İstanbul Üniv. Fen Fak. Mat Der. **59** (2000), 29-45

COMMUTATIVITY OF RINGS WITH CONSTRAINTS ON COMMUTATORS

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Abstract. In this paper, we study the commutativity of a ring R satisfying the polynomial identity $x^t[x^n, y]y^r = \pm[x, y^m]y^s$ (resp. $x^t[x^n, y]y^r = \pm y^s[x, y^m]$), for all $x, y \in R$, where m, n, r, s and t are some non-negative integers such that $m > 0, n > 0$, and $m = n$ if $n + t \neq 1$ and $m + s \neq r + 1$. The main results of the present paper assert that a semiprime ring R is commutative if $(m, n, r, s, t) \neq (0, 0, 0, 0, 0)$ and commutativity of an associative ring R follows with property $Q(m)$, for $m > 1, n > 1$, that is for all $x, y \in R$, $m[x, y] = 0$ implies $[x, y] = 0$. It is also shown that the above results are true for s-unital rings. Finally, our results generalize some of the well-known commutativity theorems for rings (see [1, 2, 5, 10, 12, 15]).

AMS Subject Classifications (1991) : 16U80

Keywords and phrases : Commutativity theorems, polynomial identities, torsion-free rings, s-unital rings, zero-divisors.

1. Introduction

Throughout, R will be an associative ring (may be without unity 1), $Z(R)$ the center of $R, C(R)$ the commutator ideal of R, $D(R)$ the set of all zero-divisors of R and $N(R)$ the set of all nilpotent elements of R. For a ring R we denote by R^{opp} the opposite ring of R, that is, the ring with the same elements and addition as R, but with opposite multiplication '0' defined by $x\ 0\ y = y\ x$. We will omit the sign '0' of the opposite multiplication. For any $x, y \in R$, $[x, y] = xy - yx$. By $GF(q)$ we mean the Galois field (finite field) with q elements, and $(GF(q))_2$ the ring of all 2×2 matrices over $GF(q)$. We set $e_{11} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, e_{12} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, e_{21} = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$ and $e_{22} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$ in $(GF(p))_2$, a prime p.

In a recent paper [1], Abujabal and Peric considered s-unital (left or right s-unital) ring R in which for any pair of elements $x, y \in R$, there exist non-negative integers m, n, s, and t, $m > 0$ or $n > 0$ and $s \neq t$ if $m = n = 1$ such that $x^t\,[x^n, y] = \pm\,[x, y^m]\,x^s$ or $x^t\,[x, y^m] = \pm\,x^s[x,\ y^m]$ for all $x, y \in R$.

The objective of this paper is to investigate the commutativity of a ring R satisfying the polynomial identity

$$x^t\,[x^n,\ y]\,y^r = \pm\,[x,\ y^m]y^s \tag{1.1}$$

or

$$x^t\,[x^n,\ y]\,y^r = \pm\,y^s[x,\ y^m] \tag{1.2}$$

for some given non-negative integers m, n, r, s and t.

Remark 1.1. In the statement of (1.1) and (1.2), we consider the $\pm$ sign same for all $x, y \in R$; each of (1.1) and (1.2) represents two different identities. But if one takes the $\pm$ sign varying with x and y, then (1.1) and (1.2) are not identities (see [10, 14]).

2. Preliminary results

Definition 2.1. A ring R is called right (resp. left) s-unital if $x \in xR$ (resp. $x \in Rx$) for each $x \in R$. Further R is called s-unital if it is both right as well as left s- unital, that is, $x \in xR \cup Rx$ for each $x \in R$.

Definition 2.2. If R is an s-unital (resp. a right s-unital or a left s-unital) ring, then for any finite subset F of R, there exists an element $e \in R$ such that $xe = ex = x$(resp. $xe = x$ or $ex = x$) for all $x \in F$. Such an element e is called the pseudo (resp. pseudo right or pseudo left) identity of F in R.

