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THE MAPPING OF THE DUAL PROJECTIVE PLANE INTO THE THREE-DIMENSIONAL SPACE OF LINES

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Abstract: In this work every point of the projective plane P_2 (whose homogeneous coordinates are dual numbers) is corresponded with a single oriented straight line of the space of straight lines. As examples, the figures corresponding to classical Desargues' axiom and Pappos' Theorem are determined. In addition, the figure corresponding Pappos figure is plotted axonometrically.

1. Introduction

1.1. First, we wish to give a brief explanation about the dual quantities which find successful application fields in geometry [1] :

Let the symbols α be defined as

$$\alpha = a + \varepsilon a^*, \quad (\varepsilon^2 = 0) \tag{1.1}$$

where a, a^* are real numbers, ε is a symbol whose square is zero $(\varepsilon^2 = 0)$. Let us define the sum and the multiplication of the symbols as it is done between the two binomials of real numbers. Then it can be easily shown that the set of such symbols α form a commutative ring with zero-divisor and unit element. The ring so-defined is called the ring of dual numbers. In (1.1), a is called the real part of α and a^* is the dual part. If $a = 0$, α is called pure dual, if $a^* = 0$, α is real. The pure dual numbers εb are the zero-divisors of the ring and the division by these numbers is not defined.

If $\omega = u + \varepsilon u^*$ is a dual angle, the cosine and the sine are defined as

$$\text{Cos } \omega = 1 - \frac{\omega^2}{2!} + \frac{\omega^4}{4!} - \Lambda \quad , \quad \text{Sin } \omega = \omega - \frac{\omega^3}{3!} + \frac{\omega^5}{5!} - \Lambda \tag{1.2}$$

Therefore, we have

$$\text{Cos } \omega = \text{Cos } (u + \varepsilon u^*) = (1 - \frac{u^2}{2!} + \frac{u^4}{4!} - \Lambda) - \varepsilon u^* (u - \frac{u^3}{3!} + \frac{u^5}{5!} - \Lambda) =$$

$$= \text{Cos } u - \varepsilon u^* \text{ Sin } u \tag{1.3}$$

Now we consider a straight line in E_3, passing through the points $\vec{x}, \vec{y}$. The six coordinates a_k, a_k^* of the vectors

$$\vec{a} = \rho(\vec{y} - \vec{x}), \ \vec{a}^* = \rho(\vec{x} \wedge \vec{y}) \ , \ (\rho \neq 0) \tag{1.4}$$

which satisfy the condition

$$\vec{a} \cdot \vec{a}^* = 0 \quad (1.4')$$

are called homogeneous Plücker coordinates of the straight line. It is easy to see that for $\vec{x} \neq \vec{y}$, ρ can be chosen so that $\vec{a}^2 = 1$.

If ρ is assumed to be chosen so that $\vec{a}^2 = 1$, the unit vector $\vec{a}$ determining the direction and the sense of the straight line and the vector $\vec{a}^*$ which is the moment vector of $\vec{a}$ with respect to the origin, determine a straight line completely. The condition of a point $\vec{x}$ to be on the straight line $(\vec{a}, \vec{a}^*)$ is

$$\vec{x} \wedge \vec{a} = \vec{a}^* \quad (1.5)$$

The footpoint of the perpendicular from the origin to the straight line has the position vector

$$\vec{b} = \vec{a} \wedge \vec{a}^* \quad (1.6)$$

Let us now consider the dual vector $\vec{A}$ whose coordinates are $\alpha_k = a_k + \varepsilon a_k^*$, we have

$$\vec{A} = \vec{a} + \varepsilon \vec{a}^*, (\vec{A}^2 = 1) \quad (1.7)$$

If we use (1.4') and $\vec{a}^2 = 1$, and therefore, $\vec{A}$ is a unit vector. Thus we obtain a mapping of the directed straight lines of $\overline{E}_3$ to the dual points of the unit sphere. If $\vec{A}$ and $\vec{B} = \vec{b} + \varepsilon \vec{b}^*$ are two unit dual vectors, we have $\vec{a} \cdot \vec{b} = \text{Cos}(\vec{a}, \vec{b}) = \text{Cos}\ u$ in the scalar product

$$\vec{A} \cdot \vec{B} = \vec{a} \cdot \vec{b} + \varepsilon (\vec{a} \cdot \vec{b}^* + \vec{b} \cdot \vec{a}^*) \quad (1.8)$$

