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VEBLEN IDENTITIES AND THEIR EQUIVALENCE IN GENERALIZED FINSLER SPACES

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ABSTRACT

The generalized Riemannian space was developed by L.P. Eisenhart [1] . Some aspects of generalized Finsler spaces were studied by A.C.Shamihoke [3] and S.P.Singh [5,6]. The present author and J.K.Gatoto [7] have obtained Veblen identities in a conformal Finsler space . The objects of the present paper is to define Veblen identities and obtain the equivalence relation between **B**ianchi and Veblen identities in the generalized Finsler space GF_n .

INTRODUCTION

We consider the n-dimensional Finsler space in which the metric tensor $g_{ij}(x,\dot{x})$ is non-symmetric in general . The round and square brackets will be used to denote its symmetric and skew-symmetric parts respectively , that is

$$g_{(ij)} = \frac{1}{2}(g_{ij} + g_{ji}) \ , \quad g_{[ij]} = \frac{1}{2}(g_{ij} - g_{ji})$$

The space endowed with this metric tensor is known as generalized Finsler space and we denote it by GF_n .

The connection parameters for the locally Minkowskian and locally Euclidean GF_n are denoted by P^{*i}_{jk} and Γ^{*i}_{jk} respectively .

Let X^i be a vector field of GF_n , then the two processes of differentiation are defined as

$$(1.1) \qquad X^i_{,j} = \partial_j X^i + \partial_h X^i \partial_j \dot{x}^h + P^{*i}_{kj} X^k \ ,$$

and

(1.2) $$X^i\big|_j = \partial_j X^i - \dot{\partial}_h X^i \Gamma^h_{kj} \frac{\dot{x}^k}{F} + \Gamma^{*i}_{kj} X^k \quad ,$$

where $\Gamma^i_{jk} = \Gamma^{*i}_{jk} + C^i_{jh}\Gamma^{*h}_{rk}\dot{x}^r$, $C_{ijk} = \frac{1}{4}\dot{\partial}^3_{ijk}F^2(x,\dot{x})$, $\partial_h = \partial/\partial x^j , \dot{\partial}_h/\partial \dot{x}^h$.

The commutation formulae involving the curvature tensor fields are given as under :

(1.3) $$2X^i_{,[jk]} = X^h \widetilde{K}^i_{hkj} - 2X^i_{,h}\Delta^h_{[jk]}$$

and

(1.4) $$2X^i\big|_{[jk]} = \dot{\partial}_h X^i K^h_{0jk} F + X^h K^i_{hkj} - 2X^i\big|_h \Delta^h_{[jk]} \quad ,$$

where

(1.5) $$\Gamma^{*i}_{[jk]} = P^{*i}_{[jk]} = \Delta^i_{[jk]} \quad , \quad X^i_{,[jk]} = \frac{1}{2}\left(X^i_{,jk} - X^i_{,kj}\right)$$

and

(1.6) $$K^i_{okh} = K^i_{jkh} l^r \ .$$

The unit vector l^j satisfies the relations

(1.7) $$l^j = \frac{\dot{x}^j}{F} \ , \ l^j\big|_k = 0 \ .$$

We also have

(1.8) $$F\big|_j = 0$$

The curvature tensor fields $\widetilde{K}^{i}_{jkh}$ and K^{i}_{jkh} satisfy the following identities :

(1.9) $$\widetilde{K}^{i}_{jkh} = -\widetilde{K}^{i}_{jhk} \quad , \quad K^{i}_{jkh} = -K^{i}_{jhk}$$

(1.10) $$\widetilde{K}^{i}_{jkh} + \widetilde{K}^{i}_{khj} + \widetilde{K}^{i}_{hjk} = 2\Delta_{[j|k|h];l}\, g^{il} \quad ,$$

where ; denotes covariant derivative based upon the connection parameter given by

$$Q^{*}_{jkh} = P^{*}_{jkh} + g_{(jk),h}$$

and

(1.1) $$K^{i}_{jkh} + K^{i}_{khj} + K^{i}_{hjk} = 2\Delta_{[j|k|h]|l}\, g^{il} \; ,$$

where l denotes covariant derivative based upon the connection parameter given by $R^{*}_{jkh} = \Gamma^{*}_{jkh}$.

The Bianchi identities satisfied by these curvature tensor fields are given by

(1.12) $$\widetilde{K}^{i}_{jkh,l} + \widetilde{K}^{i}_{jhl,k} + \widetilde{K}^{i}_{jlk,h} + 2\left[\widetilde{K}^{i}_{jmk} P^{*m}_{[lh]} + \widetilde{K}^{i}_{jmh} P^{*m}_{[kl]} + \widetilde{K}^{i}_{jml} P^{*m}_{[hk]}\right] = 0$$

and

(1.13) $$\begin{aligned} &K^{i}_{jkh} + K^{i}_{jhl}\big|_{k} + K^{i}_{jlk}\big|_{h} + F\left(K^{m}_{0kh}\dot{\partial}_{m}\Gamma^{*i}_{jl} + K^{m}_{0hl}\dot{\partial}_{m}\Gamma^{*i}_{jk} + K^{m}_{0lk}\dot{\partial}_{m}\Gamma^{*i}_{jh}\right) \\ &= 2\left(K^{i}_{jml}\Delta^{m}_{[kh]} + K^{m}_{jmk}\Delta^{m}_{[hl]} + K^{i}_{jmk}\Delta^{m}_{[lk]}\right) \end{aligned}$$

