Pairwise Semiregular Properties on Generalized Pairwise Regular-Lindelöf Spaces

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Abstract. Let (X, τ_1, τ_2) be a bitopological space and $\left(X, \tau^s_{(1,2)}, \tau^s_{(2,1)}\right)$ its pairwise semiregularization. Then a bitopological property $\mathcal{P}$ is called pairwise semiregular provided that (X, τ_1, τ_2) has the property $\mathcal{P}$ if and only if $\left(X, \tau^s_{(1,2)}, \tau^s_{(2,1)}\right)$ has the same property. In this paper we study pairwise semiregular properties of a bitopological space. We prove that pairwise almost regular-Lindelöfness and pairwise weakly regular-Lindelöfness are pairwise semiregular properties.

Keywords: Bitopological space, pairwise nearly regular-Lindelöf, pairwise almost regular-Lindelöf, pairwise weakly regular-Lindelöf, (i, j)-semiregular property, pairwise semiregular property.

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1. Introduction

Semiregular properties in topological spaces have been studied by many topologists. Some of them related to this research studied by Mrsevic et al. [11, 12], and Fawakhreh and Kılıçman [2]. The purpose of this paper is to study pairwise semiregular properties on generalized pairwise regular-Lindelöf spaces, that we have studied in [10, 8], namely, pairwise nearly regular-Lindelöf, pairwise almost regular-Lindelöf and pairwise weakly regular-Lindelöf spaces.

The main results are that the (i, j)-almost regular-Lindelöf, pairwise almost reglar-Lindelöf, (i, j)-weakly regular-Lindelöf and pairwise weakly regular-Lindelöf spaces are pairwise semiregular properties. We also show that the (i, j)-nearly regular-Lindelöf and pairwise nearly regular-Lindelöf spaces are pairwise semiregular invariant properties.

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2. **Preliminaries**

Throughout this paper, all spaces (X, τ) and (X, τ_1, τ_2) (or simply X) are always mean topological spaces and bitopological spaces, respectively. If $\mathcal{P}$ is a topological property, then (τ_i, τ_j)-$\mathcal{P}$ denotes an analogue of this property for τ_i has property $\mathcal{P}$ with respect to τ_j, and p-$\mathcal{P}$ denotes the conjunction (τ_1, τ_2)-$\mathcal{P} \wedge (\tau_2, \tau_1)$-$\mathcal{P}$, i.e., p-$\mathcal{P}$ denotes an absolute bitopological analogue of $\mathcal{P}$. As we shall see below, sometimes (τ_1, τ_2)-$\mathcal{P} \iff (\tau_2, \tau_1)$-$\mathcal{P}$ (and thus $\iff p$-$\mathcal{P}$) so that it sometimes suffices to consider one of these three bitopological analogue. Also sometimes τ_1-$\mathcal{P} \iff \tau_2$-$\mathcal{P}$ and thus $\mathcal{P} \iff \tau_1$-$\mathcal{P} \wedge \tau_2$-$\mathcal{P}$, i.e., (X, τ_i) has property $\mathcal{P}$ for each $i = 1, 2$. Also note that (X, τ_i) has a property $\mathcal{P} \iff (X, \tau_1, \tau_2)$ has a property τ_i-$\mathcal{P}$.

Sometimes the prefixes (τ_i, τ_j)- or τ_i- will be replaced by (i, j)- or i- respectively, if there is no chance for confusion. By i-open cover of X, we mean that the cover of X by i-open sets in X; similar for the (i, j)-regular open cover of X etc. By i-int(A) and i-cl(A), we shall mean the interior and the closure of a subset A of X with respect to topology τ_i, respectively. In this paper we always have $i, j \in \{1, 2\}$ and $i \neq j$. The reader may consult [1] for details of notation.

The following are some basic concepts.

Definition 2.1. *Let (X, τ_1, τ_2) be a bitopological space. A subset F of X is said to be*

(i) i-open if F is open with respect to τ_i in X, and F is called open in X if it is both 1-open and 2-open in X, or equivalently, $F \in (\tau_1 \cap \tau_2)$ in X;
(ii) i-closed if F is closed with respect τ_i in X, and F is called closed in X if it is both 1-closed and 2-closed in X, or equivalently, $X \setminus F \in (\tau_1 \cap \tau_2)$ in X.

