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COMMUTATIVITY THEOREMS FOR RINGS THROUGH A STREB RESULT

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Abstract. In the present paper, we prove the commutativity of a ring with unity satisfying any one of the following properties:

$$\{1 - p(yx^m)\}\ [yx^m - x^r b(yx^m)\ x^s, x]\{1 - q(yx^m)\} = 0,$$

$$y^s[x, y^n] = g(x)[x^2 f(x), y]h(x) \text{ and } [x, y^n]\ y^t = \tilde{g}(x)[x^2 \tilde{f}(x), y]\tilde{h}(x),$$

for some $b(X) \in X^2\mathbf{Z}[X]$, and $p(X), q(X) \in X\mathbf{Z}[X]$ and $f(X), \tilde{f}(X), g(X)$ $\tilde{g}(X), h(X), \tilde{h}(X) \in \mathbf{Z}[X]$, where $m \geq 0, r \geq 0, s \geq 0, n > 0, t > 0$ are integers. Further, we extend these results to the case when integral exponents in the underlying conditions are no longer fixed, rather they depend on the pair of ring elements x, y for their values. Moreover, it is also shown that the above result is true for s-unital rings. Finally, our results generalize many known commutativity theorems.

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1. Introduction

Throughout, R will represent an associative ring (may be without unity), $N = N(R)$, the set of nilpotent elements of $R, Z = Z(R)$, the center of $R, C = C(R)$, the commutator ideal of R, and $U = U(R)$, the group of units of R. For any $x, y \in R, [x, y]$ denotes the commutator $xy - yx$. As usual $\mathbf{Z}[X]$ is the totality of polynomials in X with coefficients in $\mathbf{Z}$, the ring of integers. Consider the following ring properties :

(I) For each $x, y \in R$, there exist polynomials $b(X) \in X^2\mathbf{Z}[X]$ and $p(X), q(X) \in X\mathbf{Z}[X]$ such that

$$\{1 - p(yx^m)\}[yx^m - x^r b(yx^m)x^s, x]\{1 - q(yx^m)\} = 0,$$

where $m \geq 0, r \geq 0, s \geq 0$ are fixed integers.

(I)' For each $x, y \in R$ there exist integers $m \geq 0, r \geq 0, s \geq 0$ and polynomials $b(X) \in X^2\mathbf{Z}[X]$ and $p(X), q(X) \in X\mathbf{Z}[X]$ such that $\{1 - p(yx^m)\}$ $[yx^m - x^r b(yx^m)x^s, x]\{1 - q(yx^m)\} = 0$.

(II) For every $x, y \in R$, there exist polynomials $f(X), \tilde{f}(X), g(X), \tilde{g}(X)$, $h(X)$ and $\tilde{h}(X)$ in $\mathbf{Z}[X]$ such that

$$y^s[x, y^m] = g(x)[x^2 f(x), y]h(x)$$

and

$$y^t[x, y^n] = \tilde{g}(x)[x^2 \tilde{f}(x), y]\tilde{h}(x),$$

where $s \geq 0, t \geq 0, m > 1, n > 1$ are fixed integers with $(m, n) = 1$.

(II)' For each $x, y \in R$, there exist integers $s = s(x, y) \geq 0$, $t = t(x, y) \geq 0, m = m(x, y) > 1, n = n(x, y) > 1$ with $(m, n) = 1$ and polynomials $f(X), \tilde{f}(X), g(X), \tilde{g}(X), h(X), \tilde{h}(X) \in \mathbf{Z}[X]$ such that

$$y^s[x, y^m] = g(x)[x^2 f(x), y]h(x)$$

and

$$y^t[x, y^n] = \tilde{g}(x)[x^2 \tilde{f}(x), y]\tilde{h}(x).$$

(III) For each x, y in R, there exist polynomials $f(X), \tilde{f}(X), g(X), \tilde{g}(X)$ and $h(X), \tilde{h}(X)$ in $\mathbf{Z}[X]$ such that

$$[x, y^m]y^s = g(x)[x^2 f(x), y]h(x)$$

and

$$[x, y^n]y^t = \tilde{g}(x)[x^2 \tilde{f}(x), y]\tilde{h}(x),$$

where $s \geq 0, t \geq 0, m > 1, n > 1$ are fixed integers with $(m, n) = 1$.

