İstanbul Üniv. Fen Fak. Mat Der. **59** (2000), 1-15

A Unified Theory for S-Continuity of Multifunctions

Valeriu POPA and Takashi NOIRI

Abstract

In this paper we introduce upper/lower s-m-continuous multifunctions as multifunctions defined on a set satisfying some minimal conditions. We obtain some characterizations and several properties of such multifunctions unifying some results established in [8], [9], [12], [20], [22] and [23].

Key words: m-structure, s-m-continuous, multifunction.
AMS(2000) Math. Subject Classification: 54C08, 54C60.

1 Introduction

Semi-open sets, preopen sets, α-open sets, δ-open sets and β-open sets play an important role in the research of generalizations of continuity in topological spaces. By using these sets many authors introduced and investigated various types of non-continuous functions and multifunctions. In 1978, Kohli [9] defined a function $f : X \to Y$ to be s-continuous if for each point $x \in X$ and each open set V of Y containing $f(x)$ and having connected complement, there exists an open set U of X containing x such that $f(U) \subset V$. In 1989, Lipski [12] extended this notion in the setting of multifunctions. By replacing an open set of X with semi-open (resp. preopen, β-open) sets, Ewert and Lipski [8] (resp. Popa and Noiri [22], [23]) defined and investigated upper/lower s-quasi-continuous (resp. upper/lower s-precontinuous, upper/lower s-β-continuous) multifunctions. The analogy among their definitions and results suggests the need of formulating a unified theory of these multifunctions.

In this paper, we introduce upper/lower s-m-continuous multifunctions as multifunctions defined on a set satisfying some minimal conditions. We obtain some characterizations and several properties of such multifunctions unifying some results established in [8], [9], [12], [20], [22] and [23]. In the last section, we recall some types of modifications of open sets and point out the possibility for new forms of s-continuous multifunctions.

2 Preliminaries

Let X be a topological space and A a subset of X. The closure of A and the interior of A are denoted by $\mathrm{Cl}(A)$ and $\mathrm{Int}(A)$, respectively. A subset A is said to be *β-open* [1] or*semi-preopen* [3] (resp. *semi-open* [11], *α-open* [17], *preopen* [14]) if $A \subset \mathrm{Cl}(\mathrm{Int}(\mathrm{Cl}(A)))$ (resp. $A \subset \mathrm{Cl}(\mathrm{Int}(A))$, $A \subset \mathrm{Int}(\mathrm{Cl}(\mathrm{Int}(A)))$, $A \subset \mathrm{Int}(\mathrm{Cl}(A))$). The family of all semi-preopen (resp. preopen, semi-open) sets in X is denoted by $\mathrm{SPO}(X)$ (resp. $\mathrm{PO}(X)$, $\mathrm{SO}(X)$). The complement of a semi-preopen or β-open (resp. semi-open, α-open, preopen) set is said to be *semi-preclosed* [3] or *β-closed* [2] (resp. *semi-closed* [5], *α-closed* [15], *preclosed* [7]). The intersection of all semi-preclosed sets of X containing A is called the *semi-preclosure* [3] or *β-closure* [2] of A and is denoted by $\mathrm{spCl}(A)$ or $\beta\mathrm{Cl}(A)$. Similarly, $\mathrm{sCl}(A)$, $\mathrm{pCl}(A)$ and $\alpha\mathrm{Cl}(A)$ are defined. The union of all semi-preopen sets of X contained in A is called the *semi-preinterior* or *β-interior* of A and is denoted by $\mathrm{spInt}(A)$ or $\beta\mathrm{Int}(A)$. Similarly, $\mathrm{sInt}(A)$, $\mathrm{pInt}(A)$ and $\alpha\mathrm{Int}(A)$ are defined.

Throughout this paper, spaces X and Y always mean topological spaces and $F : X \to Y$ (resp. $f : X \to Y$) presents a multivalued (resp. single valued) function. For a multifunction $F : X \to Y$, we shall denote the upper and lower inverse of a subset B of a space Y by $F^+(B)$ and $F^-(B)$, respectively, that is,

$$F^+(B) = \{x \in X : F(x) \subset B\} \text{ and } F^-(B) = \{x \in X : F(x) \cap B \neq \emptyset\}.$$

Definition 2.1 A multifunction $F : X \to Y$ is said to be

(1) *upper s-continuous* [12] (resp. *upper s-quasi-continuous* [8], *upper s-precontinuous* [22], *upper s-β-continuous* [23]) at a point $x \in X$ if for each open set V containing $F(x)$ and having connected complement, there exists an open (resp. semi-open, preopen, β-open) set $U \subset X$ containing x such that $F(U) \subset V$,

(2) *lower s-continuous* [12] (resp. *lower s-quasi-continuous* [8], *lower s-precontinuous* [22], *lower s-β-continuous* [23]) at a point $x \in X$ if for each open set V of Y meeting $F(x)$ and having connected complement , there exists an open (resp. semi-open, preopen, β-open) set $U \subset X$ containing x

such that $F(u) \cap V \neq \emptyset$ for each $u \in U$,

(3) *upper (lower) s-continuous* (resp. *upper (lower) s-quasi-continuous. upper (lower) s-precontinuous, upper (lower) s-β-continuous*) in X if it has this property at every point of X.

