İstanbul Üniv. Fen Fak. Mat Der. **59** (2000), 103-109

LACUNARY STATISTICAL h-REGULARITY IN TOPOLOGICAL GROUPS

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Abstract

In this paper, the concept of lacunary statistical h-regularity is introduced in topological groups and equivalence of lacunary statistical generalized h-multiplicativity and lacunary statistical h-regularity is proved for triangular limitation methods.

1 Introduction

In [3], a complex number sequence $(x(n))$ is called lacunary statistically convergent to a number l if for each $\varepsilon > 0$

$$\lim_{r\to\infty}(h_r)^{-1}\left|\{n \in I_r : \varepsilon \leq dist(x(n), l)\}\right| = 0$$

where $I_r = (k_{r-1}, k_r]$ and $\theta = (k_r)$ is a lacunary sequence i.e. (k_r) is an increasing sequence of positive integers such that $k_0 = 0$ and $h_r = k_r - k_{r-1} \to \infty$ as $r \to \infty$. This concept was generalized by the author in [1]. $|A|$ denotes the number of elements of A.

The purpose of this paper is to introduce the concept of lacunary statistical h-regularity in topological groups, and to study the relationship between lacunary statistical h-regularity and lacunary statistical generalized h-mutiplicativity of [2].

Throughout this paper, X and Y will denote Abelian topological Hausdorff groups,which satisfy the first axiom of countability, both written additively. Whenever we write G,we will mean that G is either X or Y. For a subset A of G , $s(A)$will denote the class of all sequences $(x(n))$ such that $x(n) \in A$ for $n = 1, 2, \ldots$; $c(G)$, $c_0(G)$, $C(G)$ and $S(G)$ will denote the set of all convergent sequences, the set of all convergent-to-zero sequences, the set of all Cauchy sequences and the set of all statistically convergent sequences in G, respectively. To avoid the confusion recall that $s(G)$ is the set of all sequences in G.

In [1], the author defined lacunary statistical convergence in topological groups as follows:
A sequence $(x(n))$ in G is lacunary statistically convergent to an element l of G if for each neighborhood U of 0,

$$\lim_{r\to\infty} (h_r)^{-1} |\{n \in I_r : x(n) - l \notin U\}| = 0.$$

In this case, we write $S_\theta - \lim_{n\to\infty} x(n) = l$ or $x(n) \to l(S_\theta)$. By $S_\theta(G)$, we denote the set of all lacunary statistical convergent sequences in G. In particular, by $S_\theta^0(G)$, we will denote the set of all lacunary statistically convergent-to-zero sequences in G.

We note that every lacunary statistically convergent sequence has only one limit for a certain lacunary sequence $\theta = (k_r)$ that is, if a sequence is lacunary statistically convergent to l_1 and l_2 then $l_1 = l_2$.

A sequence of additive functions $f(m)$ whose domain, for each m, is some subset of $s(X)$ and contains the set $C(X)$ and whose range is contained in Y is said to be a limitation method from X to Y. The intersection of the domains of the functions $f(m)$ is called the domain of the limitation method $(f(m))$. For the sake of simplicity, we write f instead of $(f(m))$ when no confusion arises. If the domain of $(f(m))$ is all of $s(X)$ and the equality

$$f(m : x(1), x(2), \ldots, x(m), \ldots) = f(m : y(1), y(2), \ldots, y(m), \ldots)$$

holds for every positive integer m and every pair of sequences $(x(n)), (y(n))$ for which $x(n) = y(n)$ for $n = 1, 2, \ldots m$, then the limitation method $(f(m))$ is said to be triangular. In case $(f(m))$ is a triangular limitation method, we will write $f(m : x(1), x(2), \ldots, x(m))$ instead of $f(m : x(1), x(2), \ldots, x(m), \ldots)$.

$(c_0(X), S_\theta^0(Y))$ will denote the set of all limitation methods which map all convergent-to-zero sequences in X to lacunary statistically convergent-to-zero sequences in Y.

In [2], a limitation method on X, $(f(m))$, is called lacunary statistically generalized h-multiplicative if for each neighborhood of U of 0,

$$S_\theta - \lim_{m\to\infty} f(m : x(1), x(2), \ldots, x(m), \ldots) = h(\lim_{n\to\infty} x(n))$$

for any convergent sequence $(x(n))$ where h is a homomorphism on X.One can give this definition for a limitation method from X to Y.