(I11)' For each $x, y \in R$ there exist integers $s = s(x, y) \geq 0$, $t = t(x, y) \geq 0, m = m(x, y) > 1, n = n(x, y) > 1$ with $(m, n) = 1$ and polynomials $f(X), \tilde{f}(X), g(X), \tilde{g}(X), h(X), \tilde{h}(X)$ in $\mathbf{Z}[X]$ such that

$$[x, y^m]y^s = g(x)[x^2 f(x), y]h(x)$$

and

$$[x, y^n]y^t = \tilde{g}(x)[x^2 \tilde{f}(x), y]\tilde{h}(x).$$

(IV) For each $x, y \in R$, there exist $b(t), g(t)$ in $t^2\mathbf{Z}[t]$ such that $[x - g(x), y - b(y)] = 0$.

(V) For each $x, y \in R$, there exists $b(t) \in t^2\mathbf{Z}[t]$ such that $[x - b(x), y] = 0$.

Searcōid and MacHale [10] proved commutativity of a ring satisfying the condition $xy = (xy)^{n(x,y)}$ with $n(x, y) > 1$. Tominaga and Yaqub [12, Theorem 2] established that if R is a ring such that either $xy = p(xy)$ or $xy = p(yx)$, where $p(X)$ in $X^2\mathbf{Z}[X]$, then R is commutative. A nice theorem of Herstein [3] states that if R is a ring satisfying the property (V), then R is commutative. It is natural to consider the related properties; $[xy - p(xy), x] = 0$ and $[xy - q(yx), x] = 0$ for some $p(X), q(X)$ in $X^2\mathbf{Z}[X]$ depending on ring's elements x, y. Putcha and Yaqub [9] remarked that if for each $x, y \in R$, there exists a polynomial $p(X) \in X^2\mathbf{Z}[X]$ such that $xy - p(xy)$ is central, then R^2 must be central. Also the author jointly with Bell and Quadri [1, Theorem 2] obtained the commutativity of the rings with unity 1 satisfying polynomial identitites of the form $[xy - p(xy), x] = 0$ and $[xy - q(xy), x] = 0$, where $p(X), q(X)$ are considered to be fixed. Motivated by these observations, the author [5] found the commutativity of rings with unity 1 satisfying the property $[yx^m - x^n b(y)x^l, x] = 0$, where the polynomial $b(x)$ in $X^2\mathbf{Z}[X]$ depends on the pairs $x, y \in R$ and fixed non-negative integers l, m, n. Hence a natural question arises: What can we say about the commutativity of ring R, if the underlying condition is replaced by $[yx^m - x^n b(y)x^l, x] = 0$? In the present note, we not only answer this question, but also we prove rather a more general result by establishing that a ring with unity 1 satisfying the property (I) is commutative. Further, we shall consider the property (I)', where integral exponents are allowed to vary with the pair of ring's elements x, y and also the ring satisfies the Chacron's condition (IV). Our next aim is to establish commutativity of one-sided s-unital rings satisfying any one of the properties (II), (III), (II)' and (HI)'. In fact, several commutativity results can be obtained as corollaries to our results, for instance, [4, Theorem], [5, Theorems 1 and 2], [8, Theorems 1 & 2], [10, Theorem], [12, Theorem].

2. Results

We first consider the following types of rings.

(i)$_l$ $\begin{pmatrix} GF(p) & GF(p) \\ 0 & 0 \end{pmatrix}$, p a prime.

(i)$_r$ $\begin{pmatrix} 0 & GF(p) \\ 0 & GF(p) \end{pmatrix}$, p a prime.

(i) $\begin{pmatrix} GF(p) & GF(p) \\ 0 & GF(p) \end{pmatrix}$, p a prime.

(ii) $M_\sigma(F) = \left\{ \begin{pmatrix} a & b \\ 0 & \sigma(a) \end{pmatrix} \mid a, b \in F \right\}$, where F is a finite field with a non-trivial automorphism σ.

(iii) A non-commutative division ring.

(iv) $S =< 1 > +T, T$ a non-commutative radical subring of S, must be a domain.

(v) $S =< 1 > +T, T$ a non-commutative subring of S such that $T[T,T] = [T,T]T = 0$.

