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SPACES OF ANALYTIC FUNCTINOS OF FINITE TYPE OF TWO COMPLEX VARIABLES

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ABSTRACT. Let $f(z_1,z_2)=\sum_{m,n=0}^{\infty} a_{mn} z_1^m z_2^n$ be analytic for $|z_i|<1, i=1,2$. Using the order and type characterization of $f(z_1,z_2)$ in terms of coefficients $\{a_{mn}\}$, a metric is defined on the space of all functions of type less than or equal to T. The properties of this space and linear transformations have been studied. Necessary and sufficient conditions for a base to be a proper base have also been obtained .

1. Let

$$f(z_1,z_2)=\sum_{m,n=0}^{\infty} a_{mn} z_1^m z_2^n, \tag{1.1}$$

where $z_1=r_1e^{i\theta_1}, z_2=r_2e^{i\theta_2}$, $a_{mn}\in C$, be analytic for $|z_1|<1$, $|z_2|<1$. Bose and Sharma [2] obtained the growth properties of entire functions of two complex variables and defined their order and type etc. In a recent paper [1], we have considered the growth of analytic functions of two complex variables. Thus we define, following Bose and Sharma,

$$M(r_1,r_2)=\max_{|z_i|\le r_i} |f(z_1,z_2)| .$$

The order ρ of $f(z_1,z_2)$ is defined as

$$\lim_{r_1,r_2\to 1}\sup \frac{\log^+\log^+ M(r_1,r_2)}{-\log\log(r_1r_2)^{-1}}=\rho,\ 0\le\rho\le\infty, \tag{1.2}$$

and for $0<\rho<\infty$, the type T of f is defined as

$$\lim_{r_1,r_2\to 1}\sup \frac{\log^+ M(r_1,r_2)}{-\log(r_1r_2)^{-\rho}}=T \quad ,\ 0\le T\le\infty . \tag{1.3}$$

We have shown that the analytic function $f(z_1,z_2)$ is of type T if and only if

$$\lim_{m,n\to\infty}\sup \frac{(\log^+|a_{mn}|)^{\rho+1}}{(m+n)^{\rho}}=\frac{(\rho+1)^{\rho+1}}{(2\rho)^{\rho}}T,\ \ 0<\rho<\infty . \tag{1.4}$$

Let $X(\rho,T)$ denote the class of all functions $f(z_1,z_2)=\sum_{m,n=0}^{\infty} a_{mn} z_1^m z_2^n$ analytic for $|z_i|<1$ and of order less than or equal to ρ, and if of order ρ, then of type less than or equal to T, $0<T<\infty$. Under pointwise addition and scalar multiplication, the set $X(\rho,T)$ is then a linear space over the complex field C.

For any $f \in X(\rho,T)$, we have

$$\text{(1.5)} \qquad \limsup_{m,n\to\infty} \frac{(\log^+|a_{mn}|)^{\rho+1}}{(m+n)^\rho} \le \frac{(\rho+1)^{\rho+1}}{(2\rho)^\rho} T.$$

Hence for any $\varepsilon>0$, $\exists$ positive integers m_0 and n_0 such that for all $m>m_0, n>n_0$,

$$\text{(1.6)} \qquad |a_{mn}| < \exp[(m+n)^{\rho/(\rho+1)} \{C(T+\varepsilon)\}^{1/(\rho+1)}] \,,$$

where we put $(\rho+1)^{\rho+1}/(2\rho)^\rho = C$. For each $f \in X(\rho,T)$, we define

$$\|f\|_q = \sum_{m,n=0}^{\infty} |a_{mn}| \exp[-(m+n)^{\rho/(\rho+1)} \{C(T+q^{-1})\}^{1/(\rho+1)}] \,,$$

where $q=1,2,\ldots$. In view of (1.6), $\|f\|_q$ exists for each $q=1,2,\ldots$, and for $q_1 \le q_2$,

$$\|f\|_{q_1} \le \|f\|_{q_2}.$$

This norm induces a metric topology on $X(\rho,T)$. This is given by the equivalent metric

$$\lambda(f,g) = \sum_{q=1}^{\infty} 2^{-q} \frac{\|f-g\|_q}{1+\|f-g\|_q}.$$

We denote by $X_\lambda(\rho,T)$ the space $X(\rho,T)$ equipped with above metric λ.