Definition 2.3. For a ring R and a positive integer n, we say that R has the property $Q(n)$ if all commutators in R are n-torsion free, that is, if $n[x, y] = 0$ implies $[x, y] = 0$ for all $x, y \in R$.

Remark 2.1. Clearly, every n-torsion free ring R has the property $Q(n)$ and every ring R has the property $Q(1)$. If a ring R has the property $Q(n)$, then R has the property $Q(m)$ for any factors m of n.

In the proof of our results, we need the following known results.

Lemma 2.1 [8]. Let x and y be elements in a ring R. If $[x,[x,y]] = 0$, then $[x^k, y] = k\, x^{k-1}\, [x,y]$, for any positive integer k.

Lemma 2.2 [4]. Let R be a ring with 1, and let x and y be elements of R. If $dx^m[x,y] = 0$ and $d(x+1)^m[x,y] = 0$, for some integers $m \geq 1$ and $d \geq 1$, then necessarily $d[x,y] = 0$.

Lemma 2.3 [9]. Let f be a polynomial in n non- commuting indeterminates $x_1, x_2, x_3, \ldots\ldots, x_n$ with integer coefficients. Then the following statements are equivalent:

(i) For any ring R satisfying the polynomial identity $f = 0$, $C(R)$ is nil.

(ii) For every prime p, $(G(F(p))_2$ fails to satisfy $f = 0$.

(iii) Every semiprime ring satisfying $f = 0$ is commutative.

Lemma 2.4 [13]. Let R be a ring with unity 1, and let d and m be positive integers. If $(1-y^m)\, x = 0$, then $(1-y^{dm})\, x = 0$ for all $x, y \in R$.

Lemma 2.5 [5]. Let R be a ring, and let $n > 1$ be a fixed integer. If $x^n - x \in Z(R)$ for each $x \in R$, then R is commutative.

Lemma 2.6 [15]. Let R be right (resp. left) s-unital ring. If for each pair of elements $x, y \in R$, there exists a positive integer $k = k(x,y)$ and an element $e = e(x,y) \in R$ such that $ex^k = x^k$ and $ey^k = y^k$ (resp. $x^k e = x^k$ and $y^k e = y^k$), then R is an s-unital.

3. A commutativity theorem for semiprime rings

Theorem 3.1. Let m, n, r, s and t be fixed non-negative integers such that $(m,n,r,s,t) \neq (0,0,0,0,0)$. Let R be a semiprime ring satisfying the polynomial identity (1.1) (resp. (1.2)). Then R is commutative.

Proof. Let R satisfy (1.1). But $x = e_{11}$ and $y = e_{12} \in (GF(p))_2$ for a prime p, fail to satisfy (1.1). By Lemma 2.3, R is commutative.

If R satisfies (1.2), then $x = e_{22}$ and $y = e_{12} \in (GF(p))_2$ for a prime p, fail to satisfy (1.2). Hence R is commutative by Lemma 2.3.

Remark 3.1. Since there are non-commutative rings with $R^2 \subseteq Z(R)$ neither of the conditions (1.1) and (1.2) guarantees the commutativity in arbitrary rings.

One might ask a natural question: What additional conditions are needed to ensure the commutativity for arbitrary rings which satisfy (1.1) or (1.2)? To investigate the commutativity of such a ring R, we need an extra condition on R, which is given in Definition 2.3.

4. Commutativity theorems for rings with unity 1

Theorem 4.1. Let R be a ring with unity 1 satisfying the polynomial identity (1.1) (resp. (1.2)) for some given non-negative integers $m > 0, n > 0, r, s$ and t such that $n + t > 1$. Moreover, if $n + t > 1$ (resp. $m + s > 1$ for $r = 0$), and R has $Q(n)$ property for $m > 1$, $n > 1$ and $Q(t + 1)$ property for $n = 1, t > 0$, then R is commutative.

We shall prove here the following results called steps.