Definition 2.2. [3, 5] *A bitopological space (X, τ_1, τ_2) is said to be i-Lindelöf if the topological space (X, τ_i) is Lindelöf. X is called Lindelöf (or p-Lindelöf in* [5]*) if it is both 1-Lindelöf and 2-Lindelöf. Equivalently, (X, τ_1, τ_2) is Lindelöf if every i-open cover of X has a countable subcover for each $i = 1, 2$.*

Definition 2.3. [4, 16] *A subset S of a bitopological space (X, τ_1, τ_2) is said to be (i, j)-regular open (resp. (i, j)-regular closed) if i-int$(j$-cl$(S)) = S$ (resp. i-cl$(j$-int$(S)) = S$), and S is called pairwise regular open (resp. pairwise regular closed) if it is both $(1, 2)$-regular open and $(2, 1)$-regular open (resp. $(1, 2)$-regular closed and $(2, 1)$-regular closed).*

Definition 2.4. [4, 17] *A bitopological space X is said to be (i, j)-almost regular if for each $x \in X$ and for each (i, j)-regular open set V of X containing x, there is an (i, j)-regular open set U such that $x \in U \subseteq j$-cl$(U) \subseteq V$. X is said to be pairwise almost regular if it is both $(1, 2)$-almost regular and $(2, 1)$-almost regular.*

Definition 2.5. [13] *The topology generated by the (i, j)-regular open subsets of (X, τ_1, τ_2) is denoted by $\tau^s_{(i,j)}$ and it is called (i, j)-semiregularization of X. The bitopological space $\left(X, \tau^s_{(1,2)}, \tau^s_{(2,1)}\right)$ is called the pairwise semiregularization of (X, τ_1, τ_2). If $\tau_i \equiv \tau^s_{(i,j)}$, then X is said to be (i, j)-semiregular. (X, τ_1, τ_2) is called pairwise semiregular if it is both $(1, 2)$-semiregular and $(2, 1)$-semiregular, that is, whenever $\tau_i \equiv \tau^s_{(i,j)}$ for each $i, j \in \{1, 2\}$ and $i \neq j$.*

It is very clear that $\tau^s_{(i,j)} \subseteq \tau_i$, but it is not necessary $\tau_i \subseteq \tau^s_{(i,j)}$. For a better understanding, let $\mathcal{B}_1$ be the family of all $(1, 2)$-regular open subsets of X and let $\mathcal{B}_2$ be the family of all $(2, 1)$-regular open subsets of X. Since the intersection of two (i, j)-regular open subsets of X is (i, j)-regular open set, therefore $\mathcal{B}_1$ and $\mathcal{B}_2$ both generate topologies for (X, τ_1, τ_2) say $\tau^s_{(1,2)}$ and $\tau^s_{(2,1)}$ respectively. Thus with every given bitopological space (X, τ_1, τ_2) there is an associated bitopological space $\left(X, \tau^s_{(1,2)}, \tau^s_{(2,1)}\right)$ in the manner described above. Note that, the space $\left(X, \tau^s_{(1,2)}, \tau^s_{(2,1)}\right)$ is always pairwise semiregular. Singal and Arya [16], proved the following theorem.

Theorem 2.1. *If (X, τ_1, τ_2) is pairwise semiregular, then $(X, \tau_1, \tau_2) = \left(X, \tau^s_{(1,2)}, \tau^s_{(2,1)}\right)$.*

The converse of Theorem 2.1 is also true by the definitions.