2. In this section, we obtain some properties of space $X_\lambda(\rho,T)$ and linear transformations on it. First we prove

Theorem 1. The space $X_\lambda(\rho,T)$ is a Fre'chet space.

Proof. We show that the space $X_\lambda(\rho,T)$ is complete. Therefore, let $\{f_\iota\}$ be a λ-*Cauchy* sequence in $X_\lambda(\rho,T)$. Then for any $\eta>0$, $\exists$ a positive integer $m_0=m_0(\eta)$ such that

$$\text{(2.1)} \qquad \|f_\alpha - f_\beta\|_q < \eta \quad \text{for all } \alpha,\beta > m_0, \quad q \ge 1.$$

Let $f_\alpha(z_1,z_2)=\sum_{m,n=0}^{\infty} a_{mn}^{(\alpha)} z_1^m z_2^n$, $f_\beta(z_1,z_2)=\sum_{m,n=0}^{\infty} a_{mn}^{(\beta)} z_1^m z_2^n$.

Then we have

$$(2.2)\qquad \sum_{m,n=0}^{\infty}\left|a_{mn}^{(\alpha)}-a_{mn}^{(\beta)}\right|\exp[-(m+n)^{\rho/(\rho+1)}\{C(T+q^{-1})\}^{1/(\rho+1)}]<\eta,$$

for $\alpha,\beta>m_0,\quad q\geq 1$.

Hence for each fixed m,n, $\{a_{mn}^{(\alpha)}\}_{\alpha=1}^{\infty}$, being a Cauchy sequence of complex numbers. Thus, $\exists$ a sequence $\{a_{mn}\}_{m,n=0}^{\infty}$ such that

$$\lim_{\alpha\to\infty}a_{mn}^{(\alpha)}=a_{mn},\ m,n=0,1,2,\ldots\ .$$

Now, taking $\beta\to\infty$ in (2.2), we get for $\alpha\geq m_0$.

$$(2.3)\qquad \sum_{m,n=0}^{\infty}\left|a_{mn}^{(\alpha)}-a_{mn}\right|\exp[-(m+n)^{\rho/(\rho+1)}\{C(T+q^{-1})\}^{1/(\rho+1)}]<\eta.$$

Taking $\alpha=m_0$, we get for any fixed q,

$$\left|a_{mn}\right|\exp[-(m+n)^{\rho/(\rho+1)}\{C(T+q^{-1})\}^{1/(\rho+1)}]\leq$$

$$\left|a_{mn}^{(m_0)}\right|\exp[-(m+n)^{\rho/(\rho+1)}\{C(T+q^{-1})\}^{1/(\rho+1)}]+\eta.$$

Now, $f_{m_0}(z_1,z_2)=\sum_{m,n=0}^{\infty}a_{mn}^{(m_0)}z_1^m z_2^n\in X_\lambda(\rho,T)$. Hence the condition (1.6) is satisfied for $\{a_{mn}^{(m_0)}\}$. Thus for arbitrary $p>q$, we have

$$\left|a_{mn}\right|\exp[-(m+n)^{\rho/(\rho+1)}\{C(T+q^{-1})\}^{1/(\rho+1)}]$$

$$<\eta+\left|a_{mn}^{(m_0)}\right|\exp[-(m+n)^{\rho/(\rho+1)}\{C(T+q^{-1})\}^{1/(\rho+1)}]$$

$$<\eta+\exp[(m+n)^{\rho/(\rho+1)}C^{1/(\rho+1)}\{(T+p^{-1})^{1/(\rho+1)}-(T+q^{-1})^{1/(\rho+1)}\}].$$

Since $p>q$ is arbitrary, the second term on the R.H.S. approaches to zero as $(m+n)\to\infty$. Also, since $\eta>0$ was chosen arbitrarily, the sequence $\{a_{mn}\}_{m,n=0}^{\infty}$ satisfies (1.6) for any q and all sufficiently large values of m, n. Therefore $f(z_1,z_2)=\sum_{m,n=0}^{\infty}a_{mn}z_1^m z_2^n\in X_\lambda(\rho,T)$.