Step 4.1. Let R be a ring with unity 1 satisfying the polynomial identity (1.1) (resp. (1.2)) for some given non-negative integers $m > 0$, $n > 0$, r, s and t such that $n + t > 1$ and R has $Q(n)$ property for $m > 1, n > 1$. Then $N(R) \subseteq Z(R)$.

Proof. Let a be an arbitrary element in $N(R)$. Then there exists a positive integer p such that

(4.1) $\quad a^k \in Z(R)$ for all integers $k \geq p$, p minimal.

If $p = 1$, then $a \in Z(R)$. Let $p > 1$ and put $b = a^{p-1}$. By (4.1) we have

(4.2) $\quad b^k \in Z(R)$ and $b^k\,[x, b] = [x, b]\,b^k = 0$ for all $x \in R$ and $k > 1$.

Replacing x by b in (1.1) (resp. (1.2)), we get
$b^t [b^n, y] y^r = \pm [b, y^m] y^s$ (resp. $b^t [b^n, y] y^r = \pm y^s[b, y^m]$) for all $y \in R$.

Let $n + t > 1$. Then three cases arise, that is $n > 1$, $t > 1$ or $n = t = 1$. In the first two cases using (4.2), we get

$$(4.3) \qquad [b, y^m] y^s = 0 \text{ (resp. } y^s [b, y^m] = 0).$$

Let $m = 1$ in (4.3). Then we have

$$(4.4) \qquad [b, y] y^s = 0 \text{ (resp. } y^s [b, y] = 0).$$

Replacing y by $y+1$ in (4.4) and using Lemma 2.2, we obtain $[b, y] = 0$ for all $y \in R$, that is $a^{p-1} \in Z(R)$, a contradiction.

Let $m > 1$. Replace x by $1 + x$ in (1.1)(resp. (1.2)), by (4.2), and the above two identities obtained from (1.1) (resp. (1.2)) for $x = b$, we get

$$n[b, y]y^r = 0 \text{ for all } y \in R.$$

Since $n > 1$, an application of the property $Q(n)$, yields

$$[b, y]y^r = 0, \text{for all } y \in R.$$

Replacing y by $y + 1$ in the last identity and using Lemma 2.2, we get

$$[b, y] = 0, \quad \text{a contradiction.}$$

Finally, let $n = t = 1$ in (1.1) (resp. (1.2)).

Replacing x by $1 + x$, in (1.1) (resp. (1.2)), and using (1.1) (resp. (1.2)), we get

$$(1 + x)[x, y] y^r = x[x, y] y^r.$$

This implies that

$$[x, y]y^r = 0 \text{ for all } x, y \in R.$$

Replacing y by $y + 1$ in the last identity and using Lemma 2.2, we get $[x, y] = 0$ for all $x, y \in R$, that is, $N(R) \subseteq Z(R)$.

Step 4.2. Let R be a ring satisfying the polynomial identity (1.1) (resp. (1.2)) for some given non-negative integers $m > 0$, $n > 0$, r, s and t such that $n + t > 1$(resp. ($n + t > 1$ for $s = 0$ or $m + s > 1$). Then the commutator ideal $C(R)$ is nil, i.e. $C(R) \subseteq N(R)$.

Proof. Let $n+t>1$. Then the elments $x=e_{12}$ and $y=e_{11}$ in $(GF(p))_2$ show that the ring $(GF(p))_2$ fails to satisfy (1.1), and also (1.2) for $s=0$.

If $s\geq 1$, then the elements $x=e_{11}, y=e_{11}+e_{12}$ in $(GF(p))_2$ fail to satisfy (1.2).

Let $m+s>1$. Then the elments $x=e_{11}$ and $y=e_{12}$ in $(GF(p))_2$ show that the ring $(GF(p))_2$ fails to satisfy (1.2).

Hence, by Lemma 2.3, $C(R)\subseteq N(R)$.

