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ON FOUR-DIMENSIONAL SEMI-DECOMPOSABLE RIEMANNIAN SPACES

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Abstract. The object of the present paper is to study semi-decomposable semi-symmetric and Weyl-semi-symmetric Riemannian spaces.

1. **Introduction** An n-dimensional Riemannian space V_n is said to be semi-symmetric [1] its curvature tensor R_{hijk} satisfies the condition

$$R_{hijk,lm} - R_{hijk,ml} = 0 \tag{1.1}$$

where comma denotes covariant differentiation in V_n. Some authors have called such a space an s-manifold [2]. Further a Riemannian space V_n is said to be Weyl-semi-symmetric [1] if its conformal curvature tensor C_{hijk} satisfies

$$C_{hijk,lm} - C_{hijk,ml} = 0 \tag{1.2}$$

where
$$C_{hijk} = R_{hijk} - \frac{1}{n-2}(g_{ij} R_{hk} - g_{ik}R_{hj} + g_{hk}R_{ij} - g_{hj}R_{ik}) + \frac{R}{(n-1)(n-2)}(g_{hk}g_{ij} - g_{hj}g_{ik}), \tag{1.3}$$

R_{ij} is the Ricci tensor and R denotes the scalar curvature. It follows easily from (1.1) and (1.2) that every semi-symmetric Riemannian space is neecssarily Wyle-semi-symmetric, but the converse is not, in general, true. However it is known [3] that if n≥5 then (1.1) and (1.2) are equivalent. But when n = 4 this is not true. A suitable example was given in [4] (Lemma 1.1). The existence of a semi-symmetric and Weyl-semi-symmetric Riemannian spaces have been proved in [5].

An n-dimensional (n>2) Riemannian space V_n is said to be semi-decomposable [6] if in some coordinates its metric is given by

$$ds^2 = g_{ij}dx^i dx^j = \bar{g}_{ab}dx^a dx^b + \sigma \overset{*}{g}_{\alpha\beta}\, dx^\alpha dx^\beta \qquad (1.4)$$

where $i,j,k, \ldots = 1, 2 \ldots n$; $a, b, c \ldots = 1, 2, \ldots q$, $(q<n)$; $\alpha, \beta, \gamma \ldots = q + 1, q + 2, \ldots n$; $\bar{g}_{ab}$ and σ are functions of $x^1, \ldots x^q$ only and $\overset{*}{g}_{\alpha\beta}$ are functions of $x^{q+1}, \ldots x^n$ only. The two parts of (1.4) are the metrics of a V_q and a V_{n-q} which are called the decomposition spaces of V_n. Throughout this paper each object denoted by a bar is assumed to be formed from $\bar{g}_{ab}$ and each object denoted by star, from $\overset{*}{g}_{\alpha\beta}$. A comma, a dot and a semicolon shall denote covariant differentiation in V_n, V_q and V_{n-q} respectively. If in particular $\sigma = 1$, then the V_n reduces to a decomposable space [7].

The present paper deals with semi-decomposable semi-symmetric and Weyl-semi-symmetric Riemannian spaces with non-constant function σ. In section 3 it is shown that for a four-dimensional semi-decomposable semi-symmetric space, the part V_q is semi-symmetric and the part V_{n-q} is a space of constant curvature. In the last section it is proved that for a four-dimensional semi-decomposable Weyl-semi-symmetric space the part V_q is Weyl-semi-symmetric and the part V_{n-q} is a space of constant carvature.

2. Preliminaries

Let $\overline{\{{}_b{}^a{}_c\}}$, $\bar{R}_{abcd}$, $\bar{R}_{ad}$, $\bar{R}$ denote the Christoffel symbol, curvature tensor, Ricci tensor and scalar curvature respectively of the part V_q and $\overset{*}{\{{}_\beta{}^\alpha{}_\gamma\}}$, $\overset{*}{R}_{\alpha\beta\gamma\delta}$, $\overset{*}{R}_{\alpha\delta}$, $\overset{*}{R}$ denote respectively the Christoffel symbol, curvature tensor, Ricci tensor and scalar carvature of the part V_{n-q} of a semi decomposable space with non-constant function σ. Then ([8], p.16, 17)