Again from (2.3), we have for arbitrary $\varepsilon>0$ and $q=1,2,\ldots,\|f_\alpha-f\|_q<\varepsilon$. Hence

$$\lambda(f_\alpha,f)=\sum_{q=1}^{\infty}2^{-q}\frac{\|f_\alpha-f\|_q}{1+\|f_\alpha-f\|_q}$$

$$<\frac{\varepsilon}{1+\varepsilon}\sum_{q=1}^{\infty}2^{-q}=\frac{\varepsilon}{1+\varepsilon}<\varepsilon.$$

Since the above inequality holds for all $\alpha > m_0$, it follows that. $f_\alpha \to f$ as $\alpha \to \infty$. Since we have already proved that $f \in X_\lambda(\rho,T)$, this proves that $X_\lambda(\rho,T)$ is complete. This proves Theorem 1.

Now we give a characterization of linear continuous functionals on $X_\lambda(\rho,T)$. We thus have

Theorem 2. A continuous linear functional F on $X_\lambda(\rho,T)$ is of the form $F(f) = \sum_{m,n=0}^{\infty} a_{mn}c_{mn}$ if and only if

(2.4) $$|c_{mn}| \le L \exp[-(m+n)^{\rho/(\rho+1)}\{C(T+q^{-1})\}^{1/(\rho+1)}], \quad m,n \ge 1, q \ge 1,$$

where L is finite, positive number and $f(z_1,z_2) = \sum_{m,n=0}^{\infty} a_{mn}z_1^m z_2^n$.

Proof. Let $F: X_\lambda(\rho,T) \to$ C, where C is the complex field, be a linear, continuous functional. Then for any sequence $\{f_j\} \subseteq X_\lambda(\rho,T)$ with $f_j \to f$, we have $F(f_j) \to F(f)$ as $j \to \infty$. Now, let $f(z_1,z_2) = \sum_{m,n=0}^{\infty} a_{mn}z_1^m z_2^n$, where the sequence $\{a_{mn}\}$ satisfies (1.6). Then $f \in X_\lambda(\rho,T)$. Also for $j = 1,2,\ldots$, let us put $f_j(z_1,z_2) = \sum_{m,n=0}^{j} a_{mn}z_1^m z_2^n$. Then $f_j \in X_\lambda(\rho,T)$ for $j = 1,2,\ldots$. Let q be any fixed positive integer and let $0 < \varepsilon < q^{-1}$. Then from (1.6), we can find a positive integer j such that

$$|a_{mn}| < \exp[(m+n)^{\rho/(\rho+1)}\{C(T+\varepsilon)\}^{1/(\rho+1)}] \quad \forall\, m,n > j.$$

Now,

$$\|f - f_j\| = \left\| \sum_{m,m=j+1}^{\infty} a_{mn}z_1^m z_2^n \right\|_q$$

$$= \sum_{m,n=j+1}^{\infty} |a_{mn}| \exp[-(m+n)^{\rho/(\rho+1)}\{C(T+q^{-1})\}^{1/(\rho+1)}]$$

$$< \sum_{m,n=j+1}^{\infty} \exp[(m+n)^{\rho/(\rho+1)} C^{1/(\rho+1)}\{(T+\varepsilon)^{1/(\rho+1)} - (T+q^{-1})^{1/(\rho+1)}\}].$$

Since $\varepsilon < q^{-1}$, given $\delta > 0$, we get $\|f - f_j\|_q < \delta$ for all sufficiently large values of j.

Hence

$$\lambda(f,f_j)=\sum_{q=1}^{\infty}2^{-q}\frac{\|f-f_j\|_q}{1+\|f-f_j\|_q}<\sum_{q=1}^{\infty}2^{-q}(\frac{\delta}{\delta+1})<\delta .$$

Hence $f_j \to f$ in $X_\lambda(\rho,T)$ as $j\to\infty$. Therefore, by continuity of F, we have

$$\lim_{j\to\infty}F(f_j)=F(f).$$

Let us assume that $c_{mn}=F(z_1^m z_2^n)$. Then

$$F(f)=\lim_{j\to\infty}F(f_j)=\lim_{j\to\infty}\sum_{m,n=0}^{j}a_{mn}c_{mn}=\sum_{m,n=0}^{\infty}a_{mn}c_{mn} .$$

