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A Study On The Parallel Normal Deformation In A Subspace W_n Of A Weyl Space W_m

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The parallel tangent and parallel Frenet deformation in Weyl space have already been investigated [7]. In this study some problems of deformations of subspace W_n of a Weyl space W_m were investigated. The necessary conditions that the normal $\underset{\nu}{N}^a$ $(\nu = n+1,...,m)$ denoting the contravariant components of a system of unit normals to W_n be deformed parallelly and parallel tangent deformation for subspace W_n of W_m were obtained.

1. Introduction

An n- dimensional manifold W_n is said to be a Weyl space if it has a conformal metric tensor g_{ij} and a symmetric connection ∇_k satisfying the compatibility condition given by the equation

$$\nabla_k g_{ij} - 2T_k g_{ij} = 0 \qquad (1.1)$$

where T_k denotes a covariant vector field [1], [3],[4].

Under a renormalization of the fundamental tensor of the form

$$\breve{g}_{ij} = \lambda^2 g_{ij} \qquad (1.2)$$

the complementary vector field T_k is transformed by the law

$$\breve{T}_k = T_k + \partial_k \ln \lambda \qquad (1.3)$$

where λ is a scalar function defined on W_n.

A quantity A is called a satellite of weight $\{p\}$ of the tensor g_{ij}, if it admits a transformation of the form

$$\breve{A} = \lambda^p A \qquad (1.4)$$

under the renormalization (1.2) of the metric tensor g_{ij} [1], [3],[4].

The prolonged covariant derivative of a satellite A of the tensor g_{ij} of weight $\{p\}$ is defined by [1], [2],[3]

$$\dot{\nabla}_k A = \nabla_k A - p T_k A . \qquad (1.5)$$

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We note that the prolonged covariant derivative preserves the weight.

The generalized derivative of a satellite A of the tensor g_{ij} with weight $\{p\}$ is defined by [1], [3],[4]

$$\overset{\sqcap}{\partial}_k A = \partial_k A - pT_k A \ . \qquad (1.6)$$

Let $W_n(g_{ij}, T_k)$ be a subspace which is immersed in a Weyl space $W_m(g_{ab}, T_c)$. We shall denote the coordinates in W_n and W_m by $u^i (i = 1,...,n)$ and $x^a (a = 1,..,m)$; the fundamental tensors by g_{ij} and g_{ab}; the complementary vector T_k and T_c, respectively.

The coefficients of the metrics are connected by the relation

$$g_{ab} \frac{\partial x^a}{\partial u^i} \frac{\partial x^b}{\partial u^j} = g_{ij} \ . \qquad (1.7)$$

On the other hand, the complementary vectors T_c and T_k, relative to W_m and W_n are connected by the relation

$$T_k = x_k^a T_a \ . \qquad (1.8)$$

So, the prolonged covariant derivative of A, related to W_n and W_m, are defined by

$$\dot{\nabla}_k A = x_k^a \dot{\nabla}_a A \quad (k = 1,2,...,n \ ; \ a = 1,2,...,m) \qquad (1.9)$$

where x_i^a denotes the covariant derivatives of x^a with respect to u^i (in the sequel, the indices $i, j, k....$ will run from 1 to n, while the indices $a, b, c...$ will run from 1 to m).

Let $\underset{\nu}{N}^a$ $(\nu = n+1,...,m)$ be the contravariant components in W_m of a system (m-n) normals to W_n.

It is obvious that they satisfy the relations

$$g_{ab} \underset{\nu}{N}^a \underset{\mu}{N}^b = \delta_\mu^\nu \qquad (1.10)$$

and

$$g_{ab} \underset{\mu}{N}^a x_i^b = 0 \ . \qquad (1.11)$$

These $\underset{\nu}{N}^a$ $(\nu, \mu, \tau, \rho = n+1,...,m)$ vectors as seen from (1.5) are normalized by the condition

$$g_{ab} \underset{\nu}{N}^a \underset{\nu}{N}^b = 1 \ .$$

It follows immediately from $g_{ab} \underset{\nu}{N}^a \underset{\nu}{N}^b = 1$ that $\underset{\nu}{N}^a$ are satellites of g_{ab} with weight $\{-1\}$, [4].

