2-Transitive Frobenius *Q*-Groups

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Abstract. A Q-group is a finite group all of whose irreducible complex characters are rationally- valued. In this paper, we find all 2-transitive Frobenius Q-groups.

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INTRODUCTION

We use the following notations in this paper

Notation. C the field of complex numbers ; Z the ring of rational integers ; $Gen(x)$ the set of generators of cyclic group $\langle x \rangle$; $[x]$ the conjugacy class of x ; $N_G(x)$ the normalizer of x in G ; $C_G(x)$ the centralizer of x in G ; $Aut(x)$ the group of automorphisms of $\langle x \rangle$; φ Euler's function ; $\times|$ semidirect product ; Q_8 the quaternion group of order 8 ; $E(p^n)$ the elemantary abelian p-group of order p^n .

Q-GROUPS

Definition. A finite group whose complex characters are rationally-valued is called a Q-group.

For example, all of the symmetric groups and finite elemantary abelian 2-groups are Q-groups. Kletzing's lecture notes, [3] , presents a detailed investigation into the structure of Q-groups.General classification of Q-groups has not been able to be done up to now, but some special Q-groups have been classified. Frobenius Q-groups were classified by [1].

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Let G be a finite group of order n and let ξ be a primitive n-th root of unity in the field C. Then, all of the complex character values of G lie in the subring $Z[\xi]$ of C. Moreover, if G is a Q-group, these values lie in Z since $Z[\xi]\cap Q = Z$. Now, we say that a finite group is a Q-group if and only if the values of its all the irreducible complex characters lie in Z.

Another characterization of Q-groups is the following theorem.

Theorem 1. Let G be a finite group. Then, G is a Q-group if and only if for every $x \in G$, $Gen(x) \subseteq [x]$ i.e. for every $x \in G$, $N_G(x)/C_G(x) \cong Aut(x)$ [3].

Now, we can easily see the following corollary.

Corrollary 2. Let G be a Q-group. Then:

1) $[N_G(x) : C_G(x)] = |Aut(x)| = \varphi(|x|)$, for every $x \in G$.

2) If $G \neq \{1\}$, $2 \mid |G|$.

3) If N is a normal subgroup of G, then G/N is a Q-group.

FROBENIUS GROUPS

Definition. Let G be a transitive and non-regular permutation group on Ω, $|\Omega| \in IN$, $\alpha \in \Omega$, $H = G_\alpha$. Then, G is called a Frobenius group with complement H if and only if the identity element of G is unique element that fixes more than one element of Ω.

Definition. A trivial intersection set in a group G is a subset S of G such that for all $g \in G$, either $S^g = S$ or $S^g \cap S \subseteq \{1\}$.

Lemma 3. Let G be a finite group, $\{1\} \neq H$ a proper subgroup of G. Then the following are equivalent:

(a) G is a Frobenius group with complement H.

(b) H is a trivial intersection set and $H = N_G(H)$. [2]

Now we can say that a finite group G is a Frobenius group if and only if it contains a proper subgroup $H \neq \{1\}$, called a Frobenius complement, such that $H \cap H^x = \{1\}$ for all $x \notin H$.

By Frobenius Theorem [2, p.63], a Frobenius group G with complement H has a normal subgroup K, called Frobenius kernel, such that $G = K \times| H$. If $K = \{1 = x_1, ..., x_n\}$ where $n \in IN$, then we have $G = K \dot{\cup} \left[\dot{\bigcup_{i=1}^{n}} (H^{x_i} - \{1\}) \right]$, called Frobenius partition.

Definition. Let G be a group and φ be a automorphism of G. Then, φ is called fixed-point-free automorphism if $\varphi(g) \neq g$ for every $g \in G - \{1\}$.

Let G be a Frobenius group with kernel K and complement H. Then, for every $h \in H - \{1\}$, $k \mapsto h^{-1}.k.h$ is an automorphism of K fixing only the element $1 \in K$. Thus we can say that all elements of H except 1 are fixed-point-free of K. Moreover, a semi-direct group $G = K \times| H$ is a Frobenius group if h is fixed-point-free of K for every $h \in H - \{1\}$.

Lemma 4. Let G be a tranzitive permutation group on Ω, $|\Omega| \in IN - \{1\}$. Then G is 2-transitive if and only if $G = G_\alpha \dot{\cup} G_\alpha x G_\alpha$ for all $\alpha \in \Omega$ and $x \in G - G_\alpha$ [4].