Further, $|c_{mn}|=|F(z_1^m z_2^n)|$. Since F is continuous on $X_\lambda(\rho,T)$, it is continuous on $X_{\|\,\|_q}(\rho,T)$ for each $q=1,2\ldots$. Consequently, $\exists$ a positive constant L independent of q such that

$$|F(z_1^m z_2^n)|=|c_{mn}|\le L\|\delta_{mn}\|_q \;,q\ge 1 \quad ,$$

where $\delta_{mn}(z_1,z_2)=z_1^m z_2^n$. Now, using the definition of the norm for $\delta_{mn}(z_1,z_2)$, we get

$$|c_{mn}|\le L\ \exp[-(m+n)^{\rho/(\rho+1)}\{C(T+q^{-1})\}^{1/(\rho+1)}],$$

$\forall\, m,n\ge 1,\ q\ge 1$. Hence we have $F(f)=\sum_{m,n=0}^{\infty}a_{mn}c_{mn}$, where c_{mn}'s satisfy (2.4).

Conversely, suppose c_{mn}'s satisfy (2.4) and for any sequence of complex numbers $\{a_{mn}\}$, $F(f)=\sum_{m,n=0}^{\infty}a_{mn}c_{mn}$. Then for $q\ge t$, we have

$$|F(f)|\le L\ \sum_{m,n=0}^{\infty}|a_{mn}|\exp[-(m+n)^{\rho/(\rho+1)}\{C(T+q^{-1})\}^{1/(\rho+1)}],$$

or, $|F(f)|\le L\|f\|_q,\quad q\ge 1$.

Hence $F\in X'_{\|\,\|_q}(\rho,T)$ for $q=1,2,\ldots$. Since

$$\lambda(f,g)=\sum_{q=1}^{\infty}2^{-q}\frac{\|f-g\|_q}{1+\|f-g\|_q}.$$

therefore $X'_\lambda(\rho,T)=\bigcup_{q=1}^{\infty}X'_{\|\,\|_q}(\rho,T)$. Hence $F\in X'_\lambda(\rho,T)$, the dual of $X_\lambda(\rho,T)$. This proves Theorem 2.

3. In this section we shall study continuous linear transformations and proper bases in $X(\rho,T)$. Following Kamthan and Gupta [3], we give some definitions. Let

$\{\alpha_{mn}, m,n \geq 0\}$ be a double sequence of entire functions in X. The sequence $\{\alpha_{mn}\}$ is said to be linearly independent if $\sum_{m,n=0}^{\infty} a_{mn}\alpha_{mn} = 0$ implies that $a_{mn} = 0 \; \forall m,n$, for all sequences $\{a_{mn}\}$ of complex numbers for which the series $\sum_{m,n=0}^{\infty} a_{mn}\alpha_{mn}$ converges in X. A subspace X_0 of X is said to be spanned by a sequence $\{\alpha_{mn}\} \subseteq X$ if X_0 consists of all linear combinations $\sum_{m,n=0}^{\infty} a_{mn}\alpha_{mn}$ such that $\sum_{m,n=0}^{\infty} a_{mn}\alpha_{mn}$ converges in X. A sequence $\{\alpha_{mn}\} \subseteq X$ which is linearly independent and spans a subspace X_0 of X is said to be a base in X_0. Finally, a sequence $\{\alpha_{mn}\} \subseteq X$ will be called a 'proper base' if it is a base and satisfies the condition:

"For all sequences $\{a_{mn}\}$ of complex numbers, the convergence of $\sum_{m,n=0}^{\infty} a_{mn}\alpha_{mn}$ in X implies the convergence of $\sum_{m,n=0}^{\infty} a_{mn}\delta_{mn}$ in X, and conversely".

To prove our next result we define for $f \in X(\rho,T)$ and any $\delta > 0$,

$$\|f;\rho;T+\delta\| = \sum_{m,n=0}^{\infty} |a_{mn}| \exp[-(m+n)^{\rho/(\rho+1)}\{C(T+\delta)\}^{1/(\rho+1)}],$$

where $f(z_1,z_2) = \sum_{m,n=0}^{\infty} a_{mn} z_1^m z_2^n$.