On the other hand, the dual part, i.e. the coefficient of ε satisfies the relation

$$\vec{a} \cdot \vec{b}^* + \vec{b} \cdot \vec{a}^* = -(\vec{a}, \vec{b}, \overrightarrow{HK}) = -(\vec{b}, \vec{a}, \overrightarrow{KH})$$

where H, K are the foot points of the common perpendicular on the lines $(\vec{a}, \vec{a}^*)$ and $(\vec{b}, \vec{b}^*)$ respectively.

If the length of the common perpendicular is δ, we have

$$\left| \vec{a} \cdot \vec{b}^* + \vec{b} \cdot \vec{a}^* \right| = \delta\ \text{Sin}\ u, \quad (\vec{a}^2 = \vec{b}^2 = 1)$$

Now it is seen that u^* is equal to δ if $(\vec{a}, \vec{b}, \overrightarrow{HK}) > 0$ is equal to $-\delta$ if $(\vec{a}, \vec{b}, \overrightarrow{HK}) < 0$. That is, u^* is the directed length of the common perpendicular. Therefore the dual part is $-\varepsilon u^* \text{Sin}\ u$ which gives

$$\vec{A} \cdot \vec{B} = \text{Cos}\ u - \varepsilon u^* \text{Sin}\ u = \text{Cos}\ \omega, \quad (\omega = u + \varepsilon u^*) \quad (1.9)$$

Now we observe that the dual angle between the straight lines $\vec{A}, \vec{B}$ is a combination of the ordinary angle u and the directed shortest distance u^*. On the other hand, if the dual angle between $\vec{A}, \vec{B}$ is ω, the angle between $-\vec{A}$ and $\vec{B}$ (or $\vec{A}$ and $-\vec{B}$) is

$\pi - \omega = (\pi - u) + \varepsilon(-u^{*})$

As the last properties, we cite the following : If the $R(\alpha)$, $D(\alpha)$ are the real and dual parts of the dual number α, we see from (1.9) that the condition of the lines $\vec{A}, \vec{B}$ to be perpendicular is

$$R(\vec{A} \cdot \vec{B}) = 0 \tag{1.10}$$

the condition for the lines $\vec{A}, \vec{B}$ to be concurrent is

$$D(\vec{A} \cdot \vec{B}) = 0, \quad R(\vec{A} \cdot \vec{B})^2 \neq 1 \tag{1.11}$$

the condition for the lines $\vec{A}, \vec{B}$ to be perpendicularly concurrent is

$$\vec{A} \cdot \vec{B} = 0, \quad (R(\vec{A} \cdot \vec{B})^2 \neq 1) \tag{1.11'}$$

the condition for the lines $\vec{A}, \vec{B}$ to be parallel is

$$R(\vec{A} \cdot \vec{B}) = \pm 1, \text{ (where necessarily } D(\vec{A} \cdot \vec{B}) = 0) \tag{1.12}$$

Let us note that the orthogonal transformations of the unit sphere $\vec{A}^2 = 1$ corresponds to a Euclidean motion in the space of lines $\overline{E}_3$.

1.2. Now let us note some particular properties of the projective plane [2].
Between the points X^0 of P_2 and ordered triple of numbers (x_1, x_2, x_3) different from (0,0,0) a one-one correspondence can be established through an I, J, K coordinate triangle if a common proportionality factor is neglected. The triple (x_1, x_2, x_3) is called the homogeneous coordinates of the point X^0.