Since the functions x^a are invariants for transformations of u's in W_n, their first prolonged covariant derivative with respect to u's is the same as their covariant derivative or partial derivative, that is

$$\nabla_i x = x^a{}_{,i} = x^a_i = \frac{\partial x^a}{\partial u^i}. \qquad (1.12)$$

On the other hand, the weight of x^a_i is $\{0\}$, [2], [3], [5]. So

$$\overset{\square}{\nabla}_j x^a_i = \nabla_j x^a_i \; . \qquad (1.13)$$

According to (1.13) the covariant derivative of x^a_i with respect to u^j is

$$\overset{\square}{\nabla}_j x^a_i = \sum_\nu \underset{\nu}{w_{ij}} \underset{\nu}{N^a} \qquad (1.14)$$

where the coefficients $\underset{\nu}{w_{ij}}$ are the components of a symmetric covariant tensor of the second order. Using (1.10) and (1.13), it is easily seen that the prolonged covariant derivatives of $\underset{\nu}{N^a}$ with respect to u^i is [5]

$$\overset{\square}{\nabla}_i \underset{\nu}{N^a} = - \underset{\nu}{w_{ij}} g^{kj} x^a_j + \sum_\mu \underset{\mu\nu}{\theta_i} \underset{\mu}{N^a}. \qquad (1.15)$$

2. Paralel Normal Deformation

Let the point of $W_n(g_{ij},T_\kappa)$ be infinitesimally displaced in $W_m(g_{ab},T_c)$ such that the coordinates $\bar{x}^a$ of the point $\bar{P}$ (of the deformed subspace $\bar{W}_n$) corresponding to the point **P** of the subspace W_n are given by

$$\bar{x}^a = x^a + \varepsilon \lambda^a \qquad (2.1)$$

where ε is an infinitesimal constant with weight $\{1\}$ and λ^a are components of the contravariant vector with weight $\{-1\}$ [6],[7],[10]. On the other hand, let T be tensor field in W_m and let the points of the manifold undergo an infinitesimal transformation of the form (2.1). The value of the tensor field at the new points $\bar{x}^a$ will be given by

$$T(\bar{x}) = T(x) + \varepsilon \lambda^b \dot{\partial}_b T. \qquad (2.2)$$

In addition, it will be supposed that without loss of the generality that $\bar{x}^a$ are also functions of u's

$$\bar{x}^a = \bar{x}^a(u^1, u^2, \ldots, u^n). \qquad (2.3)$$

Let us give and prove the different statements of the two theorems for the parallel normal deformation.

Theorem 1: The condition that the normal $\underset{\nu}{N^a}$ of the space $W_n(g_{ij},T_\kappa)$ immersed in a Weyl space $W_m(g_{ab},T_c)$ be deformed parallely during an infinitesimal deformation defined by (2.1) is $g_{ab} \underset{\mu}{N^a} \overset{\square}{\nabla}_i \lambda^b = 0$.

Proof: $\underset{\nu}{N}^a$ $(\nu = n+1,...,m)$ at the point P of the subspace W_n denote the components of $(m-n)$ mutually orthonormals with weight $\{-1\}$ normalized by the condition

$$g_{ab}\underset{\nu}{N}^a\underset{\nu}{N}^b = 1 \ . \tag{2.4}$$

The values of covariant components of the fundamental tensor of W_m at $\overline{P}$ which is in the deformed subspace $\overline{W}_n$ are given by

$$\overline{g}_{ab} = g_{ab} + \varepsilon\,\lambda^c (g_{ad}W^d_{bc} + g_{bd}W^d_{ac}) \tag{2.5}$$

[7],[9].

On the other hand let us use the symbol $\dot{\overline{\nabla}}$ for the prolonged covariant derivative in the deformed subspace $\overline{W}_n$.