Remark 4.1. From the Steps 4.1 and 4.2, for the ring R, we get

$$C(R) \subseteq N(R) \subseteq Z(R). \tag{4.5}$$

By (4.5), R satisfies $[x,[x,y]]=0$ for all $x,y\in R$, and from Lemma 2.1 the identities (1.1) and (1.2) are equivalent and can be written in the form

$$nx^{n+t-1}\,[x,y]\,y^r = \pm my^{m+s-1}\,[x,y]. \tag{4.6}$$

Step 4.3. Let R be a ring with unity 1 satisfying the identity (4.6) for some given non-negative integers $m>0, n>0, r, s$ and t. For any $x,y\in R$, $n[x,y]=0$ if and only if $m[x,y]=0$. Moreover, R has $Q(n)$ property if and only if R has $Q(m)$ property. Let m,n be relatively prime integers. Then R has both $Q(m)$ and $Q(n)$ properties.

Proof. By hypothesis, $n[x,y]=0$ for some $x,y\in R$. Then we have

$$n\ \alpha^{n+t-1}[\alpha,\beta]\beta^r = 0 \quad for \quad \alpha\in\{x,1+x\} \text{ and } \beta\in\{y,1+y\}.$$

In view of (4.6), we have

$$m\ \beta^{m+s-1}\,[\alpha,\beta]=0. \quad for \quad \alpha\in\{x,1+x\} \text{ and } \beta\in\{y,1+y\}.$$

This implies that

$$m\ y^{m+s-1}\,[x,y]=0 \text{ and } m(y+1)^{m+s-1}\,[x,y]=0.$$

Using Lemma 2.2, we get

$$m[x,y]=0.$$

Similary, if $m[x,y]=0$, then $n[x,y]=0$.

Let R be a ring with $Q(m)$ property. If $n[x,y]=0$, for some $x,\ y\ \in R$, then $m[x,y]=0$. By $Q(m)$ property, $[x,y]=0$. Thus, R has also $Q(n)$ property.

Similarly, one can prove that if R has $Q(n)$ property, then R has also $Q(m)$ property.

Let m,n be relatively prime integers. Suppose that $m[x,y]=0$, for some $x,\ y\ \in R$. Then $n[x,y]=0$. Since m and n are relatively prime, $[x,y]=0$. Hence R has $Q(m)$ property and also $Q(n)$ property.

Proof of Theorem 4.1 . Keeping the Remark 4.1 in mind, it suffices to assume that the ring R satisfies the identity (1.1).
Replacing x by px in (1.1), we get

$$p^{n+t}\ x^t\ [x^n,y]\ y^r\ =\ \pm\ p\ [x,y^m]y^s\ \text{ for all} x,y\in\ R.$$

Combining this identity with (1.1), we get

$$(p^{n+t}-p)\ [x,y^m]\ y^s\ =\ 0.$$

In view of Lemma 2.1, one gets

$$(p^{n+t}-p)\ m\ [x,y]y^{m+s-1}=0 \text{ for all } x,y\in\ R.$$

Replace y by $y+1$ in the last expression and use Lemma 2.2, we get

$$(p^{n+t}-p)\ m\ [x,y]=0.$$

Let $d\ =\ m(p^{n+t}-p)\ >\ 1$. Then $d[x,y]\ =\ 0,$ for all $x,y\in R$. Hence $[x^d,y]=dx^{d-1}[x,y]=0$, that is,

(4.7) $\qquad x^d\in Z(R)$ for all $x\in R$ and $d=(p^{n+t}-p)\ m>1$.

Let $n>1$. Replacing x by x^n in (1.1), we get

(4.8) $$x^{nt}\ [(x^n)^n,y]y^r=\pm[x^n,y^m]y^s.$$

Now, we have

$$\begin{aligned} x^{nt}\ [(x^n)^n,\ y]\ y^r &= n\ x^{nt+n(n-1)}\ [x^n,\ y]\ y^r. \\ &= nx^{(n-1)(n+t)}(x^t[x^n, y]y^r) \\ &= \pm nx^{(n-1)(n+t)}\ [x, y^m]\ y^s \end{aligned}$$