The equation of a line A'^0 of P_2, that is the condition that a point (x_1, x_2, x_3) to be on the line A'^0 is

$$a_1' x_1 + a_2' x_2 + a_3' x_3 = 0 \tag{1.13}$$

where the triple $(a_1', a_2', a_3') \neq (0,0,0)$ is also determined uniquely of a common proportionality factor is disregarded . The ordered triple (a_1', a_2', a_3') so determined is called the homogeneous coordinates of the straight line A'^0. Therefore (1.13) is the equation of A'^0 if (a_1', a_2', a_3') is known and is the equation of the point X^0 if (x_1, x_2, x_3) is known. Summarily, (1.13) gives either the points on A'^0, or the lines passing through X^0.

2. Dual Points and Dual Lines on P_2

2.1. Let the real points of P_2 be denoted as $A^0, B^0, \ldots$ and the real lines as $A'^0, B'^0, C''^0, \ldots$ The homogeneous coordinates (x_1, x_2, x_3) of a point X^0, can be interpreted as the Cartesian coordinates of the direction vector $\vec{X}$ of a straight line in $\overline{E}_3$; since a direction in $\overline{E}_3$ is also determined by an ordered-triple of numbers different from (0,0,0) if a common proportionality factor is disregarded. Similarly the homogeneous coordinates (a'_1, a'_2, a'_3) of the real line A'^0 defines a direction vector $\vec{a}'$ in $\overline{E}_3$. Therefore the equation (1.13) can be rewritten in the form

$$\vec{a}' \cdot \vec{x} = 0 \tag{2.1}$$

We called the direction vector $\vec{a}'$ and $\vec{x}$ as the coordinate vectors of A'^0 and X^0. On the other hand, the coordinate vector $\vec{c}'$ of the real line $A^0 B^0 = C'^0$ will be in the form

$$\vec{c}' = \rho\,(\vec{a} \wedge \vec{b}), \quad (\rho \neq 0) \tag{2.2}$$

since it satisfies the properties $\vec{c}' \cdot \vec{a} = \vec{c}' \cdot \vec{b} = 0$ due to (2.1). Similarly, the coordinate vector of the real point $A'^0 B'^0 = C^0$ will be

$$\vec{c} = \rho\,(\vec{a}' \wedge \vec{b}'), \quad (\rho \neq 0) \tag{2.2'}$$

Naturally, the points on the straight line C'^0 and the straight line passing through C^0 can be respectively given by the following equations :

$$\vec{x} = a\vec{a} + b\vec{b}, \quad \vec{y}' = a\vec{a}' + b\vec{b}' \tag{2.3}$$

2.2. Now we will consider the dual points and dual lines of P_2 whose coordinates are dual numbers. But as (0,0,0) was neglected above, in the dual case, also some restrictions need to be necessarily made. The first restriction is the necessity of disregarding such triples which are purely dual as $(\varepsilon a_1, \varepsilon a_2, \varepsilon a_3)$. This must be so because according to (2.2), (2.2'), a pair of purely dual points cannot define a line, neither a pair of purely dual lines is able to define a point. In addition, straight lines determined by a purely dual point and a dual (or real) point will be purely dual. According to (2.2), two purely dual such lines passing through A should determine A, but those two lines will surely determine no point at all. The same argument can also be repeated for the purely dual straight line A'. Hence, together with the real ones, the purely dual lines and points also must be disregarded. Hence the point A and the line A' of P_2 which are neither real nor purely dual have the coordinate vectors

$$\vec{A}_1 = \vec{x} + \varepsilon \vec{x}^*, \quad \vec{A}'_1 = \vec{x}' + \varepsilon \vec{x}'^*, \quad (\vec{x} \neq 0, \vec{x}' \neq 0; \vec{x}^* \neq 0, \vec{x}'^* \neq 0) \tag{2.4}$$

Let us normalize the vectors $\vec{A}_1, \vec{A}'_1$ by $\frac{1}{|\vec{x}|}, \frac{1}{|\vec{x}'|}$ respectively. We use the notation

$$\frac{\vec{x}}{|\vec{x}|} = \vec{x}^0, \quad \frac{\vec{x}'}{|\vec{x}'|} = \vec{x}'^0$$