In a recent paper [11], Streb gave a nice classification for non-commutative rings which yields a powerful tool in obtaining a number of commutativity theorems (see [5, 6, 7]). It follows from the proof of [6, Corollary 1], that if R is a non-commutative ring with unity 1, then there exists a factorsubring of R which is of type (i), (ii), (iii), (iv) or (v). This observation gives the following proposition that plays a vital role in our subsequent discussion.

Proposition 2.1. Let P be a ring property which is inherited by factorsubrings. If no ring of type (i), (ii), (iii), (iv) or (v) satisfies P, then every ring with unity 1 and satisfying P is commutative.

We state the following known results.

Lemma 2.1 [4]. Let f be a polynomial in n non-commuting indeterminates
$x_1, x_2, \cdots, x_n$ with relatively prime integral coefficients. Then the following are equivalent.

(a) For any ring R satisfying the polynomial identity $f = 0, C$ is a nil ideal.

(b) For every prime p, $(GF(p))_2$ fails to satisfy $f = 0$.

(c) Every semiprime ring satisfying $f = 0$ is commutative.

Lemma 2.2 [7]. If R is a non-commutative ring satisfying (V), then there exists a factorsubring of R which is of type (i) or (ii).

Lemma 2.3 [3]. Let R be a ring in which for all x, y in R, there exists polynomial $f(X)$ in $X^2\mathbf{Z}[X]$ such that $[x - f(x), y] = 0$. Then R is commutative.

Now, we prove the following results called steps.

Step 2.1. Let R be a division ring satisfying the property (I). Then R is commutative.

Before proving Step 2.1, we begin with

Claim 2.1. Let R be a ring with unity 1 satisfying the property (I). If x is in U, then for each $y \in R$ there exists $q(X) \in X^2\mathbf{Z}[X]$ such that $[x, y - q(y)] = 0$.

Proof. Choose polynomials $b(X)$ in $X^2\mathbf{Z}[X]$ and $p(X), q(X)$ in $X\mathbf{Z}[X]$ such that

$$\{1 - p(yu^{-m}u^m)\}[yu^{-m}u^m - u^r b(yu^{-m}u^m)u^s, u]$$
$$\{1 - q(yu^{-m}u^m)\} = 0$$

or

$$\{1 - p(y)\}[y - u^r b(y)u^s, u]\{1 - q(y)\} = 0.$$

The above expression depends on a choice of u and y. This shows that either $1 - p(y) = 0, 1 - q(y) = 0$ or $[y - u^r b(y)u^s, u] = 0$. Clearly, in the first two cases one gets the required result. Now, we may assume that for unit $u \in U$ and arbitrary $y \in R$,

$$[y - u^r b(y) u^s, u] = 0. \tag{2.1}$$

Next, choose polynomial $b(X)$ in $X^2\mathbf{Z}[X]$ such that

$[y - u^{-r} b(y) u^{-s}, u^{-1}] = 0$. This implies that $[y - u^{-r} b(y) u^{-s}, u] = 0$,

$$[u, b(y)] = u^r [u, y] u^s. \tag{2.2}$$

In view of (2.1), choose the polynomial $c(X)$ in $X^2\mathbf{Z}[X]$ such that $[b(y) - u^r c(b(y)) u^s, u] = 0$; hence for $w(X) = c(b(X)) \in X^2\mathbf{Z}[X]$, we find that

$$[u, b(y)] = u^r\ [u, w(y)] u^s. \tag{2.3}$$

From (2.2) and (2.3), we obtain $u^r[u, y]u^s = u^r[u, w(y)]u^s$.
But $u \in U$; thus $[y - w(y), u] = 0$.

Proof of Step 2.1. For each $x, y \in R$, there exists $b(X)$ in $X^2\mathbf{Z}[X]$ such that $[x, y - q(y)] = 0$, by Claim (2.1). Hence R is a commutative ring by Lemma 2.3.

Remark 2.1. By making use of Remark 12 of [2] one can prove that if a ring R with unity satisfies the property (I), then U is commutative.

Step 2.2. Let R be a ring with unity 1 satisfying the property (II) or (III). Then $C \subseteq N$.

Proof. Let R satisfy (II). Take $(1 + y)$ for y in (II) and subtract (II) to get

$$(1 + y)^s [x, (1 + y)^m] = y^s [x, y^m].$$

As $x = e_{22} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$ and $y = e_{12} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ fail to satisfy the above polynomial identity in $(GF(p))_2, p$ a prime. Thus by Lemma 2.1, R has nil commutator ideal, that is $C \subseteq N$.