3 s-m-continuous multifunctions

Definition 3.1 A subfamily m_X of the power set $\mathcal{P}(X)$ of a nonempty set X is called a *minimal structure* (briefly *m-structure*) on X if $\emptyset \in m_X$ and $X \in m_X$. Each member of m_X is said to be *m_X-open* and the complement of a m_X-open set is said to be *m_X-closed*.

Remark 3.1 Let (X, τ) be a topological space. Then the families τ, $\mathrm{SO}(X)$, $\mathrm{PO}(X)$, $\alpha(X)$, $\mathrm{SPO}(X)$ are all m-structures on X.

Definition 3.2 Let X be a nonempty set and m_X a m-structure on X. For a subset A of X, the *m_X-closure* of A and the *m_X-interior* of A are defined in [**13**] as follows:

(1) $m_X\text{-Cl}(A) = \cap\{F : A \subset F, X - F \in m_X\}$,

(2) $m_X\text{-Int}(A) = \cup\{U : U \subset A, U \in m_X\}$.

Remark 3.2 Let (X, τ) be a topological space and A a subset of X. If $m_X = \tau$ (resp. $\mathrm{SO}(X)$, $\mathrm{PO}(X)$, $\alpha(X)$, $\mathrm{SPO}(X)$), then we have

(1) $m_X\text{-Cl}(A) = \mathrm{Cl}(A)$ (resp. $\mathrm{sCl}(A)$, $\mathrm{pCl}(A)$, $\alpha\mathrm{Cl}(A)$, $\mathrm{spCl}(A)$),

(2) $m_X\text{-Int}(A) = \mathrm{Int}(A)$ (resp. $\mathrm{sInt}(A)$, $\mathrm{pInt}(A)$, $\alpha\mathrm{Int}(A)$, $\mathrm{spInt}(A)$).

Lemma 3.1 (Maki [13]). *Let X be a nonempty set and m_X a m-structure on X. For subsets A and B of X, the following hold:*

(1) $m_X\text{-Cl}(X - A) = X - (m_X\text{-Int}(A))$ *and* $m_X\text{-Int}(X - A) = X - (m_X\text{-Cl}(A))$,

(2) If $(X - A) \in m_X$, *then* $m_X\text{-Cl}(A) = A$ *and if* $A \in m_X$, *then* $m_X\text{-Int}(A) = A$,

(3) $m_X\text{-Cl}(\emptyset) = \emptyset$, $m_X\text{-Cl}(X) = X$, $m_X\text{-Int}(\emptyset) = \emptyset$ *and* $m_X\text{-Int}(X) = X$,

(4) If $A \subset B$, *then* $m_X\text{-Cl}(A) \subset m_X\text{-Cl}(B)$ *and* $m_X\text{-Int}(A) \subset m_X\text{-Int}(B)$,

(5) $A \subset m_X\text{-Cl}(A)$ *and* $m_X\text{-Int}(A) \subset A$,

(6) $m_X\text{-Cl}(m_X\text{-Cl}(A)) = m_X\text{-Cl}(A)$ *and* $m_X\text{-Int}(m_X\text{-Int}(A)) = m_X\text{-Int}(A)$.

Lemma 3.2 *Let X be a nonempty set with a minimal structure m_X and A a subset of X. Then $x \in m_X$-Cl(A) if and only if $U \cap A \neq \emptyset$ for every $U \in m_X$ containing x.*

Proof. Necessity. Suppose that there exists $U \in m_X$ containing x such that $U \cap A = \emptyset$. Then $A \subset X - U$ and $X - (X - U) = U \in m_X$. Then m_X-Cl$(A) \subset X - U$. Since $x \in U$, we have $x \notin m_X$-Cl(A).

Sufficiency. Suppose that $x \notin m_X$-Cl(A). There exists a subset F of X such that $X - F \in m_X$, $A \subset F$ and $x \notin F$. Thus there exists $(X - F) \in m_X$ containing x such that $(X - F) \cap A = \emptyset$.

Definition 3.3 A minimal structure m_X on a nonempty set X is said to have *property* $(\mathcal{B})$ [13] if the union of any family of subsets belong to m_X belongs to m_X.

Lemma 3.3 (Popa and Noiri [24]). *For a minimal structure m_X on a nonempty set X, the following are equivalent:*

(1) m_X has property $(\mathcal{B})$;

(2) If m_X-Int$(V) = V$, then $V \in m_X$;

(3) If m_X-Cl$(F) = F$, then $X - F \in m_X$.

Lemma 3.4 *Let X be a nonempty set and m_X a minimal structure on X satisfying $(\mathcal{B})$. For a subset A of X, the following properties hold:*

(1) $A \in m_X$ if and only if m_X-Int$(A) = A$,

(2) A is m_X-closed if and only if m_X-Cl$(A) = A$,

(3) m_X-Int$(A) \in m_X$ and m_X-Cl(A) is m_X-closed.

Proof. This follows immediately from Lemmas 3.1 and 3.3.