But the normalized vectors $\frac{\vec{x}^*}{|\vec{x}|}$ and $\frac{\vec{x}'^*}{|\vec{x}'|}$ will again be denoted as $\vec{x}^*, \vec{x}'^*$. Then the normalized coordinate vectors are obtained as

$$\vec{A}_1 = \vec{x}^0 + \varepsilon \vec{x}^*, \quad \vec{A}'_1 = \vec{x}'^0 + \varepsilon \vec{x}'^* \tag{2.4'}$$

From now on, we will use the normalized coordinate vectors (2.4') as the coordinate vectors of dual points and lines of P_2.

The points (or lines) A, B of P_2, which have the normalized vector denoted by

$$\vec{A}^*_1 = \vec{a}^0 + \varepsilon \vec{a}^* \ , \ \vec{B}^*_1 = \vec{a}^0 + \varepsilon \vec{b}^*$$

both having the same real part, will be called dependent points (or lines). Two dependent points can determine a purely dual line and two dependent lines can determine a purely dual point, both of which had been necessarily disregarded above. Hence, as a second restriction, the dependent points(or lines) will not be considered as different from each other.

Let us correspond the dual unit vectors

$$\vec{A} = \vec{x}^0 + \varepsilon \vec{x}^0 \wedge \vec{x}^* \ (= \vec{a} + \varepsilon \vec{a}^*)$$
$$\vec{A}' = \vec{x}'^0 + \varepsilon \vec{x}'^0 \wedge \vec{x}'^* \ (= \vec{a}' + \vec{a}'^*) \ , \ (\vec{A}^2 = \vec{A}'^2 = 1) \tag{2.5}$$

to dual point A and the dual line A'. For this correspondence to be acceptable, the unit vector corresponding to $A(A')$ and to $\alpha A(\alpha A')$ must be the same. In fact, for $\alpha = a + \varepsilon a^*$ we have

$$\alpha \vec{A}_1 = a\vec{x} + \varepsilon (a\vec{x}^* + a^* \vec{x}) \ , \ \alpha \vec{A}'_1 = a\vec{x}' + \varepsilon (a\vec{x}'^* + a^* \vec{x}')$$

and

$$(\alpha \vec{A}_1)^* = \vec{x}^0 + \varepsilon (\frac{\vec{x}^*}{|\vec{x}|} + \frac{a^*}{a}\vec{x}^0) \ , \ (\alpha \vec{A}'^*_1) = \vec{x}'^0 + \varepsilon (\frac{\vec{x}'^*}{|\vec{x}'|} + \frac{a^*}{a}\vec{x}'^0)$$

The vectors to be corresponded to these normalized vectors are obviously the vectors $\vec{A}, \vec{A}'$ given in (2.5), according to the correspondence rule (2.5). Thus, to the dual points and dual lines of P_2, there corresponds directed lines which are the only elements which form $\overline{E}_3$. But, the correspondence should be restricted to the elements which are not excluded by the above mentioned two exceptional cases. Hence, to the all results and their dual correspondence involving the dual lines and dual points in P_2, the same result corresponds in $\overline{E}_3$. For instance, both the dual points of a dual line of P_2 and the dual lines passing through a dual poinr of P_2 have the same corresponding figure in $\overline{E}_3$. On the other hand, both the dependent lines and the dependent points obviously correspond parallel lines in $\overline{E}_3$.