$$[x^n, y^m]y^s = nx^{n-1}[x, y^m]y^s.$$

From the above, we have

$$n\ x^{n-1}\ [x, y^m]\ y^s - nx^{(n-1)(n+t)}[x, y^m]y^s = 0.$$

$$nx^{n-1}\ (1 - x^{(n-1)(n+t-1)})[x, y^m]y^s = 0.$$

In view of Lemma 2.4, we get

$$nx^{n-1}(1 - x^{d\ (n-1)(n+t-1)})[x, y^m]y^s = 0. \tag{4.9}$$

Clearly one can prove that the polynomial identity (1.1) implies that

$$x^{t'}\ [x^{n^2}, y]\ y^{r'} = [x, y^{m^2}]y^{s'}, \tag{4.10}$$

for all $x, y \in R$ and $r' = mr + r$, $s' = ms + s$ and $t' = nt + t$.

It is noticed that the ring R is isomorphic to a subdirect sum of subdirectly irreducible rings $R_i, i \in I$. As homomorphic image of R, each of the rings R_i has a unity 1 and satisfies all the identities satisfied by R. But R_i does not necessarily satisfy $Q(n)$ for $m > 1$, $n > 1$ (resp. $Q(t+1)$ for $n = 1, t > 0$).

Now, consider the ring R_i for some fixed index $i \in I$. If H is the intersection of all non-zero ideals of R_i, then $H \neq \{0\}$ and $Hc = \{0\}$ for all central zero divisors c of R_i.

If u is any zero divisor of R_i, then (4.9) can be written as

$$nu^{n-1}\ (1 - u^{d\ (n-1)(n+t-1)})[u, y^m]y^s = 0.$$

Let $nu^{n-1}[u, y^m]\ y^s \neq 0$. Then $1 - u^{d(n-1)(n+t-1)}$ will be a central zero divisor c of $R_i, i \in I$. We have

$$\{0\} = H(1 - u^{d(n-1)(n+t-1)}) = H - Hc = H.$$

This gives a contradiction because $H \neq \{0\}$. Thus $nu^{n-1}\ [u, y^m]\ y^s = 0$, and by Lemma 2.1, we obtain

$$mnu^{n-1}[u,y]y^{m+s-1} = 0 \ \text{ for all } y \in R_i \ \text{ and } u \in D(R_i).$$

Replacing y by $y+1$ in the last expression and using Lemma 2.2, we get

$$(4.11) \qquad mnu^{n-1}[u,y] = 0 \ \text{ for all } y \in R_i \ \text{ and } u \in D(R_i).$$

Combining (4.10) and (4.11), we get

$$u^{t'}\ [u^{n^2}, y]\ y^{r'} = 0 \ \text{ for all } y \in R_i \ \text{ and } u \in D(R_i).$$

Replacing y by $y+1$ in the last expression and using Lemma 2.2, we get

$$u^{t'}\ [u^{n^2}, y] = 0 \ \text{ for all } y \in R_i \ \text{ and } u \in D(R_i).$$

In view of (4.10), Lemmas 2.1 and 2.2, we get

$$m^2\ [u,y] = 0 \text{ for all } y \in R_i, u \in D(R_i).$$

This implies that $[u, y^{m^2}] = m^2\ y^{m^2-1}[u,y] = 0$ for all $y \in R_i$ and $u \in D(R_i)$. Hence

$$(4.12) \qquad [u, y^{m^2}] = 0 \text{ for all } y \in R_i,\ u \in D(R_i).$$

Let $z \in Z(R_i)$, center of R_i. Replacing x by zx in (1.1), we get

$$\begin{aligned} z^{n+t}x^t[x^n,y]y^r &= z(\pm[x,y^m]y^s) \\ &= zx^t[x^n,y]y^r. \end{aligned}$$

$$(z^{n+t}-z)x^t[x^n,y]y^r = 0.$$

Replacing y by $y+1$ and using Lemma 2.2, we get

$$(z^{n+t}-z)x^t[x^n,y] = 0.$$

By Lemma 2.1, we have

$$n(z^{n+t}-z)x^{n+t-1}[x,y] = 0.$$

Replacing x by $x+1$ and using Lemma 2.2, we get $n(z^{n+t}-z)\ [x,y] = 0$. Thus, by Lemma 2.1, we get

$$(z^{n+t}-z)[x^n,y] = n(z^{n+t}-z)x^{n-1}[x,y] = 0 \text{ for all } x,y \in R_i \text{ and } z \in Z(R_i).$$