Theorem 2.2. *A bitopological space X is (i, j)-semiregular if and only if for each $x \in X$ and for each i-open subset V of X containing x, there is an i-open set U such that $x \in U \subseteq i$-int$(j$-cl$(U)) \subseteq V$.*

Proof. Let (X, τ_1, τ_2) be an (i, j)-semiregular space, then $\tau_i = \tau^s_{(i,j)}$, i.e., τ_i is generated by (i, j)-regular open sets in (X, τ_1, τ_2). Suppose that $x \in X$ and let V be an i-open set in

(X, τ_1, τ_2) containing x. Since the family of (i,j)-regular open sets in (X, τ_1, τ_2) forms a base for τ_i, there exists an i-open sets U in (X, τ_1, τ_2) such that $x \in U \subseteq i\text{-int}\,(j\text{-cl}\,(U)) \subseteq V$. Conversely, assume the condition holds. Generally we have $\tau^s_{(i,j)} \subseteq \tau_i$. Suppose that $p \in X$ and $F_p \in \tau_i$ with $p \in F_p$. By hypothesis, there is an i-open set U_p in (X, τ_1, τ_2) such that $p \in U_p \subseteq i\text{-int}\,(j\text{-cl}\,(U_p)) \subseteq F_p$. Hence the family $\{i\text{-int}\,(j\text{-cl}\,(U_p)) : p \in X\}$ forms a base for $\tau^s_{(i,j)}$ which implies that $F_p \in \tau^s_{(i,j)}$. Therefore $\tau_i \subseteq \tau^s_{(i,j)}$ and thus (X, τ_1, τ_2) is (i,j)-semiregular. □

Corollary 2.3. *A bitopological space X is pairwise semiregular if and only if for each $x \in X$ and for each i-open subset V of X containing x, there is an i-open set U such that $x \in U \subseteq i\text{-}int\,(j\text{-}cl\,(U)) \subseteq V$ for each $i, j \in \{1, 2\}, i \neq j$.*

Khedr and Alshibani [4] use Theorem 2.2 as a definition of (i,j)-semiregular spaces. If a bitopological space X has a bitopological property $\mathcal{P}$ (see [6]), one may ask whether the pairwise semiregularization of X has the property $\mathcal{P}$. Now we introduce the concept of pairwise semiregular property.

Definition 2.6. *Let (X, τ_1, τ_2) be a bitopological space and let $\left(X, \tau^s_{(1,2)}, \tau^s_{(2,1)}\right)$ be its pairwise semiregularization. A bitopological property $\mathcal{P}$ is called pairwise semiregular provided that (X, τ_1, τ_2) has the property $\mathcal{P}$ if and only if $\left(X, \tau^s_{(1,2)}, \tau^s_{(2,1)}\right)$ has the property $\mathcal{P}$.*

Lemma 2.4. [13] *Let (X, τ_1, τ_2) be a bitopological space and let $\left(X, \tau^s_{(1,2)}, \tau^s_{(2,1)}\right)$ its pairwise semiregularization. Then*

(a) $\tau_i\text{-}int\,(C) = \tau^s_{(i,j)}\text{-}int\,(C)$ for every τ_j-closed set C;
(b) $\tau_i\text{-}cl\,(A) = \tau^s_{(i,j)}\text{-}cl\,(A)$ for every $A \in \tau_j$;
(c) the family of (τ_i, τ_j)-regular open sets of (X, τ_1, τ_2) is the same as the family of $\left(\tau^s_{(i,j)}, \tau^s_{(j,i)}\right)$-regular open sets of $\left(X, \tau^s_{(1,2)}, \tau^s_{(2,1)}\right)$;
(d) the family of (τ_i, τ_j)-regular closed sets of (X, τ_1, τ_2) is the same as the family of $\left(\tau^s_{(i,j)}, \tau^s_{(j,i)}\right)$-regular closed sets of $\left(X, \tau^s_{(1,2)}, \tau^s_{(2,1)}\right)$;
(e) $\left(\tau^s_{(i,j)}\right)^s_{(i,j)} = \tau^s_{(i,j)}$.