3. The Figures Corresponding in $\overline{E}_3$ to Dual Figures in P_2

3.1. The "Incidence" relation, that is a point being on a given lin (or a line passing through a point) is given by (2.1) as is known. The same relation can be expressed between the dual points and the dual lines of P_2 naturally in terms of the dual unit vectors as

$$\vec{A} \cdot \vec{A}' = 0 \ , \ (\vec{A} = \vec{a} + \varepsilon \vec{a}^* \ , \ \vec{A}' = \vec{a}' + \varepsilon \vec{a}'^*) \tag{3.1}$$

But this is the perpendicular concurrence condition (1.11') of the directed lines in $\overline{E}_3$ corresponding to the dual unit vectors $\vec{A}$ and $\vec{A}'$. Hence the incidence relation in P_2 between the points and lines corresponds to perpendicular concurrence in $\overline{E}_3$. As a result of this fact we have the following : The intersection point C of the lines A', B' in P_2 corresponds the common perpendicular of the lines corresponding to A', B' in $\overline{E}_3$. Likewise, to the line joining A, B in P_2, there corresponds the common perpendicular of the lines corresponding to A, B in $\overline{E}_3$. For example, to a triangle in P_2, there corresponds a figure consisting of three skew lines and their three common perpendiculars. Since there corresponds two parallel lines in $\overline{E}_3$ to two related points (lines) of P_2, obviously their common perpendicular becomes indefinite. Hence the necessity for the identification of dependent elements of P_2 in the dual generalization comes in to the scene once more, since the mapping from P_2 to $\overline{E}_3$ should be one-one

One of the most famous figures (theorems) of P_2 is the well-known Pappos figure. In this figure, every one of the three pairs of lines joining the points 1,2,3 and [1], [2], [3] on the straight lines X, Y in the form [[1] 2] = [12] ; [[2] 1] = [2 1] ; [[2] 3] = [2 3] ; [[3] 2] = [3 2] ; [[3] 1] = [3 1] ; [[1] 3] = [1 3] have three intersection points III, II, I and these three intersection points are on a common straight line [0].

The corresponding figure in $\overline{E}_3$ will be as follows :

Consider three skew lines 1,2,3 perpendicularly intersecting the straight line X, and another triple [1], [2], [3], perpendicularly intersecting the straight line Y. Let us denote the common perpendicular of [1] and 2 by [1 2] and one of [2] and 1 by [2 1]. The other common perpendiculars obtained exactly in the same way will be [2 3], [3 2] and [3 1], [1 3]. Let us call the common perpendicular of [1 2], [2 1] by III, one of [2 3], [3 2] by I and one of [3 1], [1 3] by II. Now, the theorem corresponding to Pappos' theorem in $\overline{E}_3$ is as follows : These three common perpendiculars I, II, III have the same common perpendicular [0]. The result has been plotted axonometrically in Fig. I .

The vectorial demonstration of this result in $\overline{E}_3$ can be given as follows :

Let us denote the dual unit vectors of the skew lines in $\overline{E}_3$ by $\vec{A}, \vec{B}, \vec{C}$ which perpendicularly intersects the line X, and of the ones which intersect Y, by $\vec{A}', \vec{B}', \vec{C}'$. As in the real case it can be easily shown that the necessary and sufficient condition for the three lines to intersect the same line perpendicularly is being zero of the mixed product of the dual unit vectors of three lines. Therefore we have

$$(\vec{A}, \vec{B}, \vec{C}) = (\vec{A}', \vec{B}', \vec{C}') = 0 \tag{3.2}$$

We will denote the common perpendicular of the dual unit vectors $\vec{P}, \vec{Q}$ by $\left[\vec{P}, \vec{Q}\right]$.

Thus, if we use the notation

$$\left[\vec{A}, \vec{B}'\right] = \vec{C}_1 \,,\, \left[\vec{B}, \vec{A}'\right] = \vec{C}_2 \,,\, \left[\vec{B}, \vec{C}'\right] = \vec{A}_1 \,,\, \left[\vec{C}, \vec{B}'\right] = \vec{A}_2 \,,\, \left[\vec{C}, \vec{A}'\right] = \vec{B}_1 \,,\, \left[\vec{A}, \vec{C}'\right] = \vec{B}_2$$