Definition 3.4 Let (X, m_X) be a nonempty set X with a minimal structure m_X and (Y, σ) a topological space. A multifunction $F : (X, m_X) \to (Y, \sigma)$ is said to be

(1) *upper s-m-continuous* at $x \in X$ if for each $V \in \sigma$ containing $F(x)$ and having connected complement, there exists $U \in m_X$ containing x such that $F(U) \subset V$,

(2) *lower s-m-continuous* at $x \in X$ if for each $V \in \sigma$ meeting $F(x)$ and having connected complement, there exists $U \in m_X$ containing x such that $F(u) \cap V \neq \emptyset$ for each $u \in U$,

(3) *upper/lower s-m-continuous* if it has this property at each point x of X.

Theorem 3.1 *For a multifunction $F : (X, m_X) \to (Y, \sigma)$, the following are equivalent:*
(1) F is upper s-m-continuous;
(2) $F^+(V) = m_X\text{-Int}(F^+(V))$ for each open set V of Y having connected complement;
(3) $F^-(K) = m_X\text{-Cl}(F^-(K))$ for every connected closed set K of Y;
(4) $m_X\text{-Cl}(F^-(B)) \subset F^-(\text{Cl}(B))$ for every subset B of Y having the connected closure;
(5) $F^+(\text{Int}(B)) \subset m_X\text{-Int}(F^+(B))$ for every subset B of Y such that $Y - \text{Int}(B)$ is connected.

Proof. (1) $\Rightarrow$ (2): Let V be any open set of Y having connected complement and $x \in F^+(V)$. There exists $U \in m_X$ containing x such that $F(U) \subset V$. Therefore, we have $x \in U \subset F^+(V)$ and hence $x \in m_X\text{-Int}(F^+(V))$. This shows that $F^+(V) \subset m_X\text{-Int}(F^+(V))$. By Lemma 3.1, we have $m_X\text{-Int}(F^+(V)) \subset F^+(V)$. Therefore, we obtain $F^+(V) = m_X\text{-Int}(F^+(V))$.

(2) $\Rightarrow$ (3): Let K be any connected closed set of Y. Then by Lemma 3.1, we have $X - F^-(K) = F^+(Y - K) = m_X\text{-Int}(F^+(Y - K)) = m_X\text{-Int}(X - F^-(K)) = X - m_X\text{-Cl}(F^-(K))$. Therefore, we obtain $F^-(K) = m_X\text{-Cl}(F^-(K))$.

(3) $\Rightarrow$ (4): Let B be a subset of Y having the connected closure. By Lemma 3.1, we have $F^-(B) \subset F^-(\text{Cl}(B)) = m_X\text{-Cl}(F^-(\text{Cl}(B)))$ and $m_X\text{-Cl}(F^-(B)) \subset F^-(\text{Cl}(B))$.

(4) $\Rightarrow$ (5): Let B be a subset of Y such that $Y - \text{Int}(B)$ is connected. Then by Lemma 3.1 we have

$$X - m_X\text{-Int}(F^+(B)) = m_X\text{-Cl}(X - F^+(B))$$
$$= m_X\text{-Cl}(F^-(Y - B)) \subset F^-(Y - \text{Int}(B)) \subset X - F^+(\text{Int}(B)).$$

Therefore, we obtain $F^+(\text{Int}(B)) \subset m_X\text{-Int}(F^+(B))$.

(5) $\Rightarrow$ (1): Let $x \in X$ and V be any open set of Y containing $F(x)$ and having connected complement. Then $x \in F^+(V) = F^+(\text{Int}(V)) \subset m_X\text{-Int}(F^+(V))$. There exists $U \in m_X$ containing x such that $U \subset F^+(V)$; hence $F(U) \subset V$. This shows that F is upper s-m-continuous.

Theorem 3.2 *For a multifunction $F : (X, m_X) \to (Y, \sigma)$, the following are equivalent:*
(1) F is lower s-m-continuous;
(2) $F^-(V) = m_X\text{-Int}(F^-(V))$ for each open set V of Y having connected complement;

(3) $F^+(K) = m_X\text{-Cl}(F^+(K))$ *for every connected closed set* K *of* Y*;*
(4) $m_X\text{-Cl}(F^+(B)) \subset F^+(\text{Cl}(B))$ *for every subset* B *of* Y *having the connected closure;*
(5) $F^-(\text{Int}(B)) \subset m_X\text{-Int}(F^-(B))$ *for every subset* B *of* Y *such that* $Y - \text{Int}(B)$ *is connected.*

Proof. The proof is similar to that of Theorem 3.1.

Corollary 3.1 *Let* (X, m_X) *be a nonempty set* X *with a minimal structure* m_X *satisfying* $\mathcal{B}$ *and* (Y, σ) *a topological space. For a multifunction* $F : (X, m_X) \to (Y, \sigma)$*, the following are equivalent:*
(1) F *is upper/lower s-m-continuous;*
(2) $F^+(V)/F^-(V)$ *is* m_X*-open for each open set* V *of* Y *having connected complement;*
(3) $F^-(K)/F^+(K)$ *is* m_X*-closed for every connected closed set* K *of* Y*.*

Proof. This follows from Theorems 3.1 and 3.2 and Lemma 3.4.

