMP invertible and $(a+b)^{D,\dagger} = a^{D,\dagger} + b^{D,\dagger}$.*

3. The $\ominus$-core relation

As the (j,m)-core is a generalation of the core inverse and the core partial ordering was introduced in [1], here we introduce an ordering base on the (j,m)-core.

Definition 3.1. Let a be (j,m)-core invertible and $b \in R$. Then a is below b under the $\ominus$-core relation (denoted by $a \leq^{\ominus} b$) if

$$a^{\ominus}_{j,m}a = a^{\ominus}_{j,m}b \quad and \quad aa^{\ominus}_{j,m} = ba^{\ominus}_{j,m}.$$

Lemma 3.2. *Let a be (j,m)-core invertible and $b \in R$ with $\operatorname{ind}(a) \leq \min(j,m)$. Then*

(1) $a^{\ominus}_{j,m}a = a^{\ominus}_{j,m}b \Leftrightarrow (a^j)^\dagger a = (a^j)^\dagger b \Leftrightarrow (a^j)^* a = (a^j)^* b \Leftrightarrow a^*a^j = b^*a^j \Leftrightarrow a^*a^D = b^*a^D$;

(2) $aa^{\ominus}_{j,m} = ba^{\ominus}_{j,m} \Leftrightarrow aa^D = ba^D \Leftrightarrow a^{j+1} = ba^j$.

Proof. (1). Pre-multiplying $a^{\ominus}_{j,m}a = a^{\ominus}_{j,m}b$ by $(a^j)^\dagger a^j$ yields $(a^j)^\dagger a = (a^j)^\dagger b$. Pre-multiplying $(a^j)^\dagger a = (a^j)^\dagger b$ by $a^D a$ yields $a^{\ominus}_{j,m}a = a^{\ominus}_{j,m}b$. The equivalence $(a^j)^\dagger a = (a^j)^\dagger b \Leftrightarrow (a^j)^* a = (a^j)^* b$ is obvious by $((a^j)^\dagger)^\circ = ((a^j)^*)^\circ$. The remaining is obvious.

(2). First we show that $aa^{\ominus}_{j,m} = ba^{\ominus}_{j,m} \Leftrightarrow a^2a^D = ba^Da \Leftrightarrow a^{j+1} = ba^j$. Post-multiplying $aa^{\ominus}_{j,m} = ba^{\ominus}_{j,m}$ by a^j yields $a^2a^D = ba^Da$. Post-multiplying $a^2a^D = ba^Da$ by $(a^j)^\dagger$ yields $aa^{\ominus}_{j,m} = ba^{\ominus}_{j,m}$. Post-multiplying $aa^{\ominus}_{j,m} = ba^{\ominus}_{j,m}$ by a^{2j} yields $a^{j+1} = ba^j$. Post-multiplying $a^{j+1} = ba^j$ by $(a^D)^j(a^j)^\dagger$ yields $aa^{\ominus}_{j,m} = ba^{\ominus}_{j,m}$. By post-multiplying a^D on $a^2a^D = ba^Da$, we have $aa^D = ba^D$ and $aa^D = ba^D$ implies $a^2a^D = ba^Da$ is trivial. Thus $a^2a^D = ba^Da$ if and only if $aa^D = ba^D$. □

Theorem 3.3. *Let a be (j,m)-core invertible and $b \in R$ with $\operatorname{ind}(a) \leq \min(j,m)$. Then the following statements are equivalent:*

(1) $a \leq^{\ominus} b$;

(2) $a^*a^j = b^*a^j$ *and* $a^{j+1} = ba^j$;

(3) $a^*a^D = b^*a^D$ *and* $aa^D = ba^D$;

(4) *There exists an idempotent $p \in R$ such that $a^DR = pR$, $ap = bp$ and $a^*p = b^*p$;*

(5) *There exists an Hermitian idempotent $q \in R$ such that $a^jR = qR$, $aq = bq$ and $qa = qb$.*

Proof. (1) $\Leftrightarrow$ (2) $\Leftrightarrow$ (3). These equivalences follow by Lemma 3.2.

(1) $\Rightarrow$ (4). For $p = a^Da$, first we have that $a^DR = a^DaR = pR$. By (1) and Lemma 3.2, we observe that $ap = (aa^D)a = ba^Da = bp$ and $a^*p = (a^*a^D)a = b^*a^Da = b^*p$.

(4) $\Rightarrow$ (1). Assume that there exists an idempotent $p \in R$ such that $a^DR = pR$, $ap = bp$ and $a^*p = b^*p$. Then $a^D = pa^D$ gives $a^*a^D = (a^*p)a^D = b^*pa^D = b^*a^D$ and $aa^D = (ap)a^D = bpa^D = ba^D$. Using Lemma 3.2, we deduce that (1) holds.