we can write

$$\begin{aligned} &\vec{C}_1 = \frac{\vec{A} \wedge \vec{B}'}{\operatorname{Sin} \gamma_1}, \vec{C}_2 = \frac{\vec{B} \wedge \vec{A}'}{\operatorname{Sin} \gamma_2} \quad ; \quad \vec{A}_1 = \frac{\vec{B} \wedge \vec{C}'}{\operatorname{Sin} \alpha_1}, \vec{A}_2 = \frac{\vec{C} \wedge \vec{B}'}{\operatorname{Sin} \alpha_2} ; \\ &\vec{B}_1 = \frac{\vec{C} \wedge \vec{A}'}{\operatorname{Sin} \beta_1}, \vec{B}_2 = \frac{\vec{A} \wedge \vec{C}'}{\operatorname{Sin} \beta_2} \end{aligned} \tag{3.3}$$

where $\alpha_i, \beta_i, \gamma_i$ (i = 1,2) are the dual angles between the correspondent dual unit vectors. These angles cannot be purely dual since the mentioned lines are skew among themselves. We will obtain

$$\begin{aligned} &\vec{P} = \frac{(\vec{B} \wedge \vec{C}') \wedge (\vec{C} \wedge \vec{B}')}{\operatorname{Sin} \alpha_1 \operatorname{Sin} \alpha_2 \operatorname{Sin} \alpha} \,, \quad \vec{Q} = \frac{(\vec{C} \wedge \vec{A}') \wedge (\vec{A} \wedge \vec{C}')}{\operatorname{Sin} \beta_1 \operatorname{Sin} \beta_2 \operatorname{Sin} \beta} \,, \\ &\vec{R} = \frac{(\vec{A} \wedge \vec{B}') \wedge (\vec{B} \wedge \vec{A}')}{\operatorname{Sin} \gamma_1 \operatorname{Sin} \gamma_2 \operatorname{Sin} \gamma} \end{aligned} \tag{3.4}$$

if we use the notation

$$\left[\vec{A}_1, \vec{A}_2\right] = \vec{P} \;,\; \left[\vec{B}_1, \vec{B}_2\right] = \vec{Q} \;,\; \left[\vec{C}_1, \vec{C}_2\right] = \vec{R}$$

Now we must show that $(\vec{P}, \vec{Q}, \vec{R}) = 0$. Since the dual scalars in the denominators (3.4) will not have any effect it will be enough to prove that

$$\left[(\vec{B}\wedge\vec{C}'')\wedge(\vec{C}'\wedge\vec{B}')\right]\cdot\left\{\left[(\vec{C}'\wedge\vec{A}')\wedge(\vec{A}\wedge\vec{C}'')\right]\wedge\left[(\vec{A}\wedge\vec{B}')\wedge(\vec{B}\wedge\vec{A}')\right]\right\}=0$$

In operations, the vector identities

$$\vec{A}\wedge(\vec{B}\wedge\vec{C})=(\vec{A}\cdot\vec{C})\vec{B}-(\vec{A}\cdot\vec{B})\vec{C} \quad , \quad (\vec{A}\wedge\vec{B})\wedge(\vec{C}\wedge\vec{D})=$$
$$=(\vec{A},\vec{B},\vec{D})\vec{C}-(\vec{A},\vec{B},\vec{C})\vec{D}=(\vec{C},\vec{D},\vec{A})\vec{B}-(\vec{C},\vec{D},\vec{B})\vec{A} \qquad (3.5)$$

will be used and it will be shown at the end that the given expression is identically zero under the conditions (3.2).

We want to show that the correspondent of Desargues figure which is taken as an axiom of P_2 is satisfied in $\overline{E}_3$ also.

But first, let us explain the meaning of the paradox encountered here just now : "The proof of an axiom in any way." The mapping we have defined above uses the homogeneous coordinates. Therefore, it has been assumed that the Desargues axiom is valid in the method used in corresponding homogeneous coordinates to points or lines in P_2 . Otherwise, the figures we call straight lines might as well be broken lines and the mentioned correspondence would have been impossible.

