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THE WIENER TYPE SPACES $W\left(B_{w,v}^{p,q}(G), L_\nu^r(G)\right)$

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Abstract. Let G be a locally compact abelian group $1 \le p, q, r < \infty$ and w, v, ν are Beurling's weights on G. We denote by $B_{w,v}^{p,q}(G)$ the vector space $L_w^p(G) \cap L_v^q(G)$ and endowed it with the sum norm $\|f\|_{w,v}^{p,q} = \|f\|_{p,w} + \|f\|_{p,v}$ [8]. Research on Wiener type spaces was initiated by N. Wiener in [9] and many authors worked on these spaces. H. Feichtinger gave a kind of generalization of the Wiener's definition in [1]. In this work we discussed Wiener type spaces $W\left(B_{w,v}^{p,q}(G), L_\nu^r(G)\right)$ using the space $B_{w,v}^{p,q}(G)$ [8] as a local component, and $L_\nu^r(G)$ as a global component.

1. Introduction.

Let G be a locally compact (non-compact, non-discrete) abelian group with Haar measure dx. We denote by $C_c(G)$ the space of all continuous, complex-valued functions on G with compact support. The space $L_{loc}^1(G)$ consists of all measurable functions f on G such that $f\chi_K \in L^1(G)$ for any compact subset $K \subset G$, where χ_K is the characteristic function of K. It is a topological vector space with the family of seminorms $f \to \|f\chi_K\|_1$. A Banach function space (shortly BF-space) on G is a Banach space $(B, \|\cdot\|_B)$ of measurable functions embedded into $L_{loc}^1(G)$, i. e. for any compact subset $K \subset G$ there exists some constant $C_K > 0$ such that $\|f\chi_K\|_1 \le C_K \|f\|_B$ for all $f \in B$. A BF-space is called solid, if $g \in B$, $f \in L_{loc}^1(G)$ and $|f(x)| \le |g(x)|$ locally almost everywhere (shortly l. a. e) implies $f \in B$ and $\|f\|_B \le \|g\|_B$. The left translation operators L_y are given by $L_y f(x) = f(x-y)$ for $x, y \in G$. $(B, \|\cdot\|_B)$ is called strongly translation invariant if one has $L_y B \subseteq B$ and $\|L_y f\|_B = \|f\|_B$ for all $f \in B$, $y \in G$. A Banach space $(B, \|\cdot\|_B)$ is called a Banach module over a Banach algebra $(A, \|\cdot\|_A)$ if B is a module over A in the algebraic sense and satisfied $\|a.b\|_B \le \|a\|_A \|b\|_B$ for all $a \in A$, $b \in B$. A triple (B^1, B^2, B^3) of BF-space will be called a Banach convolution triple (BCT), if convolution given by

$$f^1 * f^2(x) = \int_G f^1(x-y) f^2(y)\,dy$$

for $f^i \in B^i \cap C_c(G)$ (i=1, 2), extends to a continuous bilinear map from $B^1 \times B^2$ into B^3. It is known that a Banach space B is Banach module over the Banach algebra A if (A, B, B) is a BCT [1]. The Fourier algebra $A(G)$ is defined by $\left\{\hat{f} \,\middle|\, f \in L^1(\hat{G})\right\}$. It is a Banach algebra with respect to pointwise multiplication and the norm $\|\hat{f}\|_A = \|f\|_1$, here $\hat{f}$ is the Fourier transform of $f \in L^1(\hat{G})$. Throughout this work, we also will use Beurling weights, i. e. real-valued, measurable and locally bounded functions w on a locally compact abelian group G which satisfy $1 \le w(x)$, $w(x+y) \le w(x)w(y)$ for $1 \le p < \infty$, we set