2. VEBLEN IDENTITIES AND THEIR EQUIVALENCE

Theorem 2.1 [Shamihoke [3] . In the generalized Finsler space with the connection parameters P^{*i}_{jk} and the curvature tensor field $\widetilde{K}^{i}_{jkh}$, the Bianchi identities are expressed as

(2.1) $$\frac{1}{3}\widetilde{B}^{i}_{jkhl} = \widetilde{K}^{i}_{j[kh,l]} + 2K^{i}_{jm|k}P^{*i}_{l\bar{h}]} \quad ,$$

where the two indices together with bar represents the skew-symmetric part of that tensor in two indices and the symbol $[khr]$ gives the skew-symmetric part of the tensor in three indices.

We consider the following lemma which is very useful in deriving Veblen identities and proving the equivalence of Bianchi and Veblen identities. Similar lemma was used by C.I.Ispas [4] for determining Veblen identities in Finsler space .

Lemma Let us assume that the quantities $\widetilde{J}_{jkhl}$ satisfy the following conditions :

(2.2) $$\widetilde{J}^{i}_{jkhl} = -\widetilde{J}^{i}_{jhkl} \quad ,$$

and

(2.3) $$\widetilde{J}^{i}_{[jkh]l} = 0 \quad ,$$

Now if we put

(2.4) $$\widetilde{B}_{jkhl} \equiv 3\widetilde{J}_{j[khl]} \quad ,$$

and

(2.5) $$\widetilde{V}_{jkhl} \equiv \widetilde{J}_{jkhl} + \widetilde{J}_{hjlk} + \widetilde{J}_{lhkj} + \widetilde{J}_{kljh}$$

then they satisfy the relations

(2.6) $$\widetilde{V}_{jkhl} = \widetilde{B}_{kjhl} + \widetilde{B}_{hlkj}$$

and

$$(2.7) \qquad 2\widetilde{B}_{jkhl} = 3\widetilde{V}_{j[khl]} \; .$$

Proof . Using the definition (2.4) in the right hand side of (2.6) , we get

$$(2.8) \qquad \widetilde{B}_{kjhl} + \widetilde{B}_{hlkj} = \widetilde{J}_{kjhl} + \widetilde{J}_{khlj} + \widetilde{J}_{kljh} + \widetilde{J}_{hlkj} + \widetilde{J}_{hkjl} + \widetilde{J}_{hjlk}$$

.

In view of (2.3) , the equation (2.8) assumes the form

$$(2.9) \qquad \widetilde{B}_{kjhl} + \widetilde{B}_{hlkj} = -\widetilde{J}_{jhkl} - \widetilde{J}_{lkhj} + \widetilde{J}_{kljh} + \widetilde{J}_{hjlk} \; .$$

Applying (2.2) and (2.5) in the above equation , it yields (2.6) which proves the first part of the lemma .

Now by considering the right hand side of (2.7),we have

$$(2.10) \qquad \begin{aligned} 3\widetilde{V}_{j[khl]} &= \widetilde{J}_{jkhl} + \widetilde{J}_{hjlk} + \widetilde{J}_{lhkj} + \widetilde{J}_{kljh} + \widetilde{J}_{jhlk} \\ &+ \widetilde{J}_{ljkh} + \widetilde{J}_{klhj} + \widetilde{J}_{hkjl} + \widetilde{J}_{jlkh} + \widetilde{J}_{kjhl} + \widetilde{J}_{hklj} + \widetilde{J}_{lhjk} \end{aligned}$$

in view of (2.5) .

By virtue of (2.3) and (2.4) , the equation (2.10) becomes

$$(2.11) \qquad \begin{aligned} 3\widetilde{V}_{j[khl]} &= \widetilde{B}_{jkhl} + 3\widetilde{J}_{[lhk]j} + 3\widetilde{J}_{[hjl]k} - \widetilde{J}_{jlhk} \\ &+ 3\widetilde{J}_{[klj]h} - \widetilde{J}_{jklh} + 3\widetilde{J}_{[hkj]l} - \widetilde{J}_{jhkl} . \end{aligned}$$

On account of (2.3) , it reduces to

$$(2.12) \qquad 3\widetilde{V}_{j[khl]} = \widetilde{B}_{jkhl} - 3\widetilde{J}_{j[hkl]} \; .$$

In view of (2.2) and (2.4) , the equation (2.12) yields (2.7) which establishes the second part of the lemma .