Remark 3.3 Let $m_X = \tau$ (resp. SO(X), PO(X), SPO(X)). Then an upper/lower s-m-continuous multifunction $F : (X, m_X) \to (Y, \sigma)$ is upper/lower s-continuous (resp. upper/lower s-quasi-continuous, upper/lower s-precontinuous, upper/lower s-β-continuous). Theorems 3.1 and 3.2 establish their characterizations which are obtained in [12] (resp. [8], [22], [23]).

Corollary 3.2 *Let* $F : (X, m_X) \to (Y, \sigma)$ *be a multifunction. If for every connected set* G *of* Y $F^-(G) = m_X\text{-Cl}(F^-(G))$ *(resp.* $F^+(G) = m_X\text{-Cl}(F^+(G))$*), then* F *is upper s-m-continuous (resp. lower s-m-continuous).*

Proof. Let G be any open set of Y having connected complement. Then $Y - G$ is connected and closed. By the hypothesis $X - F^+(G) = F^-(Y - G) = m_X\text{-Cl}(F^-(Y - G)) = m_X\text{-Cl}(X - F^+(G)) = X - m_X\text{-Int}(F^+(G))$. Therefore, we have $F^+(G) = m_X\text{-Int}(F^+(G))$. By Theorem 3.1, F is upper s-m-continuous. The proof for lower s-m-continuity is entirely similar.

Remark 3.4 Let $m_X = $ PO(X) (resp. SPO(X)). Then, Corollary 3.2 establishes the results which are obtained in [22] (resp. [23]).

Definition 3.5 A function $f : (X, m_X) \to (Y, \sigma)$ is said to be *s-m-continuous* if for each point $x \in X$ and each open set V containing $f(x)$ and having connected complement, there exists $U \in m_X$ containing x such that $f(U) \subset V$.

Corollary 3.3 *For a function $f : (X, m_X) \to (Y, \sigma)$, the following are equivalent:*
(1) f is s-m-continuous;
(2) $f^{-1}(V) = m_X\text{-Int}(f^{-1}(V))$ for each open set V of Y having connected complement;
(3) $f^{-1}(K) = m_X\text{-Cl}(f^{-1}(K))$ for every connected closed set K of Y;
(4) $m_X\text{-Cl}(f^{-1}(B)) \subset f^{-1}(\text{Cl}(B))$ for every subset B of Y having the connected closure;
(5) $f^{-1}(\text{Int}(B)) \subset m_X\text{-Int}(f^{-1}(B))$ for every subset B of Y such that $Y - \text{Int}(B)$ is connected.

Remark 3.5 Let $m_X = \tau$ (resp. SO(X), PO(X), SPO(X)). Then a *s-m*-continuous function $f : (X, m_X) \to (Y, \sigma)$ is s-continuous (resp. s-quasi-continuous, s-precontinuous, s-β-continuous). Corollary 3.3 establishes the characterizations of s-continuity (resp. s-precontinuity, s-β-continuity) which are obtained in [9] (resp. [22], [23]).

Definition 3.6 A subset A of a topological space (X, τ) is said to be
(1) *α-paracompact* [10] if every cover of A by open sets of X is refined by a cover of A which consists of open sets of X and is locally finite in X,
(2) *α-regular* [27] if for each $a \in A$ and each open set U of X containing a, there exists an open set G of X such that $a \in G \subset \text{Cl}(G) \subset U$.

Lemma 3.5 (Kovačević [10]). *If A is an α-regular α-paracompact set of a space X and U is an open neighborhood of A, then there exists an open set G of X such that $A \subset G \subset \text{Cl}(G) \subset U$.*

For a multifunction $F : (X, m_X) \to (Y, \sigma)$, we define a multifunction $\text{Cl}F : (X, m_X) \to (Y, \sigma)$ as follows: $(\text{Cl}F)(x) = \text{Cl}(F(x))$ for each point $x \in X$. Similarly, we can define $\alpha\text{Cl}F$, $\text{sCl}F$, $\text{pCl}F$, $\text{spCl}F$.

Lemma 3.6 *If $F : (X, m_X) \to (Y, \sigma)$ is a multifunction such that $F(x)$ is α-paracompact α-regular for each $x \in X$, then for each open set V of Y $F^+(V) = G^+(V)$, where G denotes $\alpha\text{O1}F$, $\text{sCl}F$, $\text{pCl}F$, $\text{spCl}F$ or $\text{Cl}F$.*

Proof. The proof is similar to that of Lemma 3.3 in [21].

Theorem 3.3 *Let $F : (X, m_X) \to (Y, \sigma)$ be a multifunction such that $F(x)$ is α-regular α-paracompact for each $x \in X$. Then the following are equivalent:*

(1) F is upper s-m-continuous;
(2) $\mathrm{Cl}F$ is upper s-m-continuous;
(3) $\alpha\mathrm{Cl}F$ is upper s-m-continuous;
(4) $\mathrm{sCl}F$ is upper s-m-continuous;
(5) $\mathrm{pCl}F$ is upper s-m-continuous;
(6) $\mathrm{spCl}F$ is upper s-m-continuous.