3. **Pairwise Semiregularization of Generalized Pairwise Regular-Lindelöf Spaces**

Definition 3.1. *An i-open cover $\{U_\alpha : \alpha \in \Delta\}$ of a bitopological space X is said to be (i,j)-regular cover* [8, 10] *if for every $\alpha \in \Delta$, there exists a nonempty (j,i)-regular closed subset C_α of X such that $C_\alpha \subseteq U_\alpha$ and $X = \bigcup_{\alpha \in \Delta} i\text{-}int(C_\alpha)$. The $\{U_\alpha : \alpha \in \Delta\}$ is called pairwise regular cover if it is both $(1,2)$-regular cover and $(2,1)$-regular cover.*

Definition 3.2. *A bitopological space X is said to be (i,j)-nearly regular-Lindelöf (resp. (i,j)-almost regular-Lindelöf* [10], *(i,j)-weakly regular-Lindelöf* [8]) *if for every (i,j)-regular cover $\{U_\alpha : \alpha \in \Delta\}$ of X, there exists a countable subset $\{\alpha_n : n \in \mathbb{N}\}$ of Δ such that $X = \bigcup_{n \in \mathbb{N}} i\text{-}int(j\text{-}cl(U_{\alpha_n}))$ $\left(\text{resp. } X = \bigcup_{n \in \mathbb{N}} j\text{-}cl(U_{\alpha_n}),\ X = j\text{-}cl\left(\bigcup_{n \in \mathbb{N}} (U_{\alpha_n})\right)\right)$. X is called pairwise nearly regular-Lindelöf (resp. pairwise almost regular-Lindelöf, pairwise weakly regular-Lindelöf) if it is both $(1,2)$-nearly regular-Lindelöf (resp. $(1,2)$-almost regular-Lindelöf, $(1,2)$-weakly regular-Lindelöf) and $(2,1)$-nearly regular-Lindelöf (resp. $(2,1)$-almost regular-Lindelöf, $(2,1)$-weakly regular-Lindelöf).*

Suppose that $\{U_\alpha : \alpha \in \Delta\}$ is an (i,j)-regular cover of a bitopological space X. If for every $\alpha \in \Delta$, U_α is an (i,j)-regular open subset of X, then $\{U_\alpha : \alpha \in \Delta\}$ is called (i,j)-regular cover of X by (i,j)-regular open subsets of X. By using this concept, we have the following theorem for the (i,j)-nearly regular-Lindelöf spaces

Theorem 3.1. *A bitopological space X is (i,j)-nearly regular-Lindelöf if and only if every (i,j)-regular cover $\{U_\alpha : \alpha \in \Delta\}$ of X by (i,j)-regular open subsets of X has a countable subcover.*

Proof. Straightforward by the definitions. □

Corollary 3.2. *A bitopological space X is pairwise nearly regular-Lindelöf if and only if every (i,j)-regular cover $\{U_\alpha : \alpha \in \Delta\}$ of X by (i,j)-regular open subsets of X has a countable subcover for each $i,j \in \{1,2\}, i \neq j$.*

The following theorem and corollary proves that (i,j)-nearly regular-Lindelöf property as well as pairwise nearly regular-Lindelöf property is pairwise semiregular invariant property. We cannot say the (i,j)-nearly regular-Lindelöf property or pairwise nearly regular-Lindelöf property is pairwise semiregular property because we do not know yet whether the (i,j)-nearly regular-Lindelöf property and pairwise nearly regular-Lindelöf property is bitopological property or not.

Theorem 3.3. *Let (X,τ_1,τ_2) be a bitopological space. Then (X,τ_1,τ_2) is (τ_i,τ_j)-nearly regular-Lindelöf if and only if $\left(X,\tau^s_{(1,2)},\tau^s_{(2,1)}\right)$ is $\left(\tau^s_{(i,j)},\tau^s_{(j,i)}\right)$-nearly regular-Lindelöf.*