Theorem 2.2 In the generalized Finsler space GF_n , the curvature tensor field $\widetilde{K}^i_{jkh}$ satisfies the Veblen identities

$$\widetilde{V}^i_{jkhl} \equiv \widetilde{J}^i_{jkh,l} + \widetilde{J}^i_{hjl,k} + \widetilde{J}^i_{lhk,j} + \widetilde{J}^i_{klj,h} = 0 \quad , \tag{2.13}$$

where

$$\widetilde{J}^i_{jkh,l} \equiv \widetilde{K}^i_{jkh,l} + \frac{1}{2}\left(\widetilde{K}^i_{jmk} P^{*m}_{lh} - \widetilde{K}^i_{jmh} P^{*m}_{lk}\right) . \tag{2.14}$$

Proof. Interchanging the indices k and h in (2.14) , we get

$$\widetilde{J}^i_{jkh,l} = -\widetilde{J}^i_{jhk,l} \tag{2.15}$$

in view of (1.9) .
Taking cyclic permutation of the indices j,k,h and noting (1.10) in (2.14) , we obtain

$$\begin{aligned} \widetilde{J}_{[jkh],l} &= \frac{2}{3}\Delta_{[j|k|h];m,l}\, g^{im} + \frac{2}{3}\Delta_{[j|k|h];m}\, g^{im}_{,l} \\ &+ \frac{1}{3}\left(\widetilde{K}^i_{[j|m|k]} P^{*m}_{lh} + \widetilde{K}^i_{[k|m|h]} P^{*m}_{lj} + \widetilde{K}^i_{[h|m|j]} P^{*m}_{lk}\right) \end{aligned} \tag{2.16}$$

In view of (1.9) and (2.1) , the cyclic permutation of the indices k,h,l yields

$$3\widetilde{J}^i_{j[kh,l]} = \widetilde{B}^i_{jkhl} \quad . \tag{2.17}$$

On account of $(2.1), (2.15), (2.16), (2.17)$ and the lemma , we get (2.13) which completes the proof .

Remark 2.1 In a generalized Finsler space GF_n , the curvature tensor $\widetilde{K}^i_{jkh}$ satisfies the Veblen identities

$$(2.18)\qquad \widetilde{V}^{i}_{jkhl} \equiv \widetilde{K}^{i}_{jkh,l} + \widetilde{K}^{i}_{hjl,k} + \widetilde{K}^{i}_{lhk,j} + \widetilde{K}^{i}_{klj,h} + \frac{1}{2}\Big[\widetilde{K}^{i}_{jmk} P^{*i}_{lh} + \widetilde{K}^{i}_{hmj} P^{*i}_{kl}$$

$$+ \widetilde{K}^{i}_{lmh} P^{*m}_{jk} + \widetilde{K}^{l}_{kml} P^{*m}_{hj} - \widetilde{K}^{i}_{jmh} P^{*m}_{lk} - \widetilde{K}^{i}_{hml} P^{*m}_{kj} - \widetilde{K}^{i}_{lmk} P^{*m}_{jh} - \widetilde{K}^{i}_{kmj} P^{*m}_{hl}\Big] = 0$$

Theorem 2.1 In a generalized Finsler space GF_n ,the Bianchi and Veblen identities for the curvature tensor $\widetilde{K}^{i}_{jkh}$ are related as under :

$$(2.19)\qquad \widetilde{V}^{i}_{jkhl} = \widetilde{B}^{i}_{jkhl} + \widetilde{B}^{i}_{hljk} \quad ,$$

and

$$(2.20)\qquad 2\widetilde{B}^{i}_{jkhl} = 3\widetilde{V}^{i}_{j[khl]} \quad .$$

Proof . It is evident from the lemma .

Remark 2.2 In similar way as above ,we find that the curvature tensor K^{i}_{jkh} satisfies the Veblen identities

(2.21)

$$V^{i}_{jkhl} = K^{i}_{jkh}\big|_{l} + K^{i}_{hjl}\big|_{k} + K^{i}_{lhk}\big|_{j} + K^{i}_{klj}\big|_{h} + F\left(K^{m}_{0kh}\dot{\partial}_{m}\Gamma^{*i}_{[jl]} + K^{m}_{0jl}\dot{\partial}_{m}\Gamma^{*i}_{[hk]}\right)$$

$$-2\left(K^{i}_{[j|m|l]}\Delta^{m}_{[kh]} + K_{[h|m|k]}\Delta^{m}_{[jl]}\right) - \frac{F}{2}\Big\{K^{m}_{0lk}\dot{\partial}_{m}\Gamma^{*i}_{jk} + K^{m}_{0kl}\dot{\partial}_{m}\Gamma^{*i}_{hj} + K^{m}_{0jk}\dot{\partial}_{m}\Gamma^{*i}_{lh}$$

$$+ K^{m}_{0hj}\dot{\partial}_{m}\Gamma^{*i}_{kl}\Big\} + \left[K^{i}_{jmk}\Delta^{m}_{[lh]} + K^{i}_{hmj}\Delta^{m}_{[kl]} + K^{i}_{lmh}\Delta^{m}_{[jk]} + K^{i}_{kml}\Delta^{m}_{[hj]}\right] = 0$$

in view of (1.9) and (1.13) .

Remark 2.3 Theorem 2.3 also holds good in case of the relation between Bianchi and Veblen identities for the curvature tensor K^{i}_{jkh} in GF_n .

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