(4.13) $\quad (z^{n+t} - z)[x,^n\ y] = 0$ for all $x, y \in R_i$ and $z \in Z(R_i)$.

Clearly, from (4.7) and (4.13), we find

(4.14) $\quad (y^{d(n+t)} - y^d)\ [x^n, y] = 0$ for all x and y in R_i.

Now, let $y \in R_i$. If $[x^{m^2 n}, y] = 0$, then one can write

$$[x^{m^2 n}, y^q - y] = 0 \text{ for all positive integers } q > 1.$$

Let $[x^{m^2 n}, y] \neq 0$. Then $[x^n, y] \neq 0$. Since $[x^n, y] \neq 0$, by (4.12), $y^{d(n+t)} - y^d \in D(R_i)$, so $y^{d(n+t-1)+1} - y$ is also in $D(R_i)$. In view of (4.12), we have

(4.15)
$[x^{m^2 n}, y^p - y] = 0$ for all x, y in R_i and $p = d(n + t - 1) + 1 > 1$.

Since R_i, $i \in I$ satisfies (4.15), the original ring R also satisfies (4.15). Therefore, R has $Q(n)$ property, and by Step 4.3, also $Q(m)$ property. Combining (4.14) along with Lemmas 2.1 and 2.2, we finally get $[x, y^p - y] = 0$ for all $x, y \in R$ and some positive integer $p > 1$. Hence R is commutative by Lemma 2.5.

If $n = 1, t > 0$, then the identity (1.1), by Lemma 2.1, gives

$$[x^{t+1}, y]y^{r+mt} = \pm[y^{m(t+1)}, x]y^s.$$

By an application of the $Q(t+1)$ property, R is commutative.

5. Commutativity theorems for s-unital rings

Theorem 5.1. Let R be a right (resp. left) s-unital ring satisfying the hypothesis of Theorem 4.1. Then R is commutative.

We begin with

Step 5.1. Let R be a right (resp. left) s-unital ring satisfying the polynomial identity (1.1) (resp. (1.2)) for some non-negative integers $m > 0, n > 0, r, s$ and t such that $n + t > 1$ (resp. $m + s > 1$ for $r = 0$). Then R is s-unital.

Proof. Let R be a right (resp. left) s-unital ring, x, y arbitrary elements of R, and e an element of R such that $xe = x$ and $ye = y$ (resp. $ex = x$ and $ey = y$).

Let R be a right s-unital ring satisfying (1.1) for some non-negative integers $m > 0, n > 0, r, s$ and t such that $n + t > 1$.
Replacing y by e in (1.1), we get

$$x^{n+t}e^{r+1} \mp (x - e^m x). \tag{5.1}$$

If $n + t > 1$ or $t > 1$, then (5.1) gives $x = e^m x$, i.e., s-unital ring.

Let $t = 0$. Then $n > 1$, and hence (5.1) becomes

$$x = e^m x \pm x^n \mp ex^n.$$

This implies that R is s-unital ring.

Let R be a left s-unital ring satisfying (1.2) for some given non-negative integers $m > 0, n > 0, r, s$ and t such that $m + s > 1$ for $r = 0$.
Replace x by e in (1.2) one gets

$$y^{r+1} - ye^n y^r \mp (y^{m+s} - y^{m+s}e). \tag{5.2}$$

If $r > 0$, then (5.2) gives $y^{m+s} = y^{m+s}e$.

Similarly, for $r > 0$ one gets $x^{m+s} = x^{m+s}e$ when $m + s > 0$. By Lemma 2.6, R is s-unital ring.