Similar arguments can be used to obtain the result if R satisfies (III).

Remark 2.2. In the hypothesis of Step 2.2, the coefficients of f(x) and g(x) are not relatively prime.

Step 2.3. Let B be a factorsubring of R. If a ring B is of type $(i)_l$ or (ii), then B does not satisfy (II).

Proof. Let B be of type $(i)_l$. Taking $x = e_{12}$ and $y = e_{11} + e_{12}$ in (II), we have

$$g(e_{12})[e_{12}^2 f(e_{12}, e_{11} + e_{12}]h(e_{12}) - (e_{11} + e_{12})^s[e_{12}, (e_{11} + e_{12})^n] = e_{12} \neq 0,$$

for some integers $m > 1, s \geq 0$ and polynomials $f(X), g(X), h(X)$ in $\mathbf{Z}[X]$. This implies that B does not satisfy (II). Suppose that $B = M_\sigma(F)$ is a ring of type (ii).
It is noticed that $N(B) = Fe_{12}$. On the contrary, suppose that B satisfies (II). Then for any $b \in N(B)$ and arbitrary unit u, there exist integers $m = m(b,u) > 1, n = n(b,u) > 1, s = s(b,u) \geq 0, t = t(b,u) \geq 0$ with the condition that m and n are relatively prime, and polynomials $f(X), \tilde{f}(X), g(X), \tilde{g}(X), h(X), \tilde{h}(X)$ in $\mathbf{Z}[X]$ such that

$$u^s[b, u^m] = g(b)[b^2 f(b), u]h(b)$$

and

$$u^t[b, u^n] = \tilde{g}(b)[b^2 \tilde{f}(b), u]\tilde{h}(b).$$

But $b^2 = 0$ and u is a unit. Then the last two equations imply that $[b, u^m] = 0$ and $[b, u^n] = 0$. The relative primeness of m and n show that $[b, u] = 0$. Since the non-central element $b = e_{12}$, this yields that e_{12} is central, a contradiction. Hence B does not satisfy (II).

Remark 2.3. If a ring B is of type $(i)_r$ or (ii), then using similar arguments as Step 2.1 with the choice of $x = e_{12}, y = e_{12} + e_{22}$ in (III), one can prove that B does not satisfy (III).

3. Commutativity of rings with unity 1

Theorem 3.1. Let R be a ring with unity 1 satisfying (I). Then R is commutative.

Proof Let R be a ring of the type (i).Suppose that R satisfies (I). Then in $(GF(p))_2, p$ a prime, we get
$\{1-p(e_{12}e_{22}^m)\}[e_{12}e_{22}^m - e_{22}^r b(e_{12}e_{22}^m)e_{22}^s, e_{22}]\{1-q(e_{12}e_{22}^m)\} = e_{12} \neq 0$,
for some $b(X) \in X^2\mathbf{Z}[X]$ and $p(X),\ q(X) \in X\mathbf{Z}[X]$. Thus, we get a contradiction and hence, no ring of type (i) satisfies (I).
Further, consider the ring $R = M_\sigma(F)$. Let R satisfy (I). Then take

$$x = \begin{pmatrix} \alpha & 0 \\ 0 & \sigma(\alpha) \end{pmatrix}, (\alpha \neq \sigma(\alpha)), \text{ and } y = e_{12}$$

such that

$$\{1-p(yx^m)\}\ [yx^m - x^r b(yx^m)x^s, x]\{1-q(yx^m)\} = (\alpha - \sigma(\alpha))\sigma(\alpha)^m e_{12} \neq 0,$$