Proof. Let (X,τ_1,τ_2) be a (τ_i,τ_j)-nearly regular-Lindelöf and let $\{U_\alpha : \alpha\in\Delta\}$ be a $\left(\tau^s_{(i,j)},\tau^s_{(j,i)}\right)$ -regular cover of $\left(X,\tau^s_{(1,2)},\tau^s_{(2,1)}\right)$ by $\left(\tau^s_{(i,j)},\tau^s_{(j,i)}\right)$-regular open subsets of $\left(X,\tau^s_{(1,2)},\tau^s_{(2,1)}\right)$. By Lemma 2.4(c), $\{U_\alpha : \alpha\in\Delta\}$ is also a (τ_i,τ_j)-regular cover of (X,τ_1,τ_2) by (τ_i,τ_j)-regular open subsets of (X,τ_1,τ_2). Since (X,τ_1,τ_2) is (τ_i,τ_j) -nearly regular -Lindelöf, $\{U_\alpha : \alpha\in\Delta\}$ has a countable subcover. It follows by Lemma 2.4 and Theorem 3.1, $\left(X,\tau^s_{(1,2)},\tau^s_{(2,1)}\right)$ is $\left(\tau^s_{(i,j)},\tau^s_{(j,i)}\right)$- nearly regular -Lindelöf. Conversely suppose that $\left(X,\tau^s_{(1,2)},\tau^s_{(2,1)}\right)$ is $\left(\tau^s_{(i,j)},\tau^s_{(j,i)}\right)$- nearly regular- Lindelöf and let $\{V_\alpha : \alpha\in\Delta\}$ be a (τ_i,τ_j)-regular cover of (X,τ_1,τ_2) by (τ_i,τ_j)-regular open subsets of (X,τ_1,τ_2). Lemma 2.4(c) implies that $\{V_\alpha : \alpha\in\Delta\}$ is also a $\left(\tau^s_{(i,j)},\tau^s_{(j,i)}\right)$-regular cover of $\left(X,\tau^s_{(1,2)},\tau^s_{(2,1)}\right)$ by $\left(\tau^s_{(i,j)},\tau^s_{(j,i)}\right)$-regular open subsets of $\left(X,\tau^s_{(1,2)},\tau^s_{(2,1)}\right)$. Since $\left(X,\tau^s_{(1,2)},\tau^s_{(2,1)}\right)$ is $\left(\tau^s_{(i,j)},\tau^s_{(j,i)}\right)$-nearly regular-Lindelöf, $\{V_\alpha : \alpha\in\Delta\}$ has a countable subcover. It follows by Lemma 2.4(c) and Theorem 3.1, (X,τ_1,τ_2) is (τ_i,τ_j)-nearly regular-Lindelöf. □

Corollary 3.4. *Let (X,τ_1,τ_2) be a bitopological space. Then (X,τ_1,τ_2) is pairwise nearly regular-Lindelöf if and only if $\left(X,\tau^s_{(1,2)},\tau^s_{(2,1)}\right)$ is pairwise nearly regular-Lindelöf.*

Proposition 3.5. *Let (X,τ_1,τ_2) be a (τ_i,τ_j)-almost regular space. Then (X,τ_1,τ_2) is (τ_i,τ_j)-nearly regular-Lindelöf if and only if $\left(X,\tau^s_{(1,2)},\tau^s_{(2,1)}\right)$ is $\tau^s_{(i,j)}$-Lindelöf.*

Proof. Let (X,τ_1,τ_2) be a (τ_i,τ_j)-nearly regular-Lindelöf and let $\{U_\alpha : \alpha\in\Delta\}$ be a $\tau^s_{(i,j)}$-open cover of $\left(X,\tau^s_{(1,2)},\tau^s_{(2,1)}\right)$. For each $x\in X$, there exists $\alpha_x\in\Delta$ such that $x\in U_{\alpha_x}$ and since for each $\alpha_x\in\Delta, U_{\alpha_x}\in\tau^s_{(i,j)}$, there exists a (τ_i,τ_j)-regular open set V_{α_x} in (X,τ_1,τ_2) such that $x\in V_{\alpha_x}\subseteq U_{\alpha_x}$. Since (X,τ_1,τ_2) is (τ_i,τ_j)-almost regular, there is a (τ_i,τ_j)-regular open set C_{α_x} in (X,τ_1,τ_2) such that $x\in C_{\alpha_x}\subseteq\tau_j\text{-cl}\,(C_{\alpha_x})\subseteq V_{\alpha_x}$. Hence $X=\bigcup_{x\in X}C_{\alpha_x}\subseteq\bigcup_{x\in X}\tau_j\text{-cl}\,(C_{\alpha_x})\subseteq\bigcup_{x\in X}V_{\alpha_x}$ and since $\tau_j\text{-cl}\,(C_{\alpha_x})$ is a (τ_j,τ_i)-regular closed subset of X such that $X=\bigcup_{x\in X}\tau_i\text{-int}\,(\tau_j\text{-cl}\,(C_{\alpha_x}))$, the family $\{V_{\alpha_x} : x\in X\}$ forms a (τ_i,τ_j)-regular cover of (X,τ_1,τ_2) by (τ_i,τ_j)-regular open subsets of X. Since (X,τ_1,τ_2) is (τ_i,τ_j)-nearly regular-Lindelöf, there exists a countable subset of points $x_1,\ldots,x_n,\ldots$