$$\left.\begin{array}{l} g_{ab} = \bar{g}_{ab},\ g_{\alpha\beta} = \sigma \overset{*}{g}_{\alpha\beta},\ g^{ab} = \bar{g}^{ab},\ g^{\alpha\beta} = \dfrac{1}{\sigma}\overset{*}{g}{}^{\alpha\beta} \\ g_{a\alpha} = 0,\ g^{a\alpha} = 0 \end{array}\right] \qquad (2.1)$$

and the only Christoffel symbols which are not identically zero are as follows :

$$\{^{c}_{a\ b}\} = \{_{a}\ {}^{\bar{c}}_{b}\},\ \{^{\alpha}_{\beta\ \gamma}\} = \{_{\beta}\ {}^{\overset{*}{\alpha}}_{\gamma}\}$$

$$\{^{a}_{\alpha\ \beta}\} = -\frac{1}{2}\ g^{ab}\sigma_{.b}\ \overset{*}{g}_{\alpha\beta},\ \{^{\alpha}_{a\ \beta}\} = \frac{1}{2}\ \sigma_{.a}\ \delta^{\alpha}_{\beta} \qquad (2.2)$$

Further

$$R_{\alpha\beta\gamma\delta} = 0 \qquad (2.3)$$

$$R_{a\alpha} = 0 \qquad (2.4)$$

$$R_{abcd} = \bar{R}_{abcd} \qquad (2.5)$$

$$R_{abcd,e} = \bar{R}_{abcd\cdot e} \qquad (2.6)$$

$$R_{abcd,ef} = \bar{R}_{abcd\cdot ef} \qquad (2.7)$$

$$R_{ab} = \bar{R}_{ab} + \frac{n-q}{2\sigma}\ T_{ab} \qquad (2.8)$$

$$R_{ab,cd} = \bar{R}_{ab.cd} + \frac{n-q}{2\sigma}\ T_{ab.cd} - \frac{n-q}{2\sigma^2}\ \sigma_{.cd.}T_{ab}$$

$$-\frac{n-q}{2\sigma^2}\sigma_{.d}\ T_{ab.c} - \frac{n-q}{2\sigma^2}\sigma_{.c}\ T_{ab.d} + \frac{n-q}{\sigma^3}\ \sigma_{.c}\sigma_{.d}\ T_{ab.} \qquad (2.9)$$

where $$T_{ab} = \sigma_{.ab} - \frac{1}{2\sigma}\ \sigma_{.a}\ \sigma_{.b} \qquad (2.10)$$

3. Semi-decomposable semi-symmetric space with non-constant function σ.

Let V_n be a semi-decomposable semi-symmetric space.

From (1.1) and (2.7) we obtain

$$\bar{R}_{abcd.ef} - \bar{R}_{abcd.fe} = 0$$

that is, V_q is a semi-symmetric space. .... (A)

Now $R_{\alpha\beta\gamma\delta,c\alpha}$ can be expressed in the form ([8], p. 19)

$$R_{\alpha\beta\gamma\delta,e\alpha} = \frac{3}{4} - \frac{\sigma_{.a}\ \sigma_{.e}}{\sigma}\ \overset{*}{R}_{\alpha\beta\gamma\delta} + A_{ae}\ (\overset{*}{g}_{\beta\gamma}\ \overset{*}{g}_{\alpha\delta} - \overset{*}{g}_{\alpha\gamma}\ \overset{*}{g}_{\beta\delta}) \qquad (3.1)$$

Again, it can be shown that

$$R_{\alpha\beta\gamma\delta,ae} = -\frac{1}{2\sigma}\,\sigma_{.ae}\,\overset{*}{R}_{\alpha\beta\gamma\delta} - B_{ae}\,(\overset{*}{g}_{\beta\gamma}\,\overset{*}{g}_{\alpha\delta} - \overset{*}{g}_{\alpha\gamma}\,\overset{*}{g}_{\beta\delta}) \qquad (3.2)$$

where $\sigma_{.b} = \sigma_{.a}\, g^{\overline{ab}}$, $\Delta_1\sigma = \sigma_{.a}\sigma^{.a}$ $T_a^{\ b} = T_{ae}\ \overline{g}^{eb}$

$$A_{ae} = (\frac{\sigma_{.a}\,\sigma_{.e}}{2\sigma^2}\ \Delta_1\sigma - \frac{\sigma_{.a}\,(\Delta_1\sigma)_{.e}}{8\sigma} - \frac{3}{16\sigma}\sigma_{.e}\,(\Delta_1\sigma)_{.a} + \frac{1}{4}\,\sigma^{.m}\ T_{am.e}$$

$$B_{ae} = \frac{1}{4}\ (\frac{1}{2\sigma}(\Delta_1\sigma)\sigma_{.ae} - \frac{1}{2\sigma^2}\ \Delta_1\sigma\sigma_{.a}\,\sigma_{.e} + \frac{1}{4\sigma}\,(\Delta_1\sigma)_{.e}\,\sigma_{.a} + \sigma_{b}\cdot T_{ab.e} + T^{b}_{\ a}\sigma_{.be}).$$