Proof of Theorem 5.1. In view of Step 5.1 R is s-unital and, by the Proposition 1 of [7], we may assume that R has unity 1. Hence R is commutative by Theorem 4.1.

In particular, for $r = 0$, we have the following:

Theorem 5.2. Let R be a right (resp. left) s-unital ring satisfying (1.1) (resp. (1.2)) for some given non-negative integers $m > 0, n > 0, r, s$ and t such that $n + t > 1$ (resp. $m + s > 1$). If $r = 0, m + s > 0$ and R has $Q(m)$ property for $m > 1, n > 1$, and $Q(s + 1)$ property for $m = 1, s > 0$, then R is commutative.

Proof If $r = 0$ and R satisfies (1.1) (resp. (1.2)), then the ring R itself (resp. the opposite ring R^{opp} of R) satisfies the polynomial identity

$$[y^m, x]y^s = \pm x^t[y, x^n] \text{ (resp. } [y^m, x]y^s = \pm[y, x^n]y^s).$$

Hence, the ring R in Theorem 5.2 is commutative by Theorem 5.1.

Theorem 5.3. Let R be a right (resp. left) s-unital ring satisfying the polynomial identity (1.1) (resp. (1.2)) for some given non-negative integers $m > 0, n > 0, r, s$ and t such that $n + t > 1$ (resp. $m + s > 1$). If $r = 0$, and m, n are relatively prime integers, then R in commutative.

Proof. In view of Step 5.1, R is s-unital. Now, we may assume that R is a ring with unity 1. From Theorems 5.1 and 5.2, it is enough to prove that R has the properties $Q(m)$ and $Q(n)$.
Taking b as in the proof of Step 4.1 and $r = 0$, we have

$$n[b, y] = 0 \text{ for all } y \in R. \tag{5.3}$$

Let $r = 0$. Using the same arguments as above, we get

$$m[x, b] = 0 \text{ for all } x \in R. \tag{5.4}$$

Since m, n are relatively prime integers, by (5.3) and (5.4) we get $[x, b] = 0$ for all $x \in R$, that is, $b \in Z(R)$. Hence $N(R) \subseteq Z(R)$ and by Step 4.2, R satisfies (4.6) and also (4.7). Hence, in view of Step 4.3, $m[x, y] = 0$ is equivalent to $n[x, y] = 0$ for all $x, y \in R$. Clearly, m, n are relatively prime, R has both $Q(m)$ and $Q(n)$ properties.

The following results are immediate consequences of the above results.

Corollary 5.1. Let $m \geq n \geq 1$ be fixed integers with $m, n > 1$ and let R be a right (resp. left) s-unital ring satisfying the polynomial identity $[xy, x^n \pm y^m]$ for all $x, y \in R$. Then R is commutative if R satisfies one of the following conditions:

(i) R has the property $Q(m)$;

(ii) R has the property $Q(n)$;

(iii) m, n are relatively prime.

Proof. By hypothesis, we have

$$x[x^n, y] = \pm[x, y^m]y. \tag{5.4}$$

If R is a right s-unital ring , then Corollary 5.1 follows from Theorems 5.1 and 5.2. Clearly, the identity (5.4) is s-unital when R is a left s-unital ring.

Finally, if $m = n = 1$, then the Corollary 5.1 shows that a right (resp. left) s-unital ring R having the product of two elements with their sum (or difference) is necessarily commutative.

Corollary 5.2. [**2, Theorem**] Let m, n, r and t be fixed non-negative integers such that $m > 0$ or $n > 0$, and $r = 0$ or $t > 0$ if $m = n = 1$. If R is a ring which satisfies the polynomail identity $x^t[x^n, y]y^r = \pm[x, y^m]$, then R is commutative provided that one of the following additional conditions is fulfilled:

(i) $m = 0$, and R is an s-unital (resp. a right s-unital for $t = 0$, or a left s-unital for $r = 0$) ring with the property $Q(n)$;

(ii) $n = 0$, and R is a right or left s-unital ring with the property $Q(m)$.;

(iii) $m = 1, n \geq 1$, or $m > 1, n = 1$ and $r = t = 0$;

(iv) $m > 1, n > 1$, and R is a right or left s-unital ring with the property $Q(m)$;

(v) $m > 1, n = 1, r + t > 0$, and R is a right or left s-unital ring with the property $Q(m \pm 1)$ for $t = 0$).