of X such that $X = \bigcup_{n\in\mathbb{N}} V_{\alpha_{x_n}} \subseteq \bigcup_{n\in\mathbb{N}} U_{\alpha_{x_n}}$. This shows that $\left(X, \tau^s_{(1,2)}, \tau^s_{(2,1)}\right)$ is $\tau^s_{(i,j)}$-Lindelöf. Conversely, let $\left(X, \tau^s_{(1,2)}, \tau^s_{(2,1)}\right)$ be a $\tau^s_{(i,j)}$-Lindelöf and let $\{U_\alpha : \alpha \in \Delta\}$ be a (τ_i, τ_j)-regular cover of (X, τ_1, τ_2) by (τ_i, τ_j)-regular open subsets of (X, τ_1, τ_2). By Lemma 2.4(c), each U_α is also a $\left(\tau^s_{(i,j)}, \tau^s_{(j,i)}\right)$-regular open which is also $\tau^s_{(i,j)}$-open subsets of $\left(X, \tau^s_{(1,2)}, \tau^s_{(2,1)}\right)$. Since $\left(X, \tau^s_{(1,2)}, \tau^s_{(2,1)}\right)$ is $\tau^s_{(i,j)}$-Lindelöf, $\{U_\alpha : \alpha \in \Delta\}$ has a countable subcover. It follows by Lemma 2.4(c) and Theorem 3.1 that (X, τ_1, τ_2) is (τ_i, τ_j)-nearly regular-Lindelöf. □

Corollary 3.6. *Let (X, τ_1, τ_2) be a pairwise almost regular space. Then (X, τ_1, τ_2) is pairwise nearly regular-Lindelöf if and only if $\left(X, \tau^s_{(1,2)}, \tau^s_{(2,1)}\right)$ is Lindelöf.*

Theorem 3.7. *Let (X, τ_1, τ_2) be a bitopological space. Then (X, τ_1, τ_2) is (τ_i, τ_j)-almost regular-Lindelöf if and only if $\left(X, \tau^s_{(1,2)}, \tau^s_{(2,1)}\right)$ is $\left(\tau^s_{(i,j)}, \tau^s_{(j,i)}\right)$-almost regular-Lindelöf.*

Proof. The proof is similar to the proof of Theorem 3.3, thus we choose to omit the details. □

Corollary 3.8. *Let (X, τ_1, τ_2) be a bitopological space. Then (X, τ_1, τ_2) is pairwise almost regular-Lindelöf if and only if $\left(X, \tau^s_{(1,2)}, \tau^s_{(2,1)}\right)$ is pairwise almost regular-Lindelöf.*

Note that, the (i, j)-almost regular-Lindelöf property and the pairwise almost regular-Lindelöf property are bitopological properties (see [14, 15]). Utilizing this fact, Theorem 3.7 and Corollary 3.8, we easily obtain the following corollary.

Corollary 3.9. *The (i, j)-almost regular-Lindelöf property and the pairwise almost regular-Lindelöf property are pairwise semiregular properties.*

Theorem 3.10. *Let (X, τ_1, τ_2) be a bitopological space. Then (X, τ_1, τ_2) is (τ_i, τ_j)-weakly regular-Lindelöf if and only if $\left(X, \tau^s_{(1,2)}, \tau^s_{(2,1)}\right)$ is $\left(\tau^s_{(i,j)}, \tau^s_{(j,i)}\right)$-weakly regular-Lindelöf.*