for all $b(X) \in X^2\mathbf{Z}[X]$ and $p(X), q(X) \in X\mathbf{Z}[X]$. Thus, R is not of type (ii).
If R is of type (iii) and satisfies (I), then by Step 2.1, we get a contradiction.
Suppose that R is of type (iv). Let R satisfy (I). Then a careful scrutiny of the proof of Step 2.1 gives that there exist $u \in U$ and arbitrary $y \in R$ such that $y - y\ p(y) = 0, y - yq(y) = 0$ or $[u, y - b(y)] = 0$ for some $b(X) \in X^2\mathbf{Z}[X]$ and $p(X), q(X) \in X\mathbf{Z}[X]$. But in the present case if $t_1, t_2 \in T$, then $u = 1 + t_1$ is a unit and there exist $b(X) \in X^2\mathbf{Z}[X]$ and $p(X), q(X) \in X\mathbf{Z}[X]$ such that $t_2 - t_2 p(t_2) = 0, t_2 - t_2 q(t_2) = 0$ or $[t_2 - b(t_2), 1 + t_1] = 0$.
Thus, T is commutative by Lemma 2.3, a contradiction.
Finally, let R be of type (v). Let $t_1, t_2 \in T$ such that $[t_1, t_2] \neq 0$. Suppose that R satisfies (I). Then there exist polynomials $b(X)$ in $X^2\mathbf{Z}[X]$ and $p(X), q(X)$ in $X\mathbf{Z}[X]$ such that
$\{1 - p(t_2(1+t_1)^m)\}[t_2(1+t_1)^m - (1+t_1)^r b(t_2(1+t_1)^m)$
$(1+t_1)^s, 1+t_1]\{1 - q(t_2(1+t_1)^m)\} = 0$.
Using the above property T [T,T] = 0 = [T,T] T continuously, we get

$$\{1 - p(t_2(1+t_1)^m)\}[t_2(1+t_1)^m, 1+t_1]\{1 - q(t_2(1+t_1)^m)\} = 0$$

or

$$\{1 - p(t_2(1+t_1)^m)\}[t_2, t_1]\{1 - q(t_2(1+t_1)^m)\} = 0.$$

This implies that

$$[t_2, t_1] = 0.$$

Therefore T is commutative. This is a contradiction.

Hence we observe that no ring of type (i), (ii), (hi), (iv) or (v) satisfies (I) and by Proposition 2.1, R is a commutative ring.

Corollary 3.1. Let l, m, n be fixed non-negative integers and let R be a ring with unity 1. If for each $x, y \in R$, there exist a polynomial $b(X)$ in $X^2\mathbf{Z}[X]$ such that $[yx^m - x^n b(y)x^l, x] = 0$, then R is commutative.

Remark 3.1. Given the integral exponents m, r, s in the property (I) which is allowed to vary with the pair of ring's elements x and y, that is, if R satisfies either of the property (I)', then a careful scurtiny of the proof of Theorem 3.1 asserts that R has no factorsubring of type (i) or (ii). Further, if R satisfies the property (IV), then in view of Lemma 2.3, we get the following.

Theorem 3.2. Suppose that R is a ring with unity 1 satisfying (IV). Moreover, if R satisfies the property (I)', then R is commutative (and conversely).

4. Commutativity of one sided s-unital rings

Since there are non-commutative rings with R^2 being central, neither of these conditions guarantees the commutativity in arbitrary rings. Following [3] , a ring R is called left (resp. right) s-unital ring if $x \in Rx$ (resp. $x \in xR$). A ring R is called s-unital if and only if $x \in xR \cap Rx$ for all $x \in R$. If R is s-unital (resp. left or right s-unital), then for any finite subset F of R there exists an element $e \in R$ such that $ex = xe = x$ (resp. $ex = x$ or $xe = x$) for all $x \in F$. Such an element e will be called a pseudo (resp. a pseudo left or a pseudo right) identity of F in R.

We state the following lemma.

Lemma 4.1.[7] Let R be a left (resp. right) s-unital not a right (resp. left) s-unital, then R has a factorsubring of type $(i)_l$ (resp. $(i)_r$).

Theorem 4.1. Let R be a left s-unital ring with unity 1 satisfying (II). Then R is commutative (and conversely).

Proof It suffices to show that no ring of type $(i)_l$, (ii), (iii), or (iv) satisfies (II). Step 2.3 shows that no ring of type $(i)_l$ and type (ii) satisfies (II), and hence, by Lemma 4.1, R is also s-unital ring. Thus , by [7, Proposition 1], we can assume that R has unity 1. Applications of Step 2.2 and Lemma 2.1 give that no ring of type (iii) satisfies (II).