$\underset{\mu}{\overline{N}}^a$ $(\mu = n+1,..,m)$ at the point $\overline{P}$ of the deformed subspace $\overline{W}_n$ denote the components of $(m-n)$ mutually orthogonal normals with weight $\{-1\}$ normalized by the condition

$$\overline{g}_{ab}\underset{\mu}{\overline{N}}^a\underset{\mu}{\overline{N}}^b = 1 \tag{2.6}$$

and

$$\overline{g}_{ab}\underset{\mu}{\overline{N}}^a\dot{\overline{\nabla}}_i\overline{x}^b = 0 \ . \tag{2.7}$$

$\underset{\nu}{\overline{N}}^a$ $(\nu = n+1,..,m)$ at the point $\overline{P}$ of the deformed subspace $\overline{W}_n$ denote the components of $(m-n)$ mutually orthonormals with weight $\{-1\}$ normalized by the condition $\overline{g}_{ab}\underset{\mu}{N}^a\underset{\mu}{N}^b = 1$ and $\underset{\mu}{\overline{\overline{N}}}^a$ is a vector with weight $\{-1\}$ normalized by the condition $\overline{g}_{ab}\underset{\nu}{\overline{\overline{N}}}^a\underset{\nu}{\overline{\overline{N}}}^b = 1$ and parallel to unit normal $\underset{\mu}{N}^a$ at P.

By using (1.5), (1.6)

$$\dot{\overline{\nabla}}_i\overline{x}^a = \frac{\partial \dot{\overline{x}}}{\partial u^i} = \overline{\nabla}_i x^a = \frac{\partial \overline{x}}{\partial u^i} \tag{2.8}$$

can be written.

By using [7],[8] and [10] we can define the vectors $\delta\underset{\nu}{N}^a$ and $\Delta\underset{\nu}{N}^a$, respectively as

$$\delta\underset{\nu}{N}^a = \underset{\nu}{\overline{N}}^a - \underset{\nu}{\overline{\overline{N}}}^a \tag{2.9}$$

and

$$\Delta\underset{\nu}{N}^a = \lim_{\delta s\to 0}\frac{\delta\underset{\nu}{N}^a}{\delta s} \tag{2.10}$$

where δs define as

$$\delta s = \sqrt{g_{ab}\,\delta x^a \delta x^b} = \varepsilon\lambda \ . \tag{2.11}$$

On the other hand, the condition that the unit normal $\underset{\nu}{N}^a$ be deformed parallely is written by using [7] and [8] as

$$\Delta \underset{\nu}{N}^{a} = 0 \qquad (2.12)$$

According to (2.10)

$$\delta \underset{\nu}{N}^{a} = 0 \qquad (2.13)$$

and according to (2.9), it is seen that

$$\underset{\nu}{\overline{N}}^{a} = \underset{\nu}{\overline{\overline{N}}}^{a}. \qquad (2.14)$$

Then, from [7],

$$\underset{\nu}{\overline{\overline{N}}}^{a} = \underset{\nu}{N}^{a} - W_{bc}^{a} \underset{\nu}{N}^{b} \cdot \varepsilon \lambda^{c}. \qquad (2.15)$$

By using (2.14) in (2.15) it follows that,

$$\underset{\nu}{\overline{N}}^{a} = \underset{\nu}{N}^{a} - \varepsilon W_{bc}^{a} \underset{\nu}{N}^{b} \lambda^{c}. \qquad (2.16)$$

From (2.1) and (2.8), it follows that

$$\frac{\dot{\partial} \overline{x}^{a}}{\partial u^{i}} = \frac{\dot{\partial} x^{a}}{\partial u^{i}} + \varepsilon \frac{\dot{\partial} \lambda^{a}}{\partial u^{i}}. \qquad (2.17)$$

or

$$\dot{\overline{\nabla}}_{i} x^{a} = \dot{\nabla}_{i} x^{a} + \varepsilon \frac{\dot{\partial} \lambda^{a}}{\partial u^{i}} \qquad (2.18)$$