Theorem 3. A necessary and sufficient condition that there exists a continuous linear transformation $F: X(\rho,T) \to X(\rho,T)$ with

$$F(\delta_{mn}) = \beta_{mn}, m,n = 0,1,2,... \quad ; \delta_{mn}(z_1,z_2) = z_1^m z_2^n ; \beta_{mn} \in X(\rho,T)$$

is that for each $\delta > 0$,

$$\text{(3.1)} \qquad \limsup_{m,n\to\infty} \frac{(m+n)^{\rho}}{\{\log^+(\|\beta_{mn};\rho;T+\delta\|^{-1})\}^{(\rho+1)}} < \frac{1}{CT}.$$

Proof. Let F be a continuous linear transformation from $X(\rho,T)$ into $X(\rho,T)$ with $F(\delta_{mn}) = \beta_{mn}, m,n = 0,1,2,...$. Then, for any given $\delta > 0$, $\exists$ a $\delta_1 = \delta_1(\delta)$ and a constant $K = K(\delta)$ such that

$$\|F(\delta_{mn});\rho;T+\delta\| \le K\|\delta_{mn};\rho;T+\delta_1\|$$

i.e. $$\|\beta_{mn};\rho;T+\delta\| \le K \ \exp[-(m+n)^{\rho/(\rho+1)}\{C(T+\delta_1)\}^{1/(\rho+1)}]$$

i.e $$\frac{(m+n)^{\rho}}{\{\log^{+}(\|\beta_{mn};\rho;T+\delta\|^{-1})\}^{(\rho+1)}} \le \frac{1}{C(T+\delta_1)} < \frac{1}{CT}$$

i.e. $$\limsup_{m,n\to\infty}\frac{(m+n)^{\rho}}{\{\log^{+}(\|\beta_{mn};\rho;T+\delta\|^{-1})\}^{(\rho+1)}} \le \frac{1}{C(T+\delta_1)} < \frac{1}{CT}.$$

Conversely, suppose that the sequence $\{\beta_{mn}\}$ satisfies (3.1). Then, for any $\eta' > 0$, $\exists N_0 = N_0(\eta')$ such that

(3.2) $$\frac{(m+n)^{\rho}}{\{\log^{+}(\|\beta_{mn};\rho;T+\delta\|^{-1})\}^{(\rho+1)}} \le \frac{1}{C(T+\eta')},\ \forall\, m,n > N_0,$$

and all $\delta > 0$. Let $f(z_1,z_2) = \sum_{m,n=0}^{\infty} a_{mn} z_1^m z_2^n \in X(\rho,T)$ and choose $0 < \eta < \eta'$. Then from (1.5), $\exists\, N_1(\eta) = N_1$ such that for all $m,n \ge N_1$

(3.3) $$|a_{mn}| < \exp[(m+n)^{\rho/(\rho+1)}\{C(T+\eta)\}^{1/(\rho+1)}].$$

Let $n_0 = \max(N_0, N_1)$. Then from (3.2) and (3.3), we have for all $m,n > n_0$,

$$|a_{mn}|\|\beta_{mn};\rho;T+\delta\| \le$$

$$\exp[(m+n)^{\rho/(\rho+1)}C^{1/(\rho+1)}\{(T+\eta)^{1/(\rho+1)} - (T+\eta')^{1/(\rho+1)}\}].$$

Since $0 < \eta < \eta'$, this inequality implies that the series $\sum_{m,n=0}^{\infty} a_{mn}\beta_{mn}$ converges absolutely in $X(\rho,T)$ and $X(\rho,T)$ being complete, we infer that this series converges to an element of $X(\rho,T)$. Hence, let us define a transformation $F: X(\rho,T) \to X(\rho,T)$ by putting $F(\alpha) = \sum_{m,n=0}^{\infty} a_{mn}\beta_{mn}$ for $\alpha \in X(\rho,T)$. We note that F is linear, $F(\delta_{mn}) = \beta_{mn}$ and, for $\delta > 0, \exists \delta' > 0$, such that