Proof. We set $G = \alpha\mathrm{ClF}$, sClF, pClF, spClF or ClF. Suppose that F is upper s-m-continuous. Let V be any open set of Y containing $G(x)$ and having connected complement. By Lemma 3.6, we have $G^+(V) = F^+(V)$ and hence there exists $U \in m_X$ containing x such that $F(U) \subset V$. Since $F(u)$ is α-paracompact and α-regular for each $u \in U$, by Lemma 3.5 there exists an open set H such that $F(u) \subset H \subset \mathrm{Cl}(H) \subset V$; hence $G(u) \subset \mathrm{Cl}(H) \subset V$ for every $u \in U$. Therefore, we obtain $G(U) \subset V$. This shows that G is upper s-m-continuous.

Conversely, suppose that G is upper s-m-continuous. Let $x \in X$ and V be any open set of Y containing $F(x)$ and having connected complement. By Lemma 3.6, we have $x \in F^+(V) = G^+(V)$ and hence $G(x) \subset V$. There exists $U \in m_X$ containing x such that $G(U) \subset V$. Therefore, we obtain $U \subset G^+(V) = F^+(V)$ and hence $F(U) \subset V$. This shows that F is upper s-m-continuous.

Lemma 3.7 *If $F : (X, m_X) \to (Y, \sigma)$ is a multifunction, then for each open set V of Y $G^-(V) = F^-(V)$, where $G = \alpha\mathrm{Cl}F$, $\mathrm{sCl}F$, $\mathrm{pCl}F$, $\mathrm{spCl}F$ or $\mathrm{Cl}F$.*

Proof. The proof is similar to that of Lemma 3.4 in [21].

Theorem 3.4 *For a multifunction $F : (X, m_X) \to (Y, \sigma)$, the following are equivalent:*
(1) F is lower s-m-continuous;
(2) $\mathrm{Cl}F$ is lower s-m-continuous;
(3) $\alpha\mathrm{Cl}F$ is lower s-m-continuous;
(4) $\mathrm{sCl}F$ is lower s-m-continuous;
(5) $\mathrm{pCl}F$ is lower s-m-continuous;
(6) $\mathrm{spCl}F$ is lower s-m-continuous.

Proof. By using Lemma 3.7 this is shown similarly as in Theorem 3.3.

Remark 3.6 Let $m_X = \mathrm{SO}(X)$ (resp. $\mathrm{PO}(X)$, $\mathrm{SPO}(X)$). Then, Theorem 3.4 establishes the results which are obtained in [20] (resp. [22], [23]).

4 Some properties

For a multifunction $F : (X, m_X) \to (Y, \sigma)$, the graph $G(F) = \{(x, F(x)) : x \in X\}$ is said to be *strongly m-closed* if for each $(x, y) \in (X \times Y) - G(F)$, there exist $U \in m_X$ containing x and an open set V of Y containing y such that $[U \times \mathrm{Cl}(V)] \cap G(F) = \emptyset$.

Lemma 4.1 *A multifunction $F : (X, m_X) \to (Y, \sigma)$ has a strongly m-closed graph if and only if for each $(x, y) \in (X \times Y) - G(F)$, there exist $U \in m_X$ containing x and an open set V of Y containing y such that $F(U) \cap \mathrm{Cl}(V) = \emptyset$.*

Proof. This proof is obvious.

Theorem 4.1 *Let (Y, σ) be a regular locally connected space. If $F : (X, m_X) \to (Y, \sigma)$ is an upper s-m-continuous multifunction such that $F(x)$ is closed for each $x \in X$, then $G(F)$ is strongly m-closed.*

Proof Let $(x, y) \in (X \times Y) - G(F)$, then $y \in Y - F(x)$. Since Y is regular, there exist disjoint open sets V_1 and V_2 of Y such that $F(x) \subset V_1$ and $y \in V_2$. Moreover, since Y is locally connected, there exists an open connected set V such that $y \in V \subset \mathrm{Cl}(V) \subset V_2$. Since F is upper *s-m*-continuous and $Y - \mathrm{Cl}(V)$ is an open set having connected complement, there exists $U \in m_X$ containing x such that $F(U) \subset Y - \mathrm{Cl}(V)$. Therefore, we have $F(U) \cap \mathrm{Cl}(V) = \emptyset$ and by Lemma 4.1 $G(F)$ is strongly m-closed.

Remark 4.1 Let $m_X = \mathrm{SO}(X)$ (resp. $\mathrm{PO}(X)$, $\mathrm{SPO}(X)$). Then, Theorem 4.1 establishes the results which are obtained in [20] (resp. [22], [23]).

Let X be a nonempty set with a minimal structure m_X and A a subset of X. The *m-frontier* of A, denoted by $\mathrm{mFr}(A)$, is defined by $\mathrm{mFr}(A) = \mathrm{m}_X\text{-}\mathrm{Cl}(A) \cap \mathrm{m}_X\text{-}\mathrm{Cl}(X - A) = \mathrm{m}_X\text{-}\mathrm{Cl}(A) - m_X\text{-}\mathrm{Int}(A)$.