$$L_w^p(G) = \left\{ f \,\middle|\, fw \in L^p(G) \right\}.$$

Under the norm $\|f\|_{p,w} = \|f.w\|_p$, this is a Banach space. When $p=1$, $L_w^1(G)$ becomes an algebra under convolution, called Beurling algebra [7]. In this paper another important tool is the space $B_{w,v}^{p,q}(G) = L_w^p(G) \cap L_v^q(G)$ with the norm $\|\cdot\|_{w,v}^{p,q} = \|\cdot\|_{p,w} + \|\cdot\|_{p,v}$ [8], where w, v are Beurling weights on G and $1 \le p,q < \infty$. The main tool is the Wiener type spaces in the sense [1]. The definition is the following: Let B be a BF-space. Assume that there exists a homogeneous Banach algebra $(A, \|\cdot\|_A)$, continuously embedded into $(C_b(G), \|\cdot\|_\infty)$, and $(B, \|\cdot\|_B)$ is continuously embedded into the topological dual space $A_C'(G) = (A(G) \cap C_C(G))'$, where $A_C'(G)$ is equipped with its weak topology $\sigma(A_C'(G), A_C(G))$. Here $A_C(G) = A(G) \cap C_C(G)$ is given inductive limit topology of its subspaces $(A_K(G), \|\cdot\|_A)$, where $K \subset G$ compact, $A_K(G) = A(G) \cap C_K(G)$. Also B is Banach module over $A(G)$ with respect to pointwise multiplication. We define B_{loc} to be the space of all elements f of $A_C'(G)$ such that $hf \in B$ for all $h \in A_C(G)$. This is a locally convex vector space together with the topology defined by the seminorm $f \to \|hf\|_B, h \in A_C(G)$. Fix an open, relatively compact set $\Omega \subset G$ and for $f \in B_{loc}$ we set $F_f(x) = \|f\|_{B(x+\Omega)}$, with

$$\|f\|_{B(x+\Omega)} = \inf\left\{ \|g\|_B \,\middle|\, g \in B, hf = hg \text{ for all } h \in A_C(G) \text{ with } \sup ph \subset x + \Omega \right\}.$$

If now C is a solid, translation invariant BF-space on G, the Wiener type space *W(B,C)* with local component B and global component C is then defined by

$$W(B,C) = \left\{ f \in B_{loc} \,\middle|\, F_f \in C \right\}.$$

The natural norm on *W(B,C)* is given by

$$\|f\|_{W(B,C)} = \|F_f\|_C \quad [1].$$

2. The Wiener Type Spaces $W\left(B_{w,v}^{p,q}(G), L_v^r(G)\right)$

We introduce the Banach spaces

$$A^{\omega}(G) = F\left(L_\omega^1(\hat{G})\right) = \left\{ \hat{f} \,\middle|\, f \in L_\omega^1(\hat{G}) \right\}$$

the norm $\|\hat{f}\|_\omega = \|f\|_{1,\omega}$ where ω is an arbitrary weight function on $\hat{G}$, and F is the classical Fourier transform. With this, $A^\omega(G)$ is a Banach algebra under pointwise multiplication [7]. We set $A_C^\omega(G) = A^\omega(G) \cap C_C(G)$, equipped with the inductive limit topology τ_ω of the subspaces $A_K^\omega(G) = A^\omega(G) \cap C_K(G)$, $K \subset G$ compact, equipped with their $\|\cdot\|_\omega$-norms and $A_C^\omega(G)'$ is the topological dual of $A_C^\omega(G)$ with the weak*-topology.

Lemma 2. 1. $B^{p,q}_{w,v}(G)$ is continuously embedded into $A'_C(G)$ with its weak*-topology.

Proof. It is clear that $B^{p,q}_{w,v}(G)$ is continuously embedded into $L^p(G)$. It is also known that $A'_C(G) = \Omega(G)$, the space of quasimeasures on G, and that $L^p(G)$ is continuously embedded into $\Omega(G)$ (with its weak topology as the dual $D(G)$ [6]. This proves our proposition.

Theorem 2. 2. Let w,v be weights on G and $1 \le p,q < \infty$. If the weight function ω on $\hat{G}$ satisfies Beurling Domar condition (shortly (BD) i. e $\sum \frac{\log w(t^n)}{n^2} < \infty, t \in \hat{G}$).

then $B^{p,q}_{w,v}(G)$ is continuously embedded into $\sigma \in \left(A^{\omega}_C(G)', A^{\omega}_C(G)\right)$.