$$\frac{(m+n)^{\rho}}{\{\log^{+}(\|\beta_{mn};\rho;T+\delta\|^{-1})\}^{(\rho+1)}} \le \frac{1}{C(T+\delta')} \quad \text{for } m,n > N(\delta,\delta')$$

i.e $$\|\beta_{mn};\rho;T+\delta\| \le h \ \exp[-(m+n)^{\rho/(\rho+1)}\{C(T+\delta')\}^{1/(\rho+1)}]$$

for all $m,n \ge 0$, $h = h(\delta)$ being a constant. Hence

$$\|F(\alpha);\rho;T+\delta\| \le \sum_{m,n=0}^{\infty} |a_{mn}| \|\beta_{mn};\rho;T+\delta\|$$

$$\le \sum_{m,n=0}^{\infty} |a_{mn}| \; h \; \exp[-(m+n)^{\rho/(\rho+1)} \{C(T+\delta')\}^{1/(\rho+1)}]$$

$$\le h' \|\alpha;\rho;T+\delta'\|,$$

where $h' = \max(h^{-1},1)$. Thus F is continuous and Theorem 3 is proved.

From (1.5), we know that $\sum c_{mn}\alpha_{mn}$ converges in $X(\rho,T)$ if and only if

(3.4) $$\limsup_{m,n\to\infty} \frac{(\log^{+}|c_{mn}|)^{\rho+1}}{(m+n)^{\rho}} \le CT \quad .$$

Now, we prove

Lemma 1. The following three conditions are equivalent:

(3.5) $$\limsup_{m,n\to\infty} \frac{(m+n)^{\rho}}{\{\log^{+}(\|\beta_{mn};\rho;T+\delta\|^{-1})\}^{(\rho+1)}} < \frac{1}{CT}, \delta > 0,$$

(3.6) For all sequences $\{a_{mn}\}$ of complex numbers, the convergence of $\sum_{m,n=0}^{\infty} a_{mn}\delta_{mn}$ in $X(\rho,T)$ implies the convergence of $\sum_{m,n=0}^{\infty} a_{mn}\beta_{mn}$ in $X(\rho,T)$,

(3.7) For all sequences $\{a_{mn}\}$ of complex numbers, the convergence of $\sum_{m,n=0}^{\infty} a_{mn}\delta_{mn}$ in $X(\rho,T)$ implies that $\lim_{m,n\to\infty} a_{mn}\beta_{mn} = 0$ in $X(\rho,T)$.

Proof. In proving the sufficiency part of Theorem 3, we have already proved that $(3.5) \Rightarrow (3.6)$. The implication $(3.6) \Rightarrow (3.7)$ is evident. Hence we have to prove that $(3.7) \Rightarrow (3.5)$.

Let (3.7) be true but for some $\delta > 0$, (3.5) be not satisfied. Then, say for $\delta = \delta'$, $\exists$ sequences $\{m_k\}, \{n_l\}$ of positive integers such that for, $m = m_k$, $n = n_l$ and $k,l = 1,2,\ldots$

(3.8) $$\frac{(m+n)^{\rho}}{\{\log^{+}(\|\beta_{mn};\rho;T+\delta\|^{-1})\}^{(\rho+1)}} > \frac{1}{C\{T+(kl)^{-1}\}}.$$

We define a sequence $\{a_{mn}\}$ as

$$a_{mn} = \begin{cases} \|\beta_{mn};\rho;T+\delta'\|^{-1}, & m=m_k,\ n=n_l \\ 0 & otherwise \end{cases}$$

Then for all large values of k and l

$$\frac{\{\log^+(|a_{m_k n_l}|)\}^{(\rho+1)}}{(m_k+n_l)^{\rho}} = \frac{\{\log^+\|\beta_{m_k n_l};\rho;T+\delta'\|^{-1})\}^{(\rho+1)}}{(m_k+n_l)^{\rho}}$$

Hence

$$\limsup_{m,n\to\infty} \frac{\{\log^+|a_{mn}|\}^{(\rho+1)}}{(m+n)^{\rho}} \le CT .$$