From (1.1) we get

$$R_{\alpha\beta\gamma\delta.ea} - R_{\alpha\beta\gamma\delta,ae} = 0. \qquad (3.3)$$

In virtue of (3.1) and (3.2) we obtain from (3.3)

$$(\frac{3}{4}\ \frac{\sigma_{.a}\,\sigma_{.e}}{\sigma} + \frac{1}{2}\,\sigma_{.ae})\,\overset{*}{R}_{\alpha\beta\gamma\delta} + (A_{ae} + B_{ae})\,(\overset{*}{g}_{\beta\gamma}\,\overset{*}{g}_{\alpha\delta} - \overset{*}{g}_{\alpha\gamma}\,\overset{*}{g}_{\beta\delta}) = 0 \qquad (3.4)$$

Transvecting (3.4) with $\overset{*}{g}^{\alpha\delta}$ we get

$$(\frac{3}{4}\ \frac{\sigma_{.a}\,\sigma_{.e}}{\sigma} + \frac{1}{2}\,\sigma_{.ae})\,\overset{*}{R}_{\beta\gamma} + (n-q-1)(A_{ae} + B_{ae})\,\overset{*}{g}_{\beta\gamma} = 0$$

Again transvecting the above expression with $\overset{*}{g}^{\beta\gamma}$ we obtain

$$(\frac{3}{4}\ \frac{\sigma_{.a}\,\sigma_{.e}}{\sigma} + \frac{1}{2}\,\sigma_{.ae})\,\overset{*}{R} + (n-q)(n-q-1)(A_{ae} + B_{ae}) = 0. \qquad (3.5)$$

From (3.4) and (3.5) we obtain

$$f_{ae}\,[\overset{*}{R}_{\alpha\beta\gamma\delta} - \frac{\overset{*}{R}}{(n-q)(n-q-1)}\,(\overset{*}{g}_{\beta\gamma}\,\overset{*}{g}_{\alpha\delta} - \overset{*}{g}_{\alpha\gamma}\,\overset{*}{g}_{\beta\delta}) \qquad (3.6)$$

where $f_{ae} = \frac{3}{4}\ \frac{\sigma_{.a}\,\sigma_{.e}}{\sigma} + \frac{1}{2}\,\sigma_{.ae}$ (3.7)

Hence if at least one of the component of f_{ae} is non-zero, then V_{n-q} is a space of constant curvature. .... (B)

In view of (A) and (B) we obtain

Theorem 1. For a semi-decomposable semi-symmetric space with non-constant function σ, the part V_q is semi-symmetric and when atleast one component of f_{ae} defind by (3.7) is non-zero, the part V_{n-q} is a space of constant curvature.

4. Semi-decomposable Weyl-semi-symmetric space with non-constant function σ.

Let V_n be semi-decomposable Weyl-semi-symmetric space.

By virtue of (1.2) $C_{abce,ef} - C_{abce,fe} = 0$. (4.1)

Since $\sigma_{.ef} = \sigma_{.fe}$ (2.9) reduces to

$$R_{ab.ef} - R_{ab,fe} = \bar{R}_{ab.ef} - \bar{R}_{ab.fe} + \frac{n-q}{2\sigma}(T_{ab.ef} - T_{ab.fe}). \tag{4.2}$$