Proof. Since $B^{p,q}_{w,v}(G)$ is continuously embedded into $\sigma \in \left(A'_C(G), A_C(G)\right)$ by the Lemma 2. 1., then if one uses the above embedding and Corollary 1. 3 in [5], easily proves the Theorem.

We assume henceforth that the weight function ω on $\hat{G}$ satisfies (B. D). Therefore $A^{\omega}(G)$ satisfies all of the properties required for the construction of Wiener type spaces in the sense of Feichtinger [1]: It is clear that $A^{\omega}(G)$ is continuously embedded into $C_b(G)$. Moreover, $A^{\omega}(G)$ is a regular Banach algebra under pointwise multiplication (Reiter, [7]) and also is homogeneous Banach space [4].

Secondly, $B^{p,q}_{w,v}(G)$ is a Banach module over $A^{\omega}(G)$ under pointwise multiplication [8] and we proved that $B^{p,q}_{w,v}(G)$ is continuously embedded into $\sigma\left(A^{\omega}_c(G)', A^{\omega}_c(G)\right)$ in Theorem 2. 2. Hence, Feichtinger's general hypotheses are satisfied. That means the Wiener type spaces $W\left(B^{p,q}_{w,v}(G), L^r_v(G)\right)$ are well defined: Given any open subset Ω of G with compact closure and $f \in \left(B^{p,q}_{w,v}(G)\right)_{loc}$, we set

$$F_f(z) = \left\| f|z+\Omega \right\|^{p,q}_{w,v} \qquad z \in G.$$

The Wiener type space $W\left(B^{p,q}_{w,v}(G), L^r_v(G)\right)$ with local component $B^{p,q}_{w,v}(G)$ and global component $L^r_v(G)$ is then defined by

$$W\left(B^{p,q}_{w,v}(G), L^r_v(G)\right) = \left\{ f \in \left(B^{p,q}_{w,v}(G)\right)_{loc} \middle| F_f \in L^r_v(G) \right\}.$$

The natural norm of $W\left(B^{p,q}_{w,v}(G), L^r_v(G)\right)$ is given by

$$\left\| f \middle| W\left(B^{p,q}_{w,v}(G), L^r_v(G)\right) \right\| = \left\| F_f \right\|_{r,v}.$$

We now proceed to the investigation of some basic properties of Wiener type spaces $W\left(B^{p,q}_{w,v}(G), L^r_v(G)\right)$ in the sense [1].

Theorem 2. 3. **(i)** The Wiener type space $W\left(B^{p,q}_{w,v}(G), L^r_v(G)\right)$ is a Banach space under the norm

$$\left\| f \right\|_{W\left(B^{p,q}_{w,v}(G), L^r_v(G)\right)} = \left\| F_f \right\|_{r,v}$$

where $f \in W\left(B_{w,v}^{p,q}(G), L_v^r(G)\right)$. It is also continuously embedded into $\left(B_{w,v}^{p,q}(G)\right)_{loc}$.

(ii) The set $A_0 = \left\{f \in B_{w,v}^{p,q}(G) \middle| \sup pf \text{ is compact}\right\}$ is continuously embedded into $W\left(B_{w,v}^{p,q}(G), L_v^r(G)\right)$.

(iii) $W\left(B_{w,v}^{p,q}(G), L_v^r(G)\right)$ is left (right) invariant,

$$|||L_x||| \leq |||L_x|||_{w,v}^{p,q} |||L_x|||_{r,v}$$

where $|||.|||, |||.|||_{w,v}^{p,q}$ and $|||.|||_{r,v}$ are operator norms on $W\left(B_{w,v}^{p,q}(G), L_v^r(G)\right)$, $B_{w,v}^{p,q}(G)$ and $L_v^r(G)$ respectively.

(iv) The translation is continuous in the Wiener type spaces $W\left(B_{w,v}^{p,q}(G), L_v^r(G)\right)$.