Thus, the sequence $\{a_{mn}\}$ as defined above satisfies (3.4) and hence $\sum a_{mn}\delta_{mn}$ converges in $X(\rho,T)$. Hence by (3.7), we have $\lim_{m,n\to\infty} a_{mn}\beta_{mn} = 0$. However

$$\|a_{m_k n_l}\beta_{m_k n_l};\rho;T+\delta'\| = |a_{m_k n_l}|\,\|\beta_{m_k n_l};\rho;T+\delta'\| = 1 .$$

Therefore $\{a_{m_k n_l}\beta_{m_k n_l}\}$ does not converge to zero in $X(\rho,T)$. This is a contradiction. Hence (3.5) must hold for all $\delta > 0$ and the proof of lemma 1 is complete.

Lemma 2. The following three conditions are equivalent:

(3.9) For all sequences $\{a_{mn}\}$ of complex numbers, $\lim_{m,n\to\infty} a_{mn}\beta_{mn} = 0$ in $X(\rho,T)$ implies that $\sum_{m,n=0}^{\infty} a_{mn}\delta_{mn}$ converges in $X(\rho,T)$,

(3.10) For all sequences $\{a_{mn}\}$ of complex numbers, convergence of $\sum_{m,n=0}^{\infty} a_{mn}\beta_{mn}$ in $X(\rho,T)$ implies that $\sum_{m,n=0}^{\infty} a_{mn}\delta_{mn}$ converges in $X(\rho,T)$,

(3.11) $$\lim_{\delta\to 0}\Big[\liminf_{m,n\to\infty} \frac{(m+n)^{\rho}}{\{\log^+(\|\beta_{mn};\rho;T+\delta\|^{-1})\}^{(\rho+1)}}\Big] \ge \frac{1}{CT} .$$

Proof. Obviously $(3.9) \Rightarrow (3.10)$. Thus we prove that $(3.10) \Rightarrow (3.11)$. Assume that (3.10) holds but (3.11) is not true. Then, we have

$$\lim_{\delta\to 0}\Big[\liminf_{m,n\to\infty} \frac{(m+n)^{\rho}}{\{\log^+(\|\beta_{mn};\rho;T+\delta\|^{-1})\}^{(\rho+1)}}\Big] < \frac{1}{CT} .$$

Hence for any $\delta > 0$,

(3.12) $$\liminf_{m,n\to\infty} \frac{(m+n)^{\rho}}{\left\{\log^{+}\left(\left\|\beta_{mn};\rho;T+\delta\right\|^{-1}\right)\right\}^{(\rho+1)}} < \frac{1}{CT}.$$

Let $\eta > 0$ be a fixed number. From (3.12), we can fined increasing sequences $\{m_k\}, \{n_l\}$ of positive integers such that

$$\frac{(m_k+n_l)^{\rho}}{\left\{\log^{+}\left(\left\|\beta_{m_k n_l};\rho;T+\delta\right\|^{-1}\right)\right\}^{(\rho+1)}} < \frac{1}{C(T+\eta)}.$$

For η_1, $0<\eta_1<\eta$, we define a sequence $\{a_{mn}\}$ as

$$a_{mn} = \begin{cases} \exp[(m+n)^{\rho/(\rho+1)}\{C(T+\eta_1)\}^{\rho/(\rho+1)}], & m=m_k,\ n=n_l \\ 0 & otherwise \end{cases}$$

Then for any $\delta > 0$, we have

(3.13) $$\sum_{m,n=0}^{\infty}|a_{mn}|\left\|\beta_{mn};\rho;T+\delta\right\| = \sum_{k,l=1}^{\infty}\left|a_{m_k n_l}\right|\left\|\beta_{m_k n_l};\rho;T+\delta\right\|.$$

Now, for any $\delta > 0$, we omit those terms on the R.H.S. series for which $\delta < (kl)^{-1}$. Then the remainder of the series (3.13) is dominated by

$$\sum_{k,l=1}^{\infty}\left|a_{m_k n_l}\right|\left\|\beta_{m_k n_l};\rho;T+(kl)^{-1}\right\|.$$