Now we give the following definition :

We call a limitation method from X to Y, $(f(m))$, lacunary statistically h-regular if for each neighborhood U of 0, there exists a constant positive integer M such that

$$\lim_{r\to\infty}(h_r)^{-1}\left|\{k\in I_r : h(x(M))-f(k:x(1),x(2),\ldots,x(k),\ldots)\notin U\}\right|=0$$

for each Cauchy sequence $(x(n))$ in X where h is a continuous function on X.

2 Summability

Here we give a lemma and prove a theorem giving necessary and sufficient conditions for a triangular limitation method from X to Y to be lacunary statistically h-regular. We obtain that a triangular limitation method is lacunary statistically h-regular if and only if it is lacunary statistically generalized h-multiplicative. First we give the following theorem whose special case, i.e. the case $X=Y$, was proved in [2] and will be used in our proofs.

Theorem 1 *A limitation method from X to Y, f , belongs to $(c_0(X), S_\theta^0(Y))$ if and only if it satisfies the following conditions:*
(1) $S_\theta-\lim_{m\to\infty} f(m:0,0,\ldots,0,x,0,\ldots)=0$ for all x in X in any fixed position,
(N3) For each neighborhood U of 0 in Y and for each positive number ε, there is a neighborhood V of 0 in X such that for all $(x(n))\in c_0(X)\cap s(V)$, there exists a positive integer M such that $m>M$ implies that

$$(h_r)^{-1}\left|\{k\in I_r : f(k:x(1),x(2),\ldots,x(k),\ldots)\notin U\}\right|<\varepsilon.$$

Now we give the following:

Lemma 1 *A limitation method from X to Y, $(f(m))$, is lacunary statistically h-regular if and only if it satisfies condition (1) and the following conditions:*
(H2) $S_\theta-\lim_{m\to\infty} f(m:x,x,\ldots,x,\ldots)=h(x)$ for all x in X.
(C3) For each neighborhood V of 0 in X such that for all $(x(n))\in C(X)\cap s(V)$, there exists a positive integer M such that $m>M$ implies that

$$(h_r)^{-1}\left|\{k\in I_r : f(k:x(1),x(2),\ldots,x(k),\ldots)\notin U\}\right|<\varepsilon.$$

Proof. Sufficiency. Let $(x(n))$ be any element of $C(X)$. Let W be any neighborhood of 0 in Y and ε a positive number. We may choose a symmetric neighborhood U of 0 in Y such that $U + U + U \subseteq W$. By condition (C3), there exists a neighborhood V_1 of 0 in X such that for all Cauchy sequences in V_1, there is a positive integer N so that $m > N$ implies that

$$(h_r)^{-1}\left|\{k \in I_r : f(k : x(1), x(2), \ldots, x(k), \ldots) \notin U\}\right| < \frac{\varepsilon}{3}.$$

Since h is continuous, there exists a neighborhood V_2 of 0 in X such that $h(V_2) \subseteq U$. Write $V = V_1 \cap V_2$. Since $(x(n))$ is Cauchy sequence in X, there is a positive integer M such that $m, n > M$ implies that $x(n) - x(m) \in V$. Then the sequence

$$(0, 0, \ldots, x(M+2) - x(M+1), x(M+3) - x(M+1), x(M+4) - x(M+1), \ldots)$$

is in $C(X) \cap s(V)$, so there exists an integer $M(1) > M$ such that $m > M(1)$ implies that

$$\begin{aligned}(h_r)^{-1}|\{k \in I_r : f(k : 0, 0, \ldots, 0, x(M+2) - x(M+1),\\ x(M+3) - x(M+1), \ldots) \notin U\}| < \frac{\varepsilon}{3}.\end{aligned}$$

By condition (1), there exists an integer $M(2) > M(1)$ such that if $m > M(2)$, then

$$\begin{aligned}(h_r)^{-1}|\{k \in I_r : f(k : x(1) - x(M+1), \ldots,\\ x(M) - x(M+1), 0, 0, \ldots) \notin U\}| < \frac{\varepsilon}{3}.\end{aligned}$$

Finally, by condition (H2), there is an integer $M(3) > M(2)$ such that if $m > M(3)$ then

$$\begin{aligned}(h_r)^{-1}|\{k \in I_r : f(k : x(M+1), x(M+1), \ldots, x(M+1), \ldots)\\ -h(x(M+1)) \notin U\}| < \frac{\varepsilon}{3}.\end{aligned}$$