(v) $W\left(B_{w,v}^{p,q}(G), L_v^r(G)\right)$ is a Banach module over $W\left(A(G), L^\infty(G)\right)$ with respect to the pointwise multiplication.

Proof. By Proposition 2. 3 in [8] the space $B_{w,v}^{p,q}(G)$ is translation invariant and translation is continuous in this space. Then if one uses Theorem 1 in [1], the proof of this theorem is completed.

Proposition 2. 4. Let w and v be weight function on G satisfying $v \prec w$ and $1 \leq p, q < \infty$. Then $W\left(B_{w,v}^{p,q}(G), L_v^r(G)\right)$ is a Banach module over $W\left(B_{w,v}^{1,q}(G), L_v^r(G)\right)$ with respect to convolution.

Proof. It is easy to show that every locally compact abelian group is a *IN* group. Moreover by Proposition 2. 13 (b) in [8] the space $B_{w,v}^{p,q}(G)$ is a Banach module $B_{w,v}^{1,q}(G)$ with respect to convolution. It is also known that $L_w^p(G)$ is a Banach module over $L_w^1(G)$ with respect to convolution [4]. Then $\left(B_{w,v}^{1,q}(G), B_{w,v}^{p,q}(G), B_{w,v}^{p,q}(G)\right)$ and $\left(L_w^1(G), L_w^p(G), L_w^p(G)\right)$ are two Banach convolution triples on G. If one uses Theorem 3 in [1] shows that $\left(W(B_{w,v}^{1,q}(G), L_w^1(G)), W(B_{w,v}^{p,q}(G), L_w^p(G)), W(B_{w,v}^{p,q}(G), L_w^p(G))\right)$ a Banach convolution triples on G. Then $W\left(B_{w,v}^{p,q}(G), L_w^p(G)\right)$ is a Banach module over $W\left(B_{w,v}^{1,q}(G), L_w^1(G)\right)$ with respect to convolution.

Theorem 2. 5. $W\left(B_{w,v}^{p,q}(G), L_v^r(G)\right)$ is a BF-space on G.

Proof. By the Theorem 2. 3. (i), $W\left(B_{w,v}^{p,q}(G), L_v^r(G)\right)$ is continuously embedded into $\left(B_{w,v}^{p,q}(G)\right)_{loc}$. That means given any $h \in A_c^w(G)$ (Thus a seminorm $P_h(f) = \|h.f\|_{w,v}^{p,q}$ on $\left(B_{w,v}^{p,q}(G)\right)_{loc}$) there exists a constant $D_h > 0$ such that

$$\|h.f\|_{w,v}^{p,q} \leq D_h \left\| f \middle| W\left(B_{w,v}^{p,q}(G), L_v^r(G)\right)\right\|$$

for all $f \in W\left(B_{w,v}^{p,q}(G), L_v^r(G)\right)$. Hence one can write

$$\|h.f\|_p \leq D_h \left\| f \middle| W\left(B_{w,v}^{p,q}(G), L_v^r(G)\right)\right\|. \tag{1}$$

Take any compact subset $K \subset G$. Since $A^w(G)$ is a regular Banach algebra with respect to pointwise multiplication, then one may choice a function

$h \in A_c^m(G) = A^m(G) \cap C_c(G)$ satisfying $0 \le h < 1$ and $h(x) = 1$ for all $x \in K$. We let $Supp h = K_0$. Then $\chi_K(x) \le h(x)$, hence $\chi_K(x)|f(x)| \le h(x)|f(x)|$ for all $x \in G$. Since L^p is continuously embedded into L^1_{loc}, then there exists $D_{K_0} > 0$ such that

$$(2) \qquad \int_{K_0} |h(x)f(x)|dx \le D_{K_0} \|hf\|_p .$$

Also one has

$$(3) \qquad \int_K |f(x)|dx \le \int_{K_0} |f(x)h(x)|dx.$$

The proof is completed combining the formulas (1), (2) and (3).