Again since $R_{.ef} = R_{.fe}$, using (1.3), (2.7) and (4.2) in (4.1) we obtain

$$\bar{R}_{abcd.ef} - \bar{R}_{abcd,fe} - \frac{1}{n-2}\Big[\bar{g}_{bc}\Big\{(\bar{R}_{ad.ef} - \bar{R}_{ad,fe}) + \frac{(n-q)}{2\sigma}(T_{ad.ef} - T_{ad,fe})\Big\}$$

$$-\bar{g}_{bd}\Big\{(\bar{R}_{ac.ef} - \bar{R}_{ac,fe}) + \frac{n-q}{2\sigma}(T_{ac.ef} - T_{ac,fe})\Big\} + \bar{g}_{ad}\Big\{(\bar{R}_{bc.ef} - \bar{R}_{bc,fe})$$

$$+\frac{n-q}{2\sigma}(T_{bc.ef} - T_{bc,fe})\Big\} - \bar{g}_{ac}\Big\{(\bar{R}_{bd.ef} - \bar{R}_{bd,fe}) + \frac{n-q}{2\sigma}(T_{bd.ef} - T_{bd,fe})\Big\}\Big] = 0. \tag{4.3}$$

Transvecting (4.3) with $\bar{g}^{bc}$ we get,

$$\bar{R}_{ab.ef} - \bar{R}_{ab,fe} - \frac{1}{n-2}\Big[(q-2)(\bar{R}_{ad.ef} - \bar{R}_{ad,fe}) + \frac{(q-2)(n-q)}{2\sigma}(T_{ad.ef} - T_{ad,fe})\Big] = 0 .$$

Hence $$\bar{R}_{ab.ef} - \bar{R}_{ab.fe} = \frac{q-2}{2\sigma}(T_{ab.cf} - T_{ab.fc}). \tag{4.4}$$

Now $C_{abcd,ef} - C_{abcd,fe}$

$$= \bar{R}_{abcd.ef} - \bar{R}_{abcd.fe} - \frac{1}{q-2}[\bar{g}_{bc}(\bar{R}_{ad.ef} - \bar{R}_{ad.fe}) - \bar{g}_{bd}(\bar{R}_{ac.ef} - \bar{R}_{ac.fe})$$

$$+ \bar{g}_{ad}(\bar{R} \quad - \bar{R}_{bc.fe}) - \bar{g}ac(\bar{R}_{bd.ef} - \bar{R}_{bd.fe})]$$

$$-\frac{q-n}{n-2}[\{\bar{g}_{bc}(\bar{R}_{ad.ef} - \bar{R}_{ad.fe})\frac{1}{q-2} - \frac{1}{2\sigma}\bar{g}_{bd}(T_{ad.ef} - T_{ad.fe})\}$$

$$-\bar{g}_{bd}\{\frac{1}{q-2}(\bar{R}_{ac.ef} - \bar{R}_{ac.fe}) - \frac{1}{2\sigma}(T_{ac.ef} - T_{ac.fe})\}$$

$$+\bar{g}_{ad}\{\frac{1}{q-2}(\bar{R}_{bc.ef} - \bar{R}_{bc.fe}) - \frac{1}{2\sigma}(T_{bc.ef} - T_{bc.fe})\}$$

$$-\bar{g}_{ac}\{\frac{1}{q-2}(\bar{R}_{bd.ef} - \bar{R}_{bd.fe}) - \frac{1}{2\sigma}(T_{bd.ef} - T_{bd.fe})\} \tag{4.5}$$

By virtue of ( ) the equation (4.5) reduces to

$$C_{abcde,ef} - C_{abcd,fe} = \bar{C}_{abcd,ef} - \bar{C}_{abcd \cdot fe}$$

Hence from (1) we obtain

$$\bar{C}_{abcd.ef} \quad \bar{C}_{abcd:fe} = 0$$

That is, V_q is Weyl-semi-symmetric space .... (C)

Now from (1 we have

$$C_{\alpha\beta\gamma\delta,\, e\alpha} - C_{\alpha\beta\gamma\delta,\alpha e} = R_{\alpha\beta\gamma\delta,\, e\alpha} - R_{\alpha\beta\gamma\delta,\alpha e} - \frac{1}{n-2}[(R_{\alpha\delta,e\alpha} - R_{\alpha\delta,\alpha e})g_{\beta\gamma}$$

$$-(R_{\alpha\gamma,e\alpha} - R_{\alpha\gamma,\alpha e})g_{\beta\delta} + (R_{\beta\gamma,e\alpha} - R_{\beta\gamma,\alpha e})g_{\alpha\delta} - (R_{\beta\delta,e\alpha} - R_{\beta\delta,\alpha e})g_{\alpha\gamma}] = 0 \tag{4.6}$$

since $R_{,\alpha e} = R_{,e\alpha} = 0$.