(1) $\Leftrightarrow$ (5). We check this part similarly as (1) $\Leftrightarrow$ (4). □

Theorem 3.4. *The $\ominus$-core relation is a pre-order on the set of all (j,m)-core invertible elements in R, where the index of these elements are less or equal $\min(j,m)$.*

Proof. Obviously, $\leq^{\ominus}$ is reflexive. To verify that $\leq^{\ominus}$ is transitive, suppose that $a, b, c \in R$ such that a and b are (j, m)-core invertible elements, $a \leq^{\ominus} b$ and $b \leq^{\ominus} c$. Using Lemma 3.2, we obtain

$$a^*a^D = b^*a^D = b^*a(a^D)^2 = b^*b(a^D)^2 = b^*b^j(a^D)^{j+1} = c^*b^j(a^D)^{j+1} = c^*a^D$$

and

$$aa^D = ba^D = ba(a^D)^2 = b^2(a^D)^2 = b^{l+1}(a^D)^{l+1} = cb^l(a^D)^{l+1} = cb(a^D)^2 = ca(a^D)^2 = ca^D.$$

Applying Theorem 3.3, we deduce that $a \leq^{\ominus} c$. □

In the following example, we show that the relation "$\leq^{\ominus}$" is not antisymmetric and so it is not a partial order on the set of all (j, m)-core invertible elements in R.

Example 3.5. Let $a = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$ and $b = \begin{bmatrix} 0 & 0 \\ 2 & 0 \end{bmatrix} \in \mathbb{C}^{2\times 2}$. Since $a^D = 0 = (a^j)^\dagger$ and $b^D = 0 = (b^j)^\dagger$, for $j \geq 2$, then $a^{\ominus}_{j,m} = 0$ and $b^{\ominus}_{j,m} = 0$, which yield $a^{\ominus}_{j,m}a = 0 = a^{\ominus}_{j,m}b = aa^{\ominus}_{j,m} = ba^{\ominus}_{j,m}$ and $bb^{\ominus}_{j,m} = 0 = ab^{\ominus}_{j,m} = b^{\ominus}_{j,m}b = 0 = b^{\ominus}_{j,m}a$. Thus, $a \leq^{\ominus} b$ and $b \leq^{\ominus} a$, but $a \neq b$.

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References

[1] O.M. Baksalary and G. Trenkler. *Core inverse of matrices*, Linear Multilinear Algebra, **58**(6), 681–697, 2010.

[2] O.M. Baksalary and G. Trenkler, *On a generalized core inverse*, Appl. Math. Comput. **236**, 450–457, 2014.

[3] M.P. Drazin, *Pseudo-inverses in associative rings and semigroup*, Amer. Math. Monthly, **65**, 506–514, 1958.

[4] R.E. Hartwig, *Block generalized inverses*, Arch. Retional Mech. Anal. **61**(3), 197–251, 1976.

[5] S.B. Malik and N. Thome, *On a new generalized inverse for matrices of an arbitrary index*, Appl. Math. Comput. **226**, 575–580, 2014.

[6] D. Mosić and D.S. Djordjević, *Moore-Penrose-invertible normal and Hermitian elements in rings*, Linear Algebra Appl. **431**, 732–745, 2009.

[7] P. Patrício and R. Puystjens, *Drazin-Moore-Penrose invertiblity in rings*, Linear Algebra Appl. **389**, 159–173, 2004.

[8] D.S. Rakić, N.Č. Dinčić and D.S. Djordjević, *Group, Moore-Penrose, core and dual core inverse in rings with involution*, Linear Algebra Appl. **463**, 115–133, 2014.

[9] H. Schwerdtfeger, *Introduction to Linear Algebra and the Theory of Matrices*, P. Noordhoff, Groningen, 1950.

[10] J. von Neumann, *On regular rings*, Proc. Nati. Acad. Sci. U.S.A. **22** (12), 707–713, 1936.

[11] H.X. Wang and X.J. Liu, *A partial order on the set of complex matrices with index one*, Linear Multilinear Algebra, **66** (1), 206–216, 2018.

[12] S.Z. Xu, J.L. Chen and J. Benítez, *EP elements in rings with involution*, Bull. Malays. Math. Sci. Soc. **42**, 3409–3426, 2019.

[13] S.Z. Xu, J.L. Chen, J. Benítez and D.G. Wang, *Generalized core inverse of matrices*, Miskolc Math. Notes, **20** (1), 565–584, 2019.

[14] S.Z. Xu, J.L. Chen and X.X. Zhang, *New characterizations for core and dual core inverses in rings with involution*, Front. Math. China. **12** (1), 231–246, 2017.