Corollary 2. 6. Let w, v be weights on G, $v<w$ and $x \in G$. Then the map $x \to |||L_x|||$ is locally bounded, where $|||\cdot|||$ denotes the operator norm on $W\left(B_{w,v}^{p,q}(G), L_v^r(G)\right)$.

Proof. By the Theorem 2. 3. (iii), one writes

$$|||L_x||| \le \|L_x\|_{w,v}^{p,q} \|L_x\|_{r,v}$$

where $|||\cdot|||, \|\cdot\|_{w,v}^{p,q}$ and $\|\cdot\|_{r,v}$ are operator norms on $W\left(B_{w,v}^{p,q}(G), L_v^r(G)\right)$, $B_{w,v}^{p,q}(G)$ and $L_v^r(G)$ respectively. It is also known that $\|L_x\|_{r,v} \le v(x)$ [4] and $\|L_x\|_{w,v}^{p,q} \le c.w(x)$ [8]. Then we have

$$|||L_x||| \le c.w(x)v(x)$$

for all $x \in G$. Since w and v are weight functions, then the function $w.v$ is locally bounded. Hence $x \to |||L_x|||$ is also locally bounded.

Proposition 2. 7. The Wiener type space $W\left(B_{w,v}^{p,q}(G), L_v^r(G)\right)$ is a Banach convolution module (left and right because G is an abelian group) over some Beurling algebra $L^1_{w_0}(G)$.

Proof. We proved in Theorem 2.5 that $W\left(B_{w,v}^{p,q}(G), L_v^r(G)\right)$ is a BF-space. Thus $W\left(B_{w,v}^{p,q}(G), L_v^r(G)\right)$ is continuously embedded into $L^1_{loc}(G)$. By Theorem 2. 3., this space is left invariant and translation operator in $W\left(B_{w,v}^{p,q}(G), L_v^r(G)\right)$ is continuous. Now if one uses Lemma 1.5 in [2] proves that $W\left(B_{w,v}^{p,q}(G), L_v^r(G)\right)$ is a Banach module over $L^1_{w_0}(G)$, where

$$w_0(x) = \max\left(1, |||L_x|||\right).$$

Corollary 2.8. $W\left(B_{w,v}^{p,q}(G), L_v^r(G)\right)$ is a left (right) Banach module over $L^1_{v_0}(G)$ if $v_0(x)$ is a weight satisfying $v_0(x) > w_0(x)$ for ail $x \in G$, where $w_0(x)$ is defined as in Proposition 2.7.

Now we will begin to discuss the inclusions between the Wiener type spaces $W\left(B_{w,v}^{p,q}(G), L_v^r(G)\right)$.

Given a weighted space $L_w^r(G)$ the associated weighted sequence space is denoted by λ_w^r and defined

$$\lambda_w^r = \left\{(a_i)_{i\in I} \in \lambda^r \,\middle|\, (a_i w(i))_{i\in I} \in \lambda^r \right\},$$

where the discrete weight w given by $w(i) = w(x_i)$ for $i \in I$. It is known that λ_w^r is a Banach space with respect to the norm

$$\|z\|_{r,w} = \left(\sum_{i\in I} |a_i w(i)|^r \right)^{\frac{1}{r}}$$

where $z = (a_i)_{i\in I}$.

It is easy to prove the following two lemmas:

Lemma 2.9. If $r_1 \le r_2$ then $\lambda_w^{r_1} \subset \lambda_w^{r_2}$.

Lemma 2.10. Let v_1, v_2 be weights on G, and $1 \le r_1, r_2 < \infty$. If $v_1 < v_2$ and $r_2 \le r_1$ then $\lambda_{v_2}^{r_2} \subset \lambda_{v_1}^{r_1}$.

Any given solid BF-space Y may be quite naturally associated with a corresponding sequence space $Y_d(x)$ (sometimes called solid BK-space).