Theorem 4.2 *The set of all points x of X at which a multifunction $F : (X, m_X) \to (Y, \sigma)$ is not upper s-m-continuous (resp. lower s-m-continuous) is identical with the union of the m-frontiers of the upper inverse (resp. lower inverse) images of open sets containing (resp. meeting) $F(x)$ and having connected complement.*

Proof. Let x be a point of X at which F is not upper s-m-continuous. Then, there exists an open set V of Y containing $F(x)$ and having connected complement such that $U \cap (X - F^+(V)) \neq \emptyset$ for every $U \in m_X$ containing x. Therefore, we have $x \in m_X\text{-Cl}(X - F^+(V))$ and hence $x \in \text{mFr}(F^+(V))$ since $x \in F^+(V) \subset m_X\text{-Cl}(F^+(V))$.

Conversely, suppose that V is an open set of Y containing $F(x)$ and having connected complement such that $x \in \text{mFr}(F^+(V))$. If F is upper s-m-continuous at x, then there exists $U \in m_X$ containing x such that $U \subset F^+(V)$; hence $x \in m_X\text{-Int}(F^+(V))$. This is a contradiction and hence F is not upper s-m-continuous at x.

The proof for lower s-m-continuity is similar.

Definition 4.1 A multifunction $F : (X, m_X) \to (Y, \sigma)$ is said to be *upper s-m-rarely continuous* at a point x of X if for each open set G of Y containing $F(x)$ and having connected complement, there exist a rare set R_G with $\text{Cl}(R_G) \cap G = \emptyset$ and a m_X-open set U containing x such that $F(U) \subset G \cup R_G$. A multifunction F is said to be *upper s-m-rarely continuous* if it has this property at each point of X.

Theorem 4.3 *Let X be a nonempty set with two minimal structures m_X^0 and m_X^1 such that $U \cap V \in m_X^1$ whenever $U \in m_X^0$ and $V \in m_X^1$. If a multifunction $F : X \to (Y, \sigma)$ satisfies the following two conditions:*

(1) $F : (X, m_X^0) \to (Y, \sigma)$ is upper s-m-rarely continuous and

(2) for each open set G containing $F(x)$ and having connected complement, $F^-(\text{Cl}(R_G))$ is a m_X^1-closed set of X, where R_G is the rare set of Definition 4.1,

then $F : (X, m_X^1) \to (Y, \sigma)$ is upper s-m-continuous.

Proof. Let $x \in X$ and G be an open set of Y containing $F(x)$ and having connected complement. Since $F : (X, m_X^0) \to (Y, \sigma)$ is upper s-m-rarely continuous, there exist $V \in m_X^0$ containing x and a rare set R_G with $\text{Cl}(R_G) \cap G = \emptyset$ such that $F(V) \subset G \cup R_G$. If we suppose that $x \in F^-(\text{Cl}(R_G))$, then $F(x) \cap \text{Cl}(R_G) \neq \emptyset$, but $F(x) \subset G$ and $G \cap \text{Cl}(R_G) = \emptyset$. This is a contradiction. Thus $x \notin F^-(\text{Cl}(R_G))$. Let $U = V \cap (X - F^-(\text{Cl}(R_G)))$. Then $U \in m_X^1$ and $x \in U$ since $x \in V$ and $x \in X - F^-(\text{Cl}(R_G))$. Let $s \in U$, then $F(s) \subset G \cup R_G$ and $F(s) \cap \text{Cl}(R_G) = \emptyset$. Therefore, we have $F(s) \cap R_G = \emptyset$ and hence $F(s) \subset G$. Since $U \in m_X^1$ containing x, it follows that $F : (X, m_X^1) \to (Y, \sigma)$ is upper s-m-continuous.

Remark 4.2 Let $m_X = \text{SPO}(X)$. Then, Theorem 4.3 establishes the result which is obtained in [23].

Definition 4.2 A multifunction $F : (X, m_X) \to (Y, \sigma)$ is said to be *lower m-continuous* if for each $x \in X$ and every open set V of Y meeting $F(x)$ there exists $U \in m_X$ containing x such that $F(u) \cap V \neq \emptyset$ for every $u \in U$.

Lemma 4.2 *A multifunction $F : (X, m_X) \to (Y, \sigma)$ is lower m-continuous if and only if $F(m_X\text{-Cl}(A)) \subset \text{Cl}(F(A))$ for every subset A of X.*

Proof. This follows from Theorem 3.2 of [18].

Theorem 4.4 *If $F : (X, m_X) \to (Y, \sigma)$ is lower s-m-continuous and $F(A)$ is connected for every subset A of X, then F is lower m-continuous.*

Proof. Let A be any subset of X. Since $\text{Cl}(F(A))$ is closed and connected, by Theorem 3.2 $F^+(\text{Cl}(F(A))) = m_X\text{-Cl}(F^+(\text{Cl}(F(A))))$ and $A \subset F^+(F(A))) \subset F^+(\text{Cl}(F(A)))$. Thus we have $F(m_X\text{-Cl}(A)) \subset \text{Cl}(F(A))$. It follows from Lemma 4.2 that F is lower m-continuous.

Remark 4.3 If $m_X = \text{SO}(X)$ (resp. $\text{PO}(X)$, $\text{SPO}(X)$), then Theorem 4.4 establishes the results which are obtained in [20] (resp. [22], [23]).