Consequently by (3.11), we obtain

$$\sum_{k,l=1}^{\infty}\left|a_{m_k n_l}\right|\left\|\beta_{m_k n_l};\rho;T+(kl)^{-1}\right\| \le$$

$$\sum_{k,l=1}^{\infty}\exp[(m_k+n_l)^{\rho/(\rho+1)}C^{1/(\rho+1)}\{(T+\eta_1)^{1/(\rho+1)}-(T+\eta)^{1/(\rho+1)}\}].$$

Since $\eta_1 < \eta$, the series on the R.H.S. is convergent. Since $a_{mn}=0$ for $m \ne m_k$ and $n \ne n_l$, the series $\sum_{m,n=0}^{\infty} a_{mn}\beta_{mn}$ converges for the above choice of $\{a_{mn}\}$. Since this is true for any $\delta > 0$, $\sum_{m,n=0}^{\infty} a_{mn}\beta_{mn}$ converges in $X(\rho,T)$. On the other hand, for this sequence $\{a_{mn}\}$, we also have

(3.14) $$\limsup_{m,n\to\infty} \frac{\left\{\log^{+}|a_{mn}|\right\}^{(\rho+1)}}{(m+n)^{\rho}} = C(T+\eta_1) > CT$$

which is a contradiction. Hence $(3.10) \Rightarrow (3.11)$. Lastly, we prove that $(3.11) \Rightarrow (3.9)$. Hence, suppose that (3.11) holds but (3.9) does not hold. Then, $\exists$ a sequence $\{a_{mn}\}$

of complex numbers for which $a_{mn}\beta_{mn} \to 0$ in $X(\rho,T)$ but $\sum_{m,n=0}^{\infty} a_{mn}\delta_{mn}$ does not converge in $X(\rho,T)$. Hence from the equivalent condition (3.4), we have

$$\limsup_{m,n\to\infty} \frac{\{\log^+|a_{mn}|\}^{(\rho+1)}}{(m+n)^{\rho}} > CT.$$

Thus, $\exists$ a positive number ε and a sequences $\{m_k\}, \{n_l\}$ of positive integers such that

$$\frac{\{\log^+|a_{m_k n_l}|\}^{(\rho+1)}}{(m_k+n_l)^{\rho}} > C(T+\varepsilon).$$

Let $0<\eta<\varepsilon/2$. From (3.11), we can fined a positive number δ such that

$$\liminf_{m,n\to\infty} \frac{(m+n)^{\rho}}{\{\log^+(\|\beta_{mn};\rho;T+\delta\|^{-1})\}^{(\rho+1)}} \geq \frac{1}{C(T+\eta)}.$$

Hence $\exists$ an integer $N=N(\eta)$ such that for $m,n\geq N$,

$$\frac{(m+n)^{\rho}}{\{\log^+(\|\beta_{mn};\rho;T+\delta\|^{-1})\}^{(\rho+1)}} \geq \frac{1}{C(T+2\eta)}.$$

Therefore,

$$\begin{aligned}
\max\|a_{mn}\beta_{mn};\rho;T+\delta\| &= \max\{|a_{mn}|\,\|\beta_{mn};\rho;T+\delta\|\} \\
&\geq \max\{|a_{m_k n_l}|\,\|\beta_{m_k n_l};\rho;T+\delta\|\} \\
&\geq \exp[(m_k+n_l)^{\rho/(\rho+1)}C^{1/(\rho+1)}\{(T+\varepsilon)^{1/(\rho+1)}-(T+2\eta)^{1/(\rho+1)}\}] \\
&> 1.
\end{aligned}$$

since $\varepsilon > 2\eta$. Hence the sequence $\{a_{mn}\beta_{mn}\}$ does not converge to zero for the δ chosen above. Hence, $\{a_{mn}\beta_{mn}\}$ does not converge to zero in $X(\rho,T)$. This is clearly contradictory to (3.11) and hence we obtain that (3.11) $\Rightarrow$ (3.9). This proves Lemma 2.

The following result, which gives a characterization of a proper base in $X(\rho,T)$, follows from Lemma 1 and Lemma 2. Thus, we have

Theorem 4. A base $\{\beta_{mn}\}$ in a closed subspace $X_0(\rho,T)$ is proper if and only if the conditions (3.5) and (3.11) stated above are satisfied.

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