Given a discrete family $x = (x_i)_{i\in I}$ in G and a solid translation invariant BF-space $(Y, \|\cdot\|_Y)$ we define the associate discrete space $Y_d(x)$ as

$$\left\{ \Lambda \,\middle|\, \Lambda = (\lambda_i)_{i\in I} \ with \sum_{i\in I} |\lambda_i| \chi_{x_i w} \in Y \right\}.$$

with natural norm

$$\|\lambda\|_{Y_d} = \sum_{i\in I} |\lambda_i| \|\chi_{x_i w}\|_Y \quad [3].$$

Using this definition, we write

$$(4) \quad \left(L^{r_1}(G) \,|\, L^{r_2}(G)\right)_d = \left\{ \lambda \,\middle|\, \lambda = (\lambda_i), \sum_{i\in I} |\lambda_i| \chi_{x_i w} \in L^{r_1}(G) \,|\, L^{r_2}(G) \right\}.$$

If we use (4) and Lemma 2.9, easily prove the following two lemmas:

Lemma 2.11. If $r_2 < r_1$ then $\left(L^{r_1}(G) \,|\, L^{r_2}(G)\right)_d = \lambda^{r_2}$.

Corollary 2.12. If $r_2 < r_1$ and $v_1 < v_2$ then

$$\left(L_{v_1}^{r_1}(G) \,|\, L_{v_2}^{r_2}(G)\right)_d = \lambda_{v_2}^{r_2}.$$

Theorem 2.13. Let u_1 and u_2 be the weight functions in construction of Wiener type spaces $W\left(B_{w_1,v_1}^{p,q}(G), L_{v_1}^{r_1}(G)\right)$ and $W\left(B_{w_2,v_2}^{p,q}(G), L_{v_2}^{r_2}(G)\right)$ respectively. Also assume that $w_1, w_2, v_1, v_2, \nu_1, \nu_2$ weights on G and $1 \le p, q, r_1, r_2 < \infty$. If $u_1 \approx u_2$ and

$$B_{w_2,v_2}^{p,q}(G) \subset B_{w_1,v_1}^{p,q}(G)$$

then $W\left(B_{w_2,v_2}^{p,q}(G), L_{v_2}^{r_2}(G)\right)$ is continuously embedded into $W\left(B_{w_1,v_1}^{p,q}(G), L_{v_1}^{r_1}(G)\right)$ if and only if $r_2 \le r_1$ and $v_1 < v_2$.

Proof. Since $B_{w_2,v_2}^{p,q}(G) \subset B_{w_1,v_1}^{p,q}(G)$, then by Proposition 2.9. in [8] there exists a constant $c > 0$ such that

$$(5) \qquad \|f\|_{w_1,v_1}^{p,q} \le c \|f\|_{w_2,v_2}^{p,q}$$

for all $f \in B^{p,q}_{w_2,v_2}(G)$. Also since $u_1 \approx u_2$ then $A^{u_1}_c(G) = A^{u_2}_c(G)$ and $(A^{u_1}_c(G))' = (A^{u_2}_c(G))'$ by Lemma 1.1 in [5]. Hence a simple calculation shows that $\left(B^{p,q}_{w_2,v_2}(G)\right)_{loc}$ is continuously embedded into $\left(B^{p,q}_{w_1,v_1}(G)\right)_{loc}$.

Now using the definition of Wiener type space and (5), $W\left(B^{p,q}_{w_2,v_2}(G), L^{r_2}_{v_2}(G)\right)$ is continuously embedded into $W\left(B^{p,q}_{w_1,v_1}(G), L^{r_2}_{v_2}(G)\right)$. Also because the Proposition 3.7 in [3], $W\left(B^{p,q}_{w_1,v_1}(G), L^{r_2}_{v_2}(G)\right)$ is continuously embedded into $W\left(B^{p,q}_{w_1,v_1}(G), L^{r_1}_{v_1}(G)\right)$ if and only if

$$(6) \qquad \left(L^{r_2}_{v_2}(G)\right)_d \subset \left(L^{r_1}_{v_1}(G)\right)_d,$$

where $\left(L^{r_2}_{v_2}(G)\right)_d$ and $\left(L^{r_1}_{v_1}(G)\right)_d$ are the discretes of the spaces $L^{r_2}_{v_2}(G)$ and $L^{r_1}_{v_1}(G)$ respectively. If we assume (5) then by Lemma 2.10 and (6), we have $r_2 \le r_1$ and $v_1 < v_2$. Conversely if $r_2 \le r_1$ and $v_1 < v_2$ then $\lambda^{r_2}_{v_2} \subset \lambda^{r_1}_{v_1}$. This completes the proof of this theorem.