Theorem 5. Let G be a Frobenius group with kernel K and complement H. Then G is 2-transitive if and only if $|K| = |H| + 1$.

Proof. By the definition of Frobenius group, G is a tranzitive permutation group on Ω, $|\Omega| \in IN$ and there is $\alpha \in \Omega$ such that $H = G_\alpha$. By Lemma 4., we know that G is 2-transitive if and only if $G = H \dot{\cup} HxH$ for every $x \in G - H$. Therefore, if G is 2-transitive, then we have

$$|G| = |H| + |H x H| = |H| + \frac{|H|.|H^x|}{|H \cap H^x|} = |H| + \frac{|H|^2}{|H \cap H^x|}$$

for every $x \in G - H$. Since H is a trivial intersection set in G and $H = N_G(H)$ by Lemma3., we have $|H \cap H^x| = 1$ for every $x \in G - H$. Thus, $|G| = |H| + |H|^2$. Also, since $|G| = |K|.|H|$ by Frobenius Theorem, we have $|K|.|H| = |G| = |H| + |H|^2$ and so $|K| = |H| + 1$. Conversely, we can easily that if $|K| = |H| + 1$, then G is 2-transitive.

Theorem 6. Let G be a Frobenius Q-group with kernel K and complement H. Then, $H \cong Z_2$ or $H \cong Q_8$. Moreover,

1) If $H \cong Z_2$, then K is an elemantary abelian 3-group and for every $t \in K$, $t^u = t^{-1}$ where $1 \neq u \in H$.

2) If $H \cong Q_8$, then K is an elemantary abelian p-group where $p = 3$ or $p = 5$. [1]

2-TRANSITIVE FROBENIUS *Q*-GROUPS

All 2-transitive Frobenius Q-groups are given by the following theorem.

Theorem. Let G be a 2-transitive Frobenius Q-group. Then, $G \cong S_3$ or $G \cong E(3^2) \times| Q_8$ where $E(3^2)$ is the 2-dimensional irreducible module of group algebra $Z_3 Q_8$.

Proof. Let G be a 2-transitive Frobenius Q-group with kernel K and complement H. By Theorem 6, we know that $H \cong Z_2$ or $H \cong Q_8$.

1) If $H \cong Z_2$, then K is an elementary abelian 3-group and for every $t \in K$, $t^u = t^{-1}$ where $1 \neq u \in H$ by Theorem 6. Moreover, since G is 2-transitive, we have $|K| = |H| + 1 = 3$ by Theorem 5. Therefore, $G \cong E(3) \times| Z_2 \cong S_3$. Conversely, we can see easily that S_3 is a 2-transitive Frobenius Q-group.

2) If $H \cong Q_8$, then K is an elementary abelian p-group where $p=3$ or $p=5$ by Theorem 6. Since G is 2-transitive, we have $|K|=|H|+1=9$ by Theorem 5. Then, K must be an elemantary abelian 3-group of order 9. Since $K \lhd G$, H acts on K by conjugation. Thus, K may be considered as a Z_3H -module, so K defines a representation of H over the field Z_3. Since $3 \nmid |H|$, we can use ordinary representation theory, so K is a direct sum of some irreducible modules of group ring Z_3H by Maschke's Theorem and Wedderburn's Theorem. H has exactly five non-isomorphic irreducible module over the field Z_3 and four of them are 1-dimensional so the other is 2-dimensional. Since $\chi(u) = 1 \in Z_3$ for every 1-dimensional representation χ of H where u is the involusion, K must be the 2-dimensional irreducible module of group ring Z_3H . Therefore, we have $G \cong E(3^2) \times| Q_8$ where $E(3^2)$ is the 2-dimensional irreducible module of group algebra Z_3Q_8. Conversely, let $G \cong E(3^2) \times| Q_8$ where $E(3^2)$ is the 2-dimensional irreducible module of group algebra Z_3Q_8. Then, G is a Frobenius group with kernel K ($\cong E(3^2)$) and complement H ($\cong Q_8$) since the involution of H is fixed-point-free of K . Moreover, since $|K| = |H|+1$, G is 2-transitive by Theorem 5. Since G is a Frobenius group, we have Frobenius partition $G = K \dot{\cup} \left[\dot{\bigcup_{g\in K}} (H^g - \{1\}) \right]$. Also, for every $g \in K$, H^g ($\cong Q_8$) is a Q-group and for every $1 \neq x \in K$, $x^u = x^2$ where u is the involusion of H . Thus, by the definition of Q-group, we can see easily that G is a Q-group.

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