5 New forms of s-continuity in topological spaces

There are many modifications of open sets in topological spaces. We shall recall the main ones. Let (X, τ) be a topological space and A a subset of X. A subset A is said to be *regular closed* (resp. *regular open*) if $\text{Cl}(\text{Int}(A)) = A$ (resp. $\text{Int}(\text{Cl}(A)) = A$).

Definition 5.1 A subset A of a topological space (X, τ) is said to be

(1) *θ-open* [26] if for each $x \in A$ there exists an open set U of X such that $x \in U \subset \text{Cl}(U) \subset A$,

(2) *δ-open* [26] if for each $x \in A$ there exists a regular open set U of X such that $x \in U \subset A$,

(3) *b-open* [4] if $A \subset \text{Int}(\text{Cl}(A)) \cup \text{Cl}(\text{Int}(A))$.

Definition 5.2 A subset A of a topological space (X, τ) is said to be

(1) *semi-θ-open* [6] if for each $x \in A$ there exists a semi-open set U of X such that $x \in U \subset \text{sCl}(U) \subset A$,

(2) *semi-regular* [6] if it is semi-open and semi-closed.

A point $x \in X$ is called a δ-*cluster* point of A if $\mathrm{Int}(\mathrm{Cl}(V)) \cap A \neq \emptyset$ for every open set V containing x. The set of all δ-cluster points of A is called the δ-*closure* of A and is denoted by ${}_\delta\mathrm{Cl}(A)$. The set $\{x \in X : x \in U \subset A$ for some regular open set U of $X\}$ is called the δ-*interior* of A and is denoted by ${}_\delta\mathrm{Int}(A)$.

Definition 5.3 A subset A of a topological space (X, τ) is said to be

(1) δ-*preopen* [25] if $A \subset \mathrm{Int}({}_\delta\mathrm{Cl}(A))$,

(2) δ-*semi-open* [19] if $A \subset \mathrm{Cl}({}_\delta\mathrm{Int}(A))$.

The family of all θ-open (resp. δ-open, b-open, semi-θ-open, semi-regular, δ-preopen, δ-semi-open) sets in a topological space X is denoted by $\theta\mathrm{O}(X)$ (resp. $\delta\mathrm{O}(X)$, $\mathrm{BO}(X)$, $\mathrm{S}\theta\mathrm{O}(X)$, $\mathrm{SR}(X)$, $\delta\mathrm{PO}(X)$, $\delta\mathrm{SO}(X)$). These families have the property of the minimal structure. Moreover, they have the following properties:

Remark 5.1 (1) $\theta\mathrm{O}(X)$, $\delta\mathrm{O}(X)$ and $\alpha(X)$ have the structure of topology,

(2) $\mathrm{BO}(X)$, $\mathrm{S}\theta\mathrm{O}(X)$, $\delta\mathrm{PO}(X)$ and $\delta\mathrm{SO}(X)$ have property $\mathcal{B}$.

For each of modifications of open sets stated above, we can define a new type of upper/lower s-continuous multifunctions and obtain their characterizations and properties from Sections 3 and 4. For example, let $m_X = \alpha(X)$, then we obtain the following definitions and characterizations.

Definition 5.4 Let (X, τ) and (Y, σ) be topological spaces. A multifunction $F : (X, \tau) \to (Y, \sigma)$ is said to be

(1) *upper s-α-continuous* at $x \in X$ if for each $V \in \sigma$ containing $F(x)$ and having connected complement, there exists an α-open set U of X containing x such that $F(U) \subset V$,

(2) *lower s-α-continuous* at $x \in X$ if for each $V \in \sigma$ meeting $F(x)$ and having connected complement, there exists an α-open set U of X containing x such that $F(u) \cap V \neq \emptyset$ for each $u \in U$,

(3) *upper/lower s-α-continuous* if it has this property at each point x of X.

Theorem 5.1 *For a multifunction $F : (X, \tau) \to (Y, \sigma)$, the following are equivalent:*
(1) F is upper s-α-continuous;
(2) $F^+(V)$ is α-open in X for each open set V of Y having connected complement;

(3) $F^-(K)$ *is* α*-closed in* X *for every connected closed set* K *of* Y*;*
(4) $\alpha\mathrm{Cl}(F^-(B)) \subset F^-(\mathrm{Gl}(B))$ *for every subset* B *of* Y *having the connected closure;*
(5) $F^+(\mathrm{Int}(B)) \subset \alpha\mathrm{Int}(F^+(B))$ *for every subset* B *of* Y *such that* $Y - \mathrm{Int}(B)$ *is connected.*

Theorem 5.2 *For a multifunction* $F : (X, \tau) \to (Y, \sigma)$*, the following are equivalent:*
(1) F *is lower s-*α*-continuous;*
(2) $F^-(V)$ *is* α*-open in* X *for each open set* V *of* Y *having connected complement;*
(3) $F^+(K)$ *is* α*-closed in* X *for every connected closed set* K *of* Y*;*
(4) $\alpha\mathrm{Cl}(F^+(B)) \subset F^+(\mathrm{Cl}(B))$ *for every subset* B *of* Y *having the connected closure;*
(5) $F^-(\mathrm{Int}(B)) \subset \alpha\mathrm{Int}(F^-(B))$ *for every subset* B *of* Y *such that* $Y - \mathrm{Int}(B)$ *is connected.*

References

[1] M. E. Abd El-Monsef, S. N. El-Deeb and R. A. Mahmoud, *β-open sets and β-continuous mappings*, Bull. Fac. Sci. Assiut Univ. **12** (1983), 77–90.