Thus if $m > M(3)$ then

$$(h_r)^{-1}\left|\{k \in I_r : f(k : x(1), x(2), \ldots, x(k), \ldots) \notin W\}\right| \leq$$

$$\begin{aligned}\leq (h_r)^{-1}|\{k \in I_r : f(k : 0, 0, \ldots, 0, x(M+2) - x(M+1),\\ x(M+3) - x(M+1), \ldots) \notin U\}|\end{aligned}$$

$$+(h_r)^{-1}|\{k \in I_r : f(k : x(1) - x(M+1), \ldots, x(M) - x(M+1), 0, 0, \ldots) \notin U\}|$$

$$+(h_r)^{-1}|\{k \in I_r : f(k : x(M+1), x(M+1), \ldots, x(M+1), \ldots) - h(x(M+1)) \notin U\}|$$

$$< \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon$$

This completes the proof of the sufficiency.
Necessity. The necessity of (H2) is obvious. We see that h may be defined as

$$h(x) = S_\theta - \lim_{m\to\infty} f(m : x, x, \ldots, x, \ldots)$$

for all x in X. Thus

$$h(x+y) = S_\theta - \lim f(m : x+y, x+y, \ldots, x+y, \ldots)$$

$$= S_\theta - \lim_{m\to\infty} [f(m : x, x, \ldots, x, \ldots) + f(m : y; y, \ldots, y, \ldots)]$$

$$= S_\theta - \lim_{m\to\infty} f(m : x, x, \ldots, x, \ldots) + S_\theta - \lim_{m\to\infty} f(m : y, y, \ldots, y, \ldots)$$

$$= h(x) + h(y)$$

and also $h(0) = 0$. Hence $f \in (c_0(X), S_\theta^0(Y))$.Thus by theorem 1, f satisfies the condition (N3) and the condition (1). This completes the proof of the necessity, hence the proof of the lemma.

Theorem 2 *Let f be triangular limitation method from X to Y satisfying the condition (1). Then the condition (N3) is equivalent to the condition (C3) or either of the following conditions:*

(T3) For each neighborhood U of 0 in Y and for each positive ε, there exists a neighborhood V of 0 in X and a positive integer N such that if m and k

are integers satisfying $m > k > N$ and if $x(n)$ is in V for $n = k, k+1, \ldots, m$ then

$$(h_r)^{-1} |\{k \in I_r : f(i : 0, 0, \ldots, x(k), x(k+1), x(k+2), \ldots, x(i)) \notin U\}| < \varepsilon,$$

(3) For each neighborhood U of 0 in Y and for each positive ε, there is a neighborhood of 0 in X such that for all $(x(n)) \in c(X) \cap s(V)$, there exists a positive integer M such that $m > M$ implies that

$$(h_r)^{-1} |\{k \in I_r : f(k : x(1), x(2), \ldots, x(k), \ldots) \notin U\}| < \varepsilon,$$

(see [2]).

Theorem 3 *A triangular limitation method from X to Y, $(f(m))$, is lacunary statistically h-regular if and only if it satisfies the condition (T3) and the following conditions:*

(T1) $S_\theta - \lim_{m\to\infty} f(m : 0, 0, \ldots, 0, x, 0, \ldots, 0) = 0$ for all x in X in any fixed position,

(TH2) $S_\theta - \lim_{m\to\infty} f(m : x, x, \ldots, x, \ldots x) = h(x)$ for all x in X.

Proof. It is clear that the conditions (T1) and (TH2) are equivalent to the conditions (1) and (H2). Then by Theorem 2, (T3) is equivalent to the condition (N3). So the conditions of this theorem are equivalent to those of Lemma 1, and hence $(f(m))$ is lacunary statistically h-regular if and only if it satisfies the conditions (T1), (TH2) and (T3).

Theorem 4 *A triangular limitation method from X to Y, $(f(m))$, is lacunary statistically generalized h-multiplicative if and only if it satisfies the conditions (T1), (TH2) and (T3). (See [2] for a proof of special case i.e. when $X = Y$).*

Corollary 1 A triangular limitation method from X to Y, f ,is lacunary statistically h-regular if and only if it is lacunary statistically generalized h-multiplicative.

References

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