Now let us consider the triangles ABC and $A_0B_0C_0$ in P_2 . Let us use the following notation :

$[AA_0] = A^*$ (we call the line AA_0 as A^*) $\quad [BB_0]=B^*$, $\quad [CC_0]=C^*$;

$[AB] = C''$, $[BC]=A'$, $[CA]=B'$; $[A_0B_0]=C''_0$, $[B_0C_0]=A'_0$, $[C_0A_0]=B'_0$

According to Desargues axiom, if the intersection points of the corresponding sides A',A'_0 ; B',B'_0 ; C'',C''_0 of two triangles denoted by

$[A',A'_0]=A_1$, $[B',B'_0]=B_1$, $[C'',C''_0]=C_1$

are on a straight line D' , the straight lines A^*,B^*,C^* connecting the corresponding vertices pass through the same point D and the converse is also true. (Fig. 2)

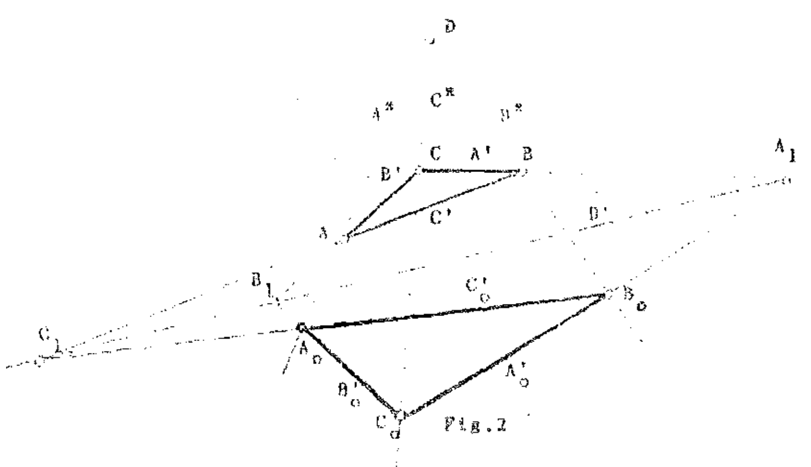

Fig.2

On the other hand we have the following configuration in $\overline{E}_3$. If we denote the dual unit vectors corresponding to dual points and lines of P_2 , similarly as we have done above,

$$\vec{A}^* = \frac{\vec{A} \wedge \vec{A}_0}{\operatorname{Sin} \alpha_1}, \quad \vec{B}^* = \frac{\vec{B} \wedge \vec{B}_0}{\operatorname{Sin} \beta_1}, \quad \vec{C}^* = \frac{\vec{C} \wedge \vec{C}_0}{\operatorname{Sin} \gamma_1};$$

$$\vec{C}' = \frac{\vec{A} \wedge \vec{B}}{\operatorname{Sin} \gamma}, \quad \vec{A}' = \frac{\vec{B} \wedge \vec{C}}{\operatorname{Sin} \alpha}, \quad \vec{B}' = \frac{\vec{C} \wedge \vec{A}}{\operatorname{Sin} \beta};$$

$$\vec{C}_0' = \frac{\vec{A}_0 \wedge \vec{B}_0}{\operatorname{Sin} \gamma_0}, \quad \vec{A}_0' = \frac{\vec{B}_0 \wedge \vec{C}_0}{\operatorname{Sin} \alpha_0}, \quad \vec{B}_0' = \frac{\vec{C}_0 \wedge \vec{A}_0}{\operatorname{Sin} \beta_0}; \tag{3.6}$$

$$\vec{A}_1 = \frac{\vec{A}' \wedge \vec{A}_0'}{\operatorname{Sin} \alpha'}, \quad \vec{B}_1 = \frac{\vec{B}' \wedge \vec{B}_0'}{\operatorname{Sin} \beta'}, \quad \vec{C}_1 = \frac{\vec{C}' \wedge \vec{C}_0'}{\operatorname{Sin} \gamma'}$$

Hence, we must show that the relations

$$(\vec{A}_1, \vec{B}_1, \vec{C}_1) = 0 \quad , \quad (\vec{A}^*, \vec{B}^*, \vec{C}^*) = 0 \tag{3.7}$$

are equivalent. This can be shown by a routine operation by using the identities (3.5).