If we use (2.5), (2.16) and (2.18) in (2.7) and neglecting the terms of the second and higher order of ε, the following equations can be written;

$$\left[g_{ab} + \varepsilon \lambda^{c} (g_{ad} W_{bc}^{d} + g_{bd} W_{ac}^{d}) \right] \left[\underset{\mu}{N}^{a} + \varepsilon \lambda^{f} W_{ef}^{a} \underset{\mu}{N}^{e} \right] \left[\dot{\nabla}_{i} x^{b} + \varepsilon \frac{\dot{\partial} \lambda^{b}}{\partial u^{i}} \right] = 0$$

$$g_{ab} \underset{\mu}{N}^{a} \frac{\partial \lambda^{b}}{\partial u^{i}} + g_{ab} W_{dc}^{b} \lambda^{c} \underset{\mu}{N}^{a} x_{i}^{d} = 0 \qquad (2.20)$$

$$\underset{\mu}{N}_{b} \left(\frac{\dot{\partial} \lambda^{b}}{\partial u^{i}} + W_{dc}^{b} \lambda^{c} x_{i}^{d} \right) = 0 \qquad (2.21)$$

and by using (1.6),

$$\underset{\mu}{N}_{b} \left(\frac{\partial \lambda^{b}}{\partial u^{i}} + T_{i} \lambda^{b} + W_{dc}^{b} \lambda^{c} x_{i}^{d} \right) = 0 \qquad (2.22)$$

and by using [9],

$$\underset{\mu}{N}_{b} (\nabla_{i} \lambda^{b} + T_{i} \lambda^{b}) = 0 \qquad (2.23)$$

and from (1.5)

$$\underset{\mu}{N}_{b} \dot{\nabla}_{i} \lambda^{b} = 0 \qquad (2.24)$$

So, the teorem is proved.

Now, let us prove the following theorem which is the another form for the condition of parallel normal deformation.

Theorem 2: The condition of parallel deformation of the normal $\underset{\mu}{N}^a$ of the subspace $W_n(g_{ij},T_k)$ immersed in a Weyl space $W_m(g_{ab},T_c)$ is

$$\underset{\mu}{p^k w_{kj}} + (\overset{\square}{\nabla}_j \underset{\mu}{c}) + \Sigma \underset{\tau}{c} \underset{\mu\tau}{\theta_j} = 0. \qquad (2.25)$$

Proof: λ^a defined in [10] as a vector in W_m being expressed linearly in terms of any m vectors not lying in the same geodesic surface can be expressed by tangential and normal component with respect to W_n.

$$\lambda^a = p^i x_i^a + \Sigma \underset{\nu}{c} \underset{\nu}{N}^a \qquad (2.26)$$

where

$$\underset{\nu}{c} = g_{ab} \lambda^a \underset{\nu}{N}^b = \lambda \cos \underset{\nu}{\theta}. \qquad (2.27)$$

Taking the prolonged covariant derivative of (2.26) with respect to u^j and using (1.14), (1.15), it follows that

$$\begin{aligned} \dot{\nabla}_j \lambda^b &= [(\overset{\square}{\nabla}_j p^k) - \underset{\mu}{\Sigma} \underset{\nu}{c} \underset{\nu}{w}_{jl} g^{lk}] x_k^a + \\ &+ \underset{\nu}{\Sigma} (p^k \underset{\nu}{w}_{kj} + \overset{\square}{\nabla}_j \underset{\nu}{c} + \underset{\mu}{\Sigma} \underset{\mu}{c} \underset{\nu\mu}{\theta}_j) \underset{\nu}{N}^a \end{aligned} \qquad (2.28)$$

by putting (2.28) in

$$\underset{\mu}{N_b} \overset{\square}{\nabla}_j \lambda^b = 0$$

and also using (1.10) and (1.11) we get

$$\underset{\mu}{p^k w_{kj}} + \overset{\square}{\nabla}_j \underset{\mu}{c} + \Sigma \underset{\tau}{c} \underset{\mu\tau}{\theta_j} = 0. \qquad (2.29)$$

So, the theorem is proved.

Parallel tangent deformation in Weyl space has been already studied [7]. By using these results as well let us prove the following theorem.