Let R be a ring of type (iv). Assume that $c, d \in T$ such that $[c, d] \neq 0$. Then there exist polynomials $f(X), \tilde{f}(X), g(X), \tilde{g}(X), h(X), \tilde{h}(X)$ in $\mathbf{Z}[X]$ such that

$$m[c,d] = (1+c)^s[(1+c)^m, d] = g(d)[d^2 f(d), c]h(d) = 0$$

and

$$n[c,d] = (1+c)^t[(1+c)^n, d] = \tilde{g}(d)[d^2 \tilde{f}(d), c]\tilde{h}(d) = 0.$$

By the relative primeness of m and n, the last two expressions give $[c, d] = 0$, a contradiction.

Hence no ring of type $(i)_l$, (ii), (iii) or (iv) satisfies (III) and in view of Proposition 2.1, R is commutative.

Theorem 4.2. Let R be a right s-unital ring satisfying (III). Then R is commutative (and conversely).

Proof Let R be a ring of type $(i)_r$. Suppose that R satisfies (IV). Then in $(GF(p))_2$, where p a prime, we have

$$g(e_{12})[e_{12}^2 f(e_{12}), e_{11} + e_{22}]h(e_{12}) - [e_{12}, (e_{11} + e_{12})^m](e_{11} + e_{22})^s = e_{12} \neq 0,$$

for some integers $m > 1, s \geq 0$ and polynomials $f(X), g(X), h(X)$ in $\mathbf{Z}[X]$. This implies that R does not satisfy (III).

Using similar arguments used to prove Theorem 4.1 with necessary variations, it can be shown that no ring of type (ii), (iii) or (iv) satisfies our hypothesis. Thus in view of Proposition 2.1, R is commutative.

Remark 4.1. Let R satisfy (IV) together with one of the properties (II)' and (III)'. Then using similar arguments as above and combining Lemma 4.1, and the proofs of Theorems 4.1 and 4.2, we get the following.

Theorem 4.3. Let R be a left (resp. right) s-unital rings satisfying (II)' (resp. (III)'). In addition, if R satisfies (IV), then R is commutative (and conversely).

Remark 4.2. The following example demonstrates that in the hypothesis of Theorems 4.1 and 4.2, the existence of both the conditions in the properties (II) and (III) is not superfluous (even if ring R has unity 1).

Example 4.1. Consider $R = \{\begin{pmatrix} a & b & c \\ 0 & a & d \\ 0 & 0 & a \end{pmatrix} \mid a, b, c, d \in GF(2)\}$. Then R is a non-commutative ring with unity satisfying the condition $y^t[x, y^4] = x^r[x^4, y]x^s$ where r, s and t may be any non-negative integers.

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References

[1] BELL, H. E., QUADRI, M. A. and KHAN, M. A., *Two commutativity theorems for rings*, Rad. Mat. **3** (1987), 255–260.

[2] CHACRON, M., *A commutativity theorem for rings*, Proc. Amer. Math. Soc. **59** (1976), 211–216.

[3] HERSTEIN, I. N., *Two remarks on commutativity of rings*, Canad. J. Math. 7 (1955), 411–412.

[4] KEZLAN, T. P., *A note on commutativity of semiprime PI- rings*, Math. Japonica **27** (1982), 267–268.

[5] KHAN, M. A., *On commutativity of right s-unital rings with polynomial constraints*, Jour. Inst. Math. & Comp. Sci. (Math. Ser.) **12** (1999), 47–51.

[6] KOMATSU, H. and TOMINAGA, H., *Chacron's condition and commutativity theorems*, Math. J. Okayama Univ. **31** (1989), 101–120.

[7] KOMATSU, H., NISHINAKA, T., and TOMINAGA, H., *On commutativity of rings*, Rad. Math. **6** (1990), 303–311.

[8] PSOMOPOULOS, E., *Commutativity theorems for rings and groups with constraints on commutators*, Internat. J. Math. and Math. Sci. 7 (1984), 513–517.

[9] PUTCHA, M. S. and YAQUB, A, *Rings satisfying polynomial constraints*, J. Math. Soc. Japan **25** (1973), 115–124.

[10] SEARCOID, M.O. and MACHALE, D, *Two elementary generalisations for boolean rings*, Amer. Math. Monthly **93** (1986), 121–122.

[11] STREB, W., *Zur Struktur nichtkommutativer ringe*, Math. J. Okayama Univ. **31** (1989), 135–140.

[12] TOMINAGA, H . and YAQUB, A., *Commutativity theorems for rings with constraints involving a commutative subset*, Resultate Math. 11 (1987), 186–192.

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