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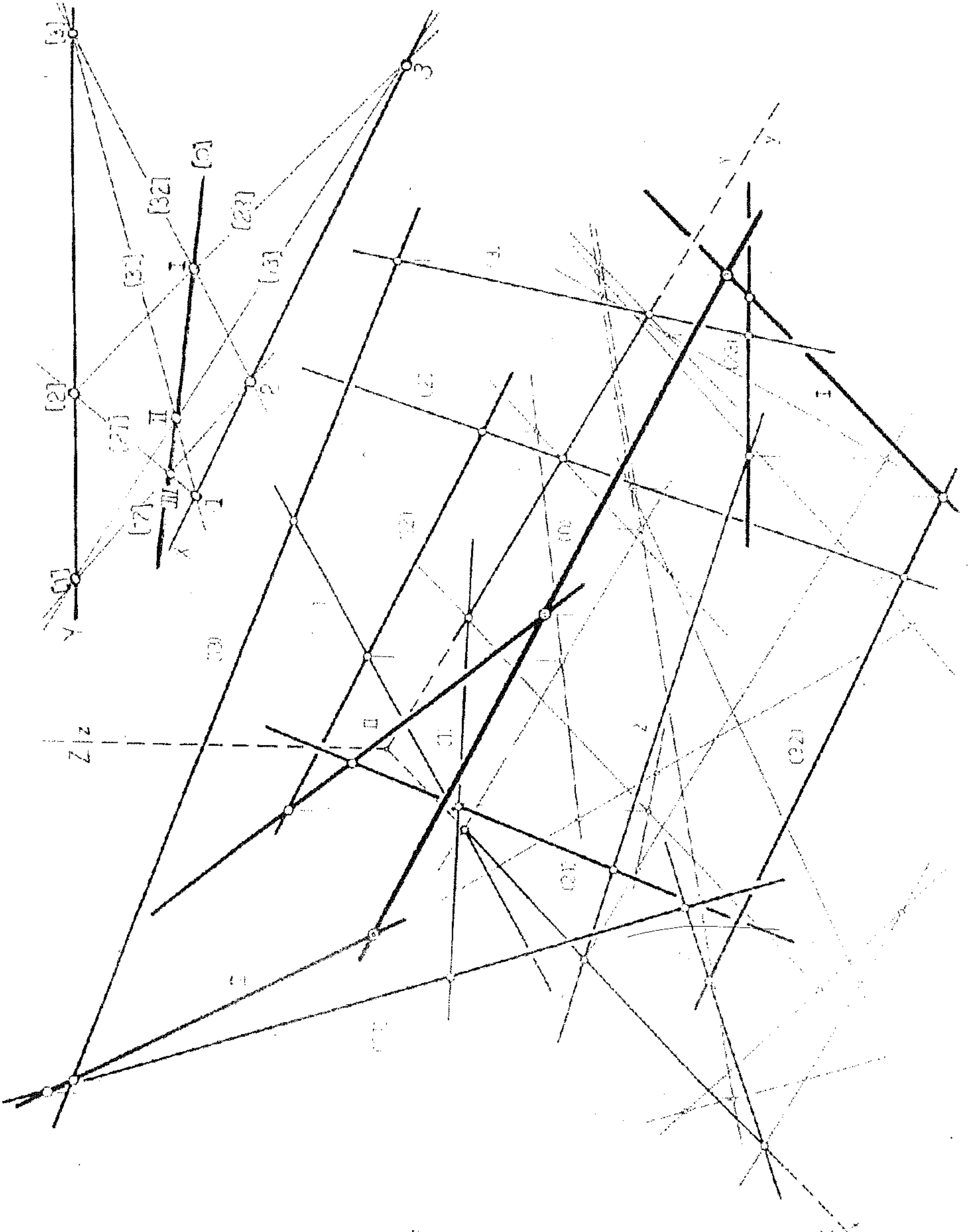

Fig. 1