[2] M. E. Abd El-Monsef, R. A. Mahmoud and E. R. Lashmi, *β-closure and β-interior*, J. Fac. Ed. Ain Shams Univ. **10** (1986), 235–245.

[3] D. Andrijević, *Semi-preopen sets*, Mat. Vesnik **38** (1986), 24–32.

[4] D. Andrijević, *On b-open sets*, Mat. Vesnik **48** (1996), 59–64.

[5] S. G. Crossley and S. K. Hildeband, *Semi-closure*, Texas J. Sci. **22** (1971), 99–112.

[6] G. Di Maio and T. Noiri, *On s-closed spaces*, Indian J. Pure Appl. Math. **18** (1987), 226–233.

[7] N. El-Deeb, I. A. Hasanein, A. S. Mashhour and T. Noiri, *On p-regular spaces*, Bull. Math. Soc. Sci. Math. R. S. Roumanie **27(75)** (1983), 311–315.

[8] J. Ewert and T. Lipski, *On s-quasi-continuous multivalued maps*, Univ. u Novom Sadu Zb. Rad. Prirod.-Mat. Fak. Ser. Mat. **20** (1990), 167–183.

[9] J. K. Kohli, *A class of mappings containing all continuous and all semiconnected mappings*, Proc. Amer. Math. Soc. **72** (1978), 175–181.

[10] I. Kovačević, *Subsets and paracompactness*, Univ. u Novom Sadu Zb. Rad. Priod.-Mat. Fak. Ser. Mat. **14** (1984), 79–87.

[11] N. Levine, *Semi-open sets and semi-continuity in topological spaces*, Amer. Math. Monthly **70** (1963), 36–41.

[12] T. Lipski, *S-continuous multivalued maps*, Math. Chronicle **18** (1989), 57–61.

[13] H. Maki, *On generalizing semi-open and preopen sets*, Report for Meeting on Topological Spaces Theory and its Applications, August 1996, Yatsushiro College of Technology, pp. 13–18.

[14] A. S. Mashhour, M. E. Abd El-Monsef and S. N. El-Deep, *On precontinuous and weak precontinuous mappings*, Proc. Math. Phys. Soc. Egypt **53** (1982), 47–53.

[15] A. S. Mashhour, I. A. Hasanein and S. N. El-Deeb, *α-continuous and α-open mappings*, Acta Math. Hungar. **41** (1983), 213–218.

[16] M. N. Mukherjee and C. K. Basu, *On semi-θ-closed sets, semi-θ-connectedness and some associated mappings*, Bull. Calcutta Math. Soc. 8**3** (1991), 227–238.

[17] O. Njåstad, *On some classes of nearly open sets*, Pacific J. Math. **15** (1965), 961–970.

[18] T. Noiri and V. Popa, *On upper and lower M-continuous multifunctions*, Filomat (Nis) **14** (2000) (to appear). .

[19] J. H. Park, B. Y. Lee and M. J. Son, *On δ-semi-open sets in topological spaces*, J. Indian Acad. Math. **19** (1997), 59–67.

[20] V. Popa and T. Noiri, *Properties of s-quasi-continuous multifunctions*, Far East J. Math. Sci. **1** (1993), 85–91.

[21] V. Popa and T. Noiri, *On upper and lower β-continuous multifunctions*, Real Anal. Exchange **22** (1996/97), 362–376.

[22] V. Popa and T. Noiri, *On s-precontinuous multifunctions*, Demonstratio Math. **33** (2000), 679–687.

[23] V. Popa and T. Noiri, *On s-β-continuous multifunctions*, J. Egypt. Math. Soc. **8** (2000), 127–137.

[24] V. Popa and T. Noiri, *On the definitions of some generalized forms of continuity under minimal conditions*, Mem. Fac. Sei. Kochi Univ. Ser. A Math. **22** (2001), 9–18.

[25] S. Raychaudhuri and M. N. Mukherjee, *On δ-almost continuity and δ-preopen sets*, Bull. Inst. Math. Acad. Sinica **21** (1993), 357–366.

[26] N. V. Veličko, *H-closed topological spaces*, Amer. Math. Soc. Transl. **78** (1968), 103–118.

[27] J. D. Wine, *Locally paracompact spaces*, Glasnik Mat. **10(30)** (1975), 351–357.

Valeriu POPA
DEPARTMENT OF MATHEMATICS
UNIVERSITY OF BACĂU
5500 BACĂU, ROMANIA
e-mail:vpopa@ub.ro

Takashi NOIRI
DEPARTMENT OF MATHEMATICS
YATSUSHIRO COLLEGE OF TECHNOLOGY
YATSUSHIRO, KUMAMOTO, 866-8501 JAPAN
e-mail:noiri@as.yatsushiro-nct.ac.jp