It is known that if $w_1 < w_2, v_1 < v_2$, then $B^{p,q}_{w_2,v_2}(G) \subset B^{p,q}_{w_1,v_1}(G)$, [8]. If one uses Theorem 2.13 easily proves the Corollary.

Corollary 2.14. If $u_1 \approx u_2, w_1 < w_2, v_1 < v_2$ then $W\left(B^{p,q}_{w_2,v_2}(G), L^{r_2}_{v_2}(G)\right)$ continuously embedded into $W\left(B^{p,q}_{w_1,v_1}(G), L^{r_1}_{v_1}(G)\right)$ if and only if $v_1 < v_2, r_2 \le r_1$.

Corollary 2.15. If $u_1 \approx u_2, w_1 \approx w_2, v_1 \approx v_2, v_1 \approx v_2$ and $r_1 = r_2$, then

$$W\left(B^{p,q}_{w_2,v_2}(G), L^{r_2}_{v_2}(G)\right) = W\left(B^{p,q}_{w_1,v_1}(G), L^{r_1}_{v_1}(G)\right).$$

Proposition 2.16. If $r_2 \le r_1$ and $v_1 < v_2$ then

$$W\left(B^{p,q}_{w,v}(G), B^{r_1,r_2}_{v_1,v_2}(G)\right) = W\left(B^{p,q}_{w,v}(G), L^{r_2}_{v_2}(G)\right).$$

Proof. Since $r_2 \le r_1$ and $v_1 < v_2$ then by Corollary 2.12 we have $\left(L^{r_1}_{v_1}(G) \mathrm{I}\ L^{r_2}_{v_2}(G)\right)_d = \lambda^{r_2}_{v_2}$. Also by Lemma 3.5 (e) in [3] we write $\left(L^{r_2}_{v_2}(G)\right)_d = \lambda^{r_2}_{v_2}$. Hence we obtain $\left(L^{r_1}_{v_1}(G) \mathrm{I}\ L^{r_2}_{v_2}(G)\right)_d = \left(L^{r_2}_{v_2}(G)\right)_d$. Consequently if we use the Proposition 3.7 in [3] we have

$$W\left(B^{p,q}_{w,v}(G), B^{r_1,r_2}_{v_1,v_2}(G)\right) = W\left(B^{p,q}_{w,v}(G), L^{r_2}_{v_2}(G)\right).$$

Theorem 2.17. If $v_1 < v_2, v_3 < v_4, v_2 < v_4, r_2 \le r_1, r_4 \le r_3, r_4 \le r_2$ and $B^{p,q}_{w_2,v_2}(G) \subset B^{p,q}_{w_1,v_1}(G)$ then $W\left(B^{p,q}_{w_2,v_2}(G), B^{r_3,r_4}_{v_3,v_4}(G)\right)$ is continuously embedded into $W\left(B^{p,q}_{w_1,v_1}(G), B^{r_1,r_2}_{v_1,v_2}(G)\right)$.

Proof. Using the proof Theorem 2.1, $W\left(B_{w_2,v_2}^{p,q}(G), L_{v_2}^{r_2}(G)\right)$ is continuously embedded into $W\left(B_{w_1,v_1}^{p,q}(G), L_{v_2}^{r_2}(G)\right)$. By Proposition 2.16 we write

$$(7)\quad W\left(B_{w_1,v_1}^{p,q}(G), L_{v_2}^{r_2}(G)\right) = W\left(B_{w_1,v_1}^{p,q}(G), B_{v_1,v_2}^{r_1,r_2}(G)\right).$$

