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ON A MINIMIZATION OF THE FIRST EIGENVALUE OF THE LAPLACE OPERATOR OVER DOMAINS

A.A Niftiyev and Y.S. Qasimov

Abstract - The problem of minimization of the first eigenvalue of Laplace operator over domains is considered in present paper. For the solving of this problem the given by authors definition of the variation of the domain is used.

The numerical algorithm is proposed for the solving of this problem in general case.

The finding of the first eigenvalue plays an important role in the spectral theory of operators. The minimization of this eigenvalue gives possibility to provide the stability and improve some characteristics of the system. But the solving of the control problem over domains meets some dificulities related with the variation of the domain.

Next problem is considered in present work

$-\Delta u = \lambda u \,, \qquad x \in D, \qquad (1)$

$u(x) = 0, \qquad x \in S_D, \qquad (2)$

where D is bounded domain from E^n, S_D -its border.

The first eigenvalue of problem (1), (2) is to be minimized over domain D.

By M we define the set of convex bounded domains. As shown in [2] the pairs of elements from M form a lineary space. Introduce next scalar product in this space. Let

$a = (A_1, A_2), \quad B = (B_1, B_2), \qquad A_i, B_i \in M, \quad i = 1, 2.$

Then

$(a, b) = \int_{S_B} P(\xi) q(\xi) d\xi,$

here $p(x) = P_{A_1}(x) - P_{A_2}(x)$, $q(x) = P_{B_1}(x) - P_{B_2}(x)$, $P_{A_i}(x), P_{B_i}(x)$ are support functions of the sets $A_i, B_i, i = 1, 2$.

Let

$K = \{D \in K_0, S_D \in C_2\},$

where K_0 is any convex underset of M. As is known [1], for fixed domain D first eigenvalue of problem (1),(2) is defined by formulae

$\lambda_1(D) = \inf I(u, D), \; u \in C_2(D)$

where

$$I(u,D)=\frac{\int\limits_D\left\|\frac{\partial u}{\partial x}\right\|^2dx}{\int\limits_D|u|^2dx}. \qquad (3)$$

Thus, we obtain next problem

$$I(u,D)\to\min,\ u\in C_2^0(D),D\in K \qquad (4)$$

Theorem. Let (u^*,D^*) be an optimal pair for problem (3). Then the relation

$$\int\limits_{S_{D^*}}\left\|\frac{\partial u}{\partial x}\right\|^2[P_D(n(\xi))-P_{D^*}(n(\xi))]d\xi\leq 0 \qquad (5)$$

holds for arbitrary $D\in K$.where u^* is a solution of problem (1),(2) by $D=D^*$, $n(\xi)$ is a normal to S_{D^*} in the point ξ.

Proof .Define

$$I_1(u,D)=\int\limits_D\left\|\frac{\partial u}{\partial x}\right\|^2dx,$$

$$I_2(u,D)=\int\limits_D\|u\|^2dx.$$

Then as follows from (3)

$$I(u,D)=\frac{I_1(u,D)}{I_2(u,D)}$$

Take arbitrary pairs $(D,u)\in K\times C_2(D),(\bar{D},\bar{u})\in K\times C_2(\bar{D})$.where $\|d\|<h,d=(\bar{D},D)$, h-is enough small number.

Consider next functional

$$J(D)=\int\limits_D f(x)dx,$$

here $f(x)$ is continous differentable on R^n function. Then using results from [5] it is not difficult to show that the first variation of $J(D)$ is calculated by formulae

$$\delta J(D,\bar{D})=\int\limits_{S_D}f(\xi)[P_{\bar{D}}(n(\xi))-P_D(n(\xi))]d\xi \qquad (6)$$

Using this formulae we calculate first variation of the functional $I(u,D)$.

It is evindent, that

$$\delta I=\frac{\delta I_1\cdot I_2-\delta I_2'\cdot I_1}{I_2^2}$$

Let $(D^*,u^*)\in K\times C_2(D^*)$ is an optimal pair for problem (3). Then

$$\delta I_1(u^*,D^*)\cdot I_2(u^*,D^*)-\delta I_2(u^*,D^*)\cdot I_1(u^*,D^*)\geq 0$$

Considering

$$\lambda_1^*=\frac{I_1(u^*,D^*)}{I_2(u^*,D^*)}$$

we have

$$\delta I_1(u^*,D^*)-\lambda_1^*\delta I_2(u^*,D^*)\geq 0 \qquad (7)$$

The first variations of functionals I_1,I_2 are defined by formulae

$$\delta I_1(u^*, D^*) = -2 \int_{D^*} \Delta u^*(x)\cdot\delta u(x)dx - \int_{S_{D^*}} \left|\frac{\partial u(\xi)}{\partial x}\right|^2 [P_D(n(\xi)) - P_{D^*}(n(\xi))]\, d\xi,$$

$$\delta I_2(u^*, D^*) = 2 \int_{D^*} u^*(x)\cdot\delta u(x)dx,$$

where

$\delta u(x) = u(x) - u^*(x), (D, u) \in K \times C_2(D)$.

Considering these relations in (7) we obtain (5).

We consider next particular cases:

1.Let K has from

$K = \{D \in M, D_0 \subset D \subset D_1\}$.

where D_0, D_1 are given convex sets. Then it is follows from (4) that $D^* = D_1$.

2.Let K be set of squares

$D = \{(x_1, x_2) : 0 < x_1 < l, 0 < x_2 < m\}$.

where $l, m \leq M < \infty$.

In this case from Theorem1 next condition of optimality for the domain D^* is obtained

$\frac{1}{l^{*3}}(l - l^*) + \frac{1}{m^{*3}}(m - m^*) \leq 0.$

This condition coincides with the condition obtained from the known formulae for the fist eigenvalue of Laplas's operator in square

$\lambda_1 = \pi^2(\frac{1}{l^2} + \frac{1}{m^2}).$

For the numerical solution of this problem in general case next algorithm is proposed.

Step1.The initial $D^{(0)} \in K$ is taken and first eigenvalue $u^{(0)}$ of problem (1), (2) is found.

Step2.Solving the problem

$$\int_{S_{D^{(0)}}} \left|\frac{\partial u^{(0)}}{\partial x}\right|^2 P_D(n(\xi))d\xi \to \min, D \in K$$

find $P(x)$;

Step3. $D^{(1)}$ is found [2] as a subgradient of the function $P(x)$ in the point $x = 0$ i.e.

$D^{(1)} = \partial P(0)$

The process continues till the satisfing of any condition of optimality.

Note : Results are generalized for the positive defined , perfectly contionous operator.

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Institute of Applied Mathematics,Baku State University

M.Rasulzadeh,Khalilov str.23, Baku 370148, Azerbaijan