Proof. The proof is quite similar to the proof of Theorem 3.3 by using the fact that

$$\begin{aligned}
\tau^s_{(j,i)}\text{-cl}\left(\bigcup_{n\in\mathbb{N}} \tau_i\text{-int}\left(\tau_j\text{-cl}\left(V_{\alpha_n}\right)\right)\right) &= \tau_j\text{-cl}\left(\bigcup_{n\in\mathbb{N}} \tau_i\text{-int}\left(\tau_j\text{-cl}\left(V_{\alpha_n}\right)\right)\right) \\
&\subseteq \tau_j\text{-cl}\left(\bigcup_{n\in\mathbb{N}} \tau_j\text{-cl}\left(V_{\alpha_n}\right)\right) \\
&\subseteq \tau_j\text{-cl}\left(\bigcup_{n\in\mathbb{N}} V_{\alpha_n}\right).
\end{aligned}$$

Thus we choose to omit the details. □

Corollary 3.11. *Let (X,τ_1,τ_2) be a bitopological space. Then (X,τ_1,τ_2) is pairwise weakly regular-Lindelöf if and only if $\left(X,\tau^s_{(1,2)},\tau^s_{(2,1)}\right)$ is pairwise weakly regular-Lindelöf.*

Note that, the (i,j)-weakly regular-Lindelöf property and the pairwise weakly regular-Lindelöf property are bitopological properties (see [14, 15]). Utilizing this fact, Theorem 3.10 and Corollary 3.11, we easily obtain the following corollary.

Corollary 3.12. *The (i,j)-weakly regular-Lindelöf property and the pairwise weakly regular-Lindelöf property are pairwise semiregular properties.*

Definition 3.3. *A bitopological space X is said to be (i,j)-almost Lindelöf* [7] *(resp. (i,j)-weakly Lindelöf* [9]*) if for every i-open cover $\{U_\alpha:\alpha\in\Delta\}$ of X, there exists a countable subset $\{\alpha_n:n\in\mathbb{N}\}$ of Δ such that*

$$X=\bigcup_{n\in\mathbb{N}} j\text{-}cl\,(U_{\alpha_n})\ \left(resp.\ X=j\text{-}cl\left(\bigcup_{n\in\mathbb{N}}(U_{\alpha_n})\right)\right).$$

X is called pairwise almost Lindelöf (resp. pairwise weakly Lindelöf) if it is both $(1,2)$-almost Lindelöf and $(2,1)$-almost Lindelöf (resp. $(1,2)$-weakly Lindelöf and $(2,1)$-weakly Lindelöf).

Proposition 3.13. *Let (X,τ_1,τ_2) be a (τ_i,τ_j)-almost regular space. Then:*
(i) (X,τ_1,τ_2) is (τ_i,τ_j)-almost regular-Lindelöf if and only if $\left(X,\tau^s_{(1,2)},\tau^s_{(2,1)}\right)$ is $\left(\tau^s_{(i,j)},\tau^s_{(j,i)}\right)$-almost Lindelöf.
(ii) (X,τ_1,τ_2) is (τ_i,τ_j)-weakly regular-Lindelöf if and only if $\left(X,\tau^s_{(1,2)},\tau^s_{(2,1)}\right)$ is $\left(\tau^s_{(i,j)},\tau^s_{(j,i)}\right)$-weakly Lindelöf.