Theorem 3: If all the tangents x_i^a of a subspace W_n immersed in W_m are displaced parallelly during an infinitesimal deformation, then so are all the normal of W_n in W_m

Proof: Let x_i^a be components of a vector at P tangential to the curve of parameter u^i in W_n. The components $\bar{x}_i^a$ of the corresponding tangential vector at $\bar{P}$ in the deformed subspace $\bar{W}_n$.

Let $\bar{\bar{x}}_i^a$ be a vector at $\bar{P}$ parallel to x_i^a.

In this case, by using [7], [8] and [10], $\dfrac{\overset{\bullet}{\partial}\, \bar{x}^a}{\partial u^i} = \dfrac{\overset{\bullet}{\partial}\, x^a}{\partial u^i} + \varepsilon \dfrac{\overset{\bullet}{\partial}\, \lambda^a}{\partial u^i}$, $\dfrac{\overset{\bullet}{\partial}\, \bar{\bar{x}}^a}{\partial u^i} = \dfrac{\overset{\bullet}{\partial}\, x^a}{\partial u^i} - W^a_{bc} x^b_i \lambda^c \varepsilon$ and $\delta x^a_i = \varepsilon (\dfrac{\overset{\bullet}{\partial}\lambda^a}{\partial u^i} + W^a_{bc} x^b_i \lambda^c)$ are written.

Then, according to, (2.9), (2.10), (2.11) and (2.12) we write $\delta x^a_i = 0$,

$$\frac{\overset{\bullet}{\partial}\, \lambda^a}{\partial u^i} + W^a_{bc} x^b_i \lambda^c = 0 \qquad (2.30)$$

i.e.

$$\overset{\bullet}{\nabla}_i \lambda^a = 0 \qquad (2.31)$$

This condition and (2.24) are combined and the theorem is proved

3. Special cases:

I. Let $W_n(g_{ij}, T_k)$ be a hypersurface which is immersed in a Weyl space $W_{n+1}(g_{ab}, T_k)$. The following equations are given in [11]

$$\lambda^b = p^k x^b_k + c N^b \qquad (3.1)$$

$$\overset{\square}{\nabla}_k x^a_i = w_{ik} N^a \qquad (3.2)$$

$$\overset{\square}{\nabla}_j \lambda^b = (\nabla_j p^m - c\, g^{km} w_{jk}) x^b_m + (w_{jm} p^m + \overset{\square}{\nabla}_j c) N^b \qquad (3.3)$$

By considering the above equations and the special case of (2.24), $N_b \overset{\square}{\nabla}_j \lambda^b = 0$ then (2.29) becomes

$$\overset{\square}{\nabla}_i c + p^j w_{ji} = 0. \qquad (3.4)$$

II. If the deformation is in W_n, that is the vector λ^a lies in W_n, λ^a is written as

$$\lambda^a = p^i x^a_i \qquad (3.5)$$

As can be seen from (2.26), all $\underset{\tau}{c}$ are zero. In this case, the condition (2.29) becomes

$$p^k \underset{\mu}{w_{kj}} = 0. \qquad (3.6)$$

If W_n is also totally geodesic according to enveloping space W_m, $\underset{\mu}{w_{kj}} = 0$.

In this case the condition (2.29) is identically satisfied.

III. $\lambda^a = \sum \underset{\nu}{c} \underset{\nu}{N^a}$, then the condition (2.27) becomes

$$\overset{\square}{\nabla}_j \underset{\mu}{c} + \sum \underset{\tau}{c} \underset{\mu\tau}{N^a} = 0 \qquad (3.7)$$

If the deformation λ^a is only along $\underset{\mu}{N^a}$, the λ^a becomes

$$\lambda^{a} = \underset{\mu}{c} \underset{\mu}{N}^{a} \qquad (3.8)$$

It is clear that $\underset{\mu}{c} = \lambda$ and $\underset{\nu}{c} = 0 \quad (\mu \neq \nu)$.

In this case, the condition (3.7) if $\underset{\mu}{c} = const.$, i.e. the deformation is of constant magnitude.

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