Hence from the formula (7), $W\left(B_{w_2,v_2}^{p,q}(G), L_{v_2}^{r_2}(G)\right)$ is continuously embedded into $W\left(B_{w_1,v_1}^{p,q}(G), B_{v_1,v_2}^{r_1,r_2}(G)\right)$. Since $r_4 \le r_3$ and $v_3 < v_4$, then because the Corollary 2.12 we have

$$(8)\qquad \left(L_{v_3}^{r_3}(G) \cap L_{v_4}^{r_4}(G)\right)_d = \lambda_{v_4}^{r_4}.$$

Also since $r_4 \le r_2$ and $v_2 < v_4$ then $\lambda_{v_4}^{r_4} \subset \lambda_{v_2}^{r_2}$ by Lemma 2.10. If one uses the formulas (8) and the equality $\left(L_{v_2}^{r_2}(G)\right)_d = \lambda_{v_2}^{r_2}$ obtains that

$$\left(B_{v_3,v_4}^{r_3,r_4}(G)\right)_d = \left(L_{v_3}^{r_3}(G) \cap L_{v_4}^{r_4}(G)\right)_d \subset \left(L_{v_2}^{r_2}(G)\right)_d.$$

Also by Proposition 3.7 in [3], $W\left(B_{w_2,v_2}^{p,q}(G), B_{v_3,v_4}^{r_3,r_4}(G)\right)$ is continuously embedded into $W\left(B_{w_2,v_2}^{p,q}(G), L_{v_2}^{r_2}(G)\right)$. Therefore $W\left(B_{w_2,v_2}^{p,q}(G), B_{v_3,v_4}^{r_3,r_4}(G)\right)$ is continuously embedded into $W\left(B_{w_1,v_1}^{p,q}(G), B_{v_1,v_2}^{r_1,r_2}(G)\right)$.

Corollary 2.18. If $w_1 < w_2, v_1 < v_2, v_1 < v_2, v_3 < v_4, v_2 < v_4, r_2 \le r_1, r_4 \le r_3$ and $r_4 \le r_2$ then $W\left(B_{w_2,v_2}^{p,q}(G), B_{v_3,v_4}^{r_3,r_4}(G)\right)$ is continuously embedded into $W\left(B_{w_1,v_1}^{p,q}(G), L_{v_1,v_2}^{r_1,r_2}(G)\right)$.

Proof. Since $w_1 < w_2, v_1 < v_2$ by Corollary 2.10 in [8],we write $B_{w_2,v_2}^{p,q}(G) \subset B_{w_1,v_1}^{p,q}(G)$. Also by Theorem 2.17, $W\left(B_{w_2,v_2}^{p,q}(G), B_{v_3,v_4}^{r_3,r_4}(G)\right)$ is continuously embedded into $W\left(B_{w_1,v_1}^{p,q}(G), B_{v_1,v_2}^{r_1,r_2}(G)\right)$.

If one uses Corollary 2.19, easily proves the following Corollary.

Corollary 2.19. If $w_1 \approx w_2, v_1 \approx v_2, v_1 \approx v_2, v_3 \approx v_4, v_2 \approx v_4$ and $r_1 = r_2 = r_3 = r_4$ then

$$W\left(B_{w_2,v_2}^{p,q}(G), B_{v_3,v_4}^{r_3,r_4}(G)\right) = W\left(B_{w_1,v_1}^{p,q}(G), B_{v_1,v_2}^{r_1,r_2}(G)\right).$$

The proof of the following Proposition is easy using the proof technique of Theorem 2.14.

Proposition 2.20. If $w_1 < v_1, w_2 < v_2, v_2 < v_1, q_1 \le p_1, q_2 \le p_2, q_1 \le q_2$ and $L_{v_1}^{r}(G) \subset L_{v_2}^{r}(G)$ then $W\left(L_{v_1}^{r}(G), B_{w_1,v_1}^{p_1,q_1}(G)\right)$ is continuously embedded into $W\left(L_{v_2}^{r}(G), B_{w_2,v_2}^{p_2,q_2}(G)\right)$.

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