Proof. The proof of each part is quite similar. We choose to prove only part (i). Let (X,τ_1,τ_2) be a (τ_i,τ_j)-almost regular-Lindelöf and let $\{U_\alpha:\alpha\in\Delta\}$ be a $\tau^s_{(i,j)}$-open cover of $\left(X,\tau^s_{(1,2)},\tau^s_{(2,1)}\right)$. Since $\tau^s_{(i,j)}\subseteq\tau_i$, $\{U_\alpha:\alpha\in\Delta\}$ is a τ_i-open cover of the (τ_i,τ_j)-almost regular-Lindelöf space (X,τ_1,τ_2). For each $x\in X$, there exists $\alpha_x\in\Delta$ such that $x\in U_{\alpha_x}$ and since for each $\alpha_x\in\Delta, U_{\alpha_x}\in\tau^s_{(i,j)}$, there exists a (τ_i,τ_j)-regular open set V_{α_x} in (X,τ_1,τ_2) such that $x\in V_{\alpha_x}\subseteq U_{\alpha_x}$. Since (X,τ_1,τ_2) is (τ_i,τ_j)-almost regular, there is a (τ_i,τ_j)-regular open set C_{α_x} in (X,τ_1,τ_2) such that $x\in C_{\alpha_x}\subseteq\tau_j\text{-cl}\,(C_{\alpha_x})\subseteq V_{\alpha_x}$. Since for each $\alpha_x\in\Delta$, there exists a (τ_j,τ_i)-regular closed set $\tau_j\text{-cl}\,(C_{\alpha_x})$ in (X,τ_1,τ_2) such that $\tau_j\text{-cl}\,(C_{\alpha_x})\subseteq V_{\alpha_x}$ and $X=\bigcup_{x\in X}C_{\alpha_x}=\bigcup_{x\in X}\tau_i\text{-int}\,(\tau_j\text{-cl}\,(C_{\alpha_x}))$, the family $\{V_{\alpha_x}:x\in X\}$ is a (τ_i,τ_j)-regular cover of (X,τ_1,τ_2). Hence there exists a countable subset of points $x_1,\ldots,x_n,\ldots$ of X such that $X=\bigcup_{n\in\mathbb{N}}\tau_j\text{-cl}\left(V_{\alpha_{x_n}}\right)$. By Lemma 2.4(b),

$X = \bigcup_{n\in\mathbb{N}} \tau^s_{(j,i)}\text{-cl}\left(V_{\alpha_{x_n}}\right) \subseteq \bigcup_{n\in\mathbb{N}} \tau^s_{(j,i)}\text{-cl}\left(U_{\alpha_{x_n}}\right)$. This shows that $\left(X, \tau^s_{(1,2)}, \tau^s_{(2,1)}\right)$ is $\left(\tau^s_{(i,j)}, \tau^s_{(j,i)}\right)$-almost Lindelöf. Conversely, let $\left(X, \tau^s_{(1,2)}, \tau^s_{(2,1)}\right)$ be a $\left(\tau^s_{(i,j)}, \tau^s_{(j,i)}\right)$-almost Lindelöf and let $\{U_\alpha : \alpha \in \Delta\}$ be a (τ_i, τ_j)-regular cover of (X, τ_1, τ_2). Since $U_\alpha \subseteq \tau_i$-int $(\tau_j\text{-cl}\,(U_\alpha))$ and $\tau_i\text{-int}\,(\tau_j\text{-cl}\,(U_\alpha)) \in \tau^s_{(i,j)}$, $\{\tau_i\text{-int}\,(\tau_j\text{-cl}\,(U_\alpha)) : \alpha \in \Delta\}$ is $\tau^s_{(i,j)}$-open cover of the $\left(\tau^s_{(i,j)}, \tau^s_{(j,i)}\right)$-almost Lindelöf space $\left(X, \tau^s_{(1,2)}, \tau^s_{(2,1)}\right)$. Then there exists a countable subset $\{\alpha_n : n \in \mathbb{N}\}$ of Δ such that $X = \bigcup_{n\in\mathbb{N}} \tau^s_{(j,i)}\text{-cl}\,(\tau_i\text{-int}\,(\tau_j\text{-cl}\,(U_{\alpha_n})))$. By Lemma 2.4(b), we have $X = \bigcup_{n\in\mathbb{N}} \tau_j\text{-cl}\,(\tau_i\text{-int}\,(\tau_j\text{-cl}\,(U_{\alpha_n}))) \subseteq \bigcup_{n\in\mathbb{N}} \tau_j\text{-cl}\,(U_{\alpha_n})$. This implies that (X, τ_1, τ_2) is (τ_i, τ_j)-almost regular-Lindelöf. □

Corollary 3.14. *Let (X, τ_1, τ_2) be a pairwise almost regular space. Then*:
(i) (X, τ_1, τ_2) is pairwise almost regular-Lindelöf if and only if $\left(X, \tau^s_{(1,2)}, \tau^s_{(2,1)}\right)$ is pairwise almost Lindelöf.
(ii) (X, τ_1, τ_2) is pairwise weakly regular-Lindelöf if and only if $\left(X, \tau^s_{(1,2)}, \tau^s_{(2,1)}\right)$ is pairwise weakly Lindelöf.

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