Corollary 5.3 [12, Theorem 2]. Let m, t be fixed non-negative integers. Suppose that R satisfies the polynomial identity $x^t[x, y] = [x, y^m]$.

(i) If R is a left s-unital, then R is commutative except for $(m, t) = (1, 0)$.

(ii) If R is right s-unital, then R is commutative except for $m = 1, t = 0$ and also $m = 0, t > 0$.

Corollary 5.4 [1, Theorem 1]. Let $m \geq n \geq 1$ be fixed integers with $m, n > 1$ and let R be a left (resp. a right) s-unital ring satisfying the polynomial identity $x\ [x^n, y] = [y^m, x]\ y$ for all $x, y \in R$. Then R is commutative.

Remark 5.1. In Corollary 5.3, if $n > 1$ and R has $Q(m)$ property, then the ring R in Corollary 5.3 is commutative by Theorem 5.1 for $m = 1$ and by Theorem 5.2 for $m > 1$.

6. Counterexamples

Example 6.1. Let F be a field. Then the non-commutative ring $R = \begin{pmatrix} 0 & F \\ 0 & F \end{pmatrix}$ (resp. $R_1 = \begin{pmatrix} F & 0 \\ F & 0 \end{pmatrix}$) has a left (resp. right) identity element and satisfies the polynomial identity $[x,y]x = 0$ (resp. $x[x,y] = 0$) for all $x, y \in R$. Further, if $m = 0$ and $n \geq 0$, then Theorem 5.1 need not be true for s-unital ring.

The following example shows that the hypothesis of R to be a right s-unital, a left s-unital or the existence of unity 1 in R is not superfluous in Theorems 4.1, 5.1 and 5.2.

Example 6.2. Let

$$A_1 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, B_1 \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, S_1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$$

be elements of the ring of all 3×3 matrices over $\mathbf{Z}_2$ the ring of integers mod 2. If R is the subring generated by the matrices A_1, B_1 and S_1, then each of the integers $n \geq 1$ and $x, y \in R$, $[x^n, y] = [x, y^n]$ holds. However R is not commutative.

Remark 6.1. In Theorem 5.2, the restriction of $Q(n)$ property is essential. To do this, we consider Example 6.2 and use Dorroh construction (with the ring of integers mod 2) to get a ring R with 1. This ring R satisfies $[x^2, y] = [x, y^2]$ for all $x, y \in R$, and is not commutative (see [3, Remark]).

In general, there are rings with unity satisfying the identity (1.1) or (1.2) which are not commutative. Now, we give an example to show that a multiplicative group which satisfies (1.1) need not be commutative.

Example 6.3. Let G be a multiplicative group with center $Z(G)$. Suppose that the group $G/Z(G)$ is a periodical group of finite period p. Then, for any $x \in G$, $x^p \in Z(G)$, and thus, such a group G satisfies the identity

$$[x^n, y] = [x, y^m] \text{ for } n = 1 \text{ and } m = p + 1. \tag{6.1}$$

Therefore, if any finite group G satisfies the hypothesis of the Example 6.3, then a group G satisfying the identity (6.1) for some given relatively prime positive integers m and n need not be commutative. Moreover, if $m = n + 1$, then G is necessarily commutative (see [12, Theorem 3]).

Acknowledgements

This paper was presented and written when the author was a visiting Mathematician at the T.I.F.R. Mumbai, India. I am greatly indebted to the referee for his valuable comments and helpful suggestions and also Professors R. Parimala and S.G. Dani, Dean Faculty of Mathematics for the hospitality.

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