Now it can be shown that

$$R_{\alpha\delta,e\alpha} = \frac{3}{4} \sigma^2 \sigma_{\cdot\,\cdot\,e} \overset{*}{R}_{\alpha\delta} + D_{ae} \overset{*}{g}_{\alpha\delta} \tag{4.7}$$

$$R_{\alpha\delta,\alpha e} = \left(\frac{\sigma_{\cdot a}\sigma_{\cdot e}}{\sigma^2} \quad - \frac{1}{\sigma} \sigma_{\cdot ae}\right) \overset{*}{R}_{\alpha\delta} + E_{ae} \overset{*}{g}_{\alpha\delta} \tag{4.8}$$

where $T = T_{ab}\, g^{ab}$, $P = \quad + \frac{n-q-1}{2\sigma} \Delta_1 \sigma$

$$D_{ae} = \frac{3}{8} \frac{\sigma_{\cdot a}\sigma_{\cdot e}}{\sigma^2} P - \frac{1}{4} \quad \frac{\sigma_{\cdot a}\sigma_{\cdot e}}{\sigma} + \frac{1}{2} \{ \bar{R}_{am.e} + \frac{n-q}{2\sigma} T_{am.e} - \frac{\sigma_{\cdot e}}{\sigma} T_{am})\} \sigma^{.m}$$

$$- \frac{\sigma_{\cdot e}}{4\sigma} (\bar{R}_{am} + \frac{1}{2} \; q \; T_{am}) \sigma^{.m}$$

and $$E_{ae} = \frac{1}{2} [(\sigma_{\cdot b} \; \bar{R}^{b}{}_{a})_{.e} - \frac{1}{\sigma} \; \sigma_{\cdot e}\sigma_{\cdot b} \; \bar{R}^{b}{}_{a} + \frac{n-q}{2\sigma} ((\sigma_{\cdot b} T^{b}{}_{a})_{.e} - \frac{\sigma_{\cdot e}}{\sigma} \sigma_{b} T^{b}{}_{a})$$

$$- \frac{P}{2\sigma} \sigma_{\cdot ae} + \frac{P}{\sigma^2} \sigma \quad \sigma_{\cdot e} - \frac{1}{2} P_{\cdot a} \sigma_{\cdot a}]$$

From (4.7) and (4.8), we get

$$R_{\alpha\delta,e\alpha} - R_{\alpha\delta,\alpha e} = \frac{1}{2\sigma} (\sigma_{\cdot ae} - \frac{1}{2\sigma} \sigma_{\cdot a}\, \sigma_{\cdot e}) \overset{*}{R}_{\alpha\delta} + (D_{ae} - E_{ae}) \overset{*}{g}_{\alpha\delta} \tag{4.9}$$

Also, it can be shown that

$$R_{\beta\gamma,\alpha e} = R_{\beta\gamma,e\alpha} = \; - \frac{3}{2\sigma} \; \sigma_{\cdot e}) \overset{*}{R}_{\alpha\delta\beta\gamma;\;\alpha} \tag{4.10}$$

Now let us suppose that at least one component of T_{ae} is zero such that $\sigma_{\cdot e}$ and $\sigma_{\cdot e}$ are non-zero. By virtue of (4.9), (4.10), (3.1) and (3.2) the equation (4.6) takes the form

$$\frac{\sigma_{\cdot a}\sigma_{\cdot e}}{\sigma} \overset{*}{R}_{\alpha\beta\gamma\delta} + M_{ae} (\overset{*}{g}_{\beta\gamma} \overset{*}{g}_{\alpha\delta} - \overset{*}{g}_{\beta\delta} \overset{*}{g}_{\alpha\gamma}) = 0$$

where $M_{ae} = A_{ae} + B_{ae} - \frac{\sigma}{n-2}(D_{ae} - E_{ae})$

Now proceeding similarly as in section 3 we obtain

$$\overset{*}{R}_{\alpha\beta\gamma\delta} = \frac{\overset{*}{R}}{(n-q)(n-q-1)} \left(\overset{*}{g}_{\beta\gamma} \overset{*}{g}_{\alpha\delta} - \overset{*}{g}_{\beta\delta} \overset{*}{g}_{\alpha\gamma} \right)$$

That is, V_{n-q} is the space of constant curvature . (D)

In view of (C) and (D) we get

Theorem 2. For a semi-decomposable Weyl-semi-symmetric space with noh constant function σ the part V_q is Weyl-semi-symmetric and the part V_{n-q} is a space of constant curvature.

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