İstanbul Üniv. Fen Fak. Mat. Dergisi **57-58** (1998-1999), 163-213

ON COMPLEX CHARACTERS OF SL(4,q)

NEŞE YELKENKAYA

Abstract

In this paper we restricted the complex characters of GL(4,q) down to SL(4,q) by using the character values on classes of GL(4,q). Using a version of Clifford's Theorem making use of properties of roots of unity, we determine which restricted characters are irreducible, which of them are split and how many parts they split up into irreducible components. Thus, we determine some of irreducible complex characters of SL(4,q) and degrees of conjugate irreducible components of all reducible restricted characters.

INTRODUCTION

Let q be a fixed prime power and GF(q) be Galois field with q elements. We regard each $GF(q^d)$ as an extension of GF(q) and we can think of $GF(q^d)$, for $1 \leq d \leq n$, as subfields of $GF(q^{n!})$.

Let GL(n,q) denote the group of all non-singular nxn matrices over GF(q) and SL(n,q) denote the group of nxn matrices over GF(q) with determinant unity. Let $A \in GL(n,q)$ have characteristic polynomial

$$f_1^{k_1} f_2^{k_2} \dots f_N^{k_N}$$

where $f_1, f_2, \dots, f_N$ are distinct irreducible polynomials over GF(q), $k_i \geq 0$ (i=1, ..., N) and $d_1, d_2, \dots d_N$ are the respective degrees of $f_1, f_2, \dots, f_N$; $\sum_{i=1}^{N} k_i d_i = n$. We will denote conjugacy class c of A by the symbol

$$c = (f_1^{\upsilon_1} f_2^{\upsilon_2} \dots f_N^{\upsilon_N})$$

where $v_1, v_2, \dots, v_N$ are certain partitions of $k_1, k_2, \dots, k_N$ respectively. Let **F** be the set of irreducible polynomials f=f(t) over GF(q), of degrees ≤ n, excepting the polynomial t. The classes c of GL(4,q) are as follows:

I) $c = (f^{\{1\}})$, $d(f) = 4$

II) $c = (f_1^{\{1\}} f_2^{\{1\}})$, $d(f_1) = 1$, $d(f_2) = 3$

III) $c = (f_1^{\{1\}} f_2^{\{1\}}),\ d(f_i) = 2\ (i = 1,2)$

IV₁) $c = (f^{\{2\}}),\ d(f) = 2$

IV₂) $c = (f^{\{1^2\}}),\ d(f) = 2$

V) $c = (f_1^{\{1\}} f_2^{\{1\}} f_3^{\{1\}}),\ d(f_1) = d(f_2) = 1,\ d(f_3) = 2$

VI₁) $c = (f_1^{\{2\}} f_2^{\{1\}}),\ d(f_1) = 1,\ d(f_2) = 2$

VI₂) $c = (f_1^{\{1^2\}} f_2^{\{1\}}),\ d(f_1) = 1,\ d(f_2) = 2$

VII) $c = (f_1^{\{1\}} f_2^{\{1\}} f_3^{\{1\}} f_4^{\{1\}}), d(f_i) = 1\ (i = 1,2,3,4)$

VIII₁) $c = (f_1^{\{1\}} f_2^{\{1\}} f_3^{\{2\}}),\ d(f_i) = 1\ (i = 1,2,3)$

VIII₂) $c = (f_1^{\{1\}} f_2^{\{1\}} f_3^{\{1^2\}}),\ d(f_i) = 1\ (i = 1,2,3)$

IX₁) $c = (f_1^{\{1\}} f_2^{\{3\}}),\ d(f_i) = 1\ (i = 1,2)$

IX₂) $c = (f_1^{\{1\}} f_2^{\{12\}}), d(f_i) = 1\ (i = 1,2)$

IX₃) $c = (f_1^{\{1\}} f_2^{\{1^3\}}),\ d(f_i) = 1\ (i = 1,2)$

X₁) $c = (f_1^{\{4\}}),\ d(f_i) = 1$

X₂) $c = (f_1^{\{13\}}),\ d(f) = 1$

X₃) $c = (f^{\{2^2\}}),\ d(f) = 1$

X₄) $c = (f^{\{1^2\}}),\ d(f) = 1$

X₅) $c = (f^{\{1^4\}}),\ d(f) = 1$

XI₁) $c = (f_1^{\{2\}} f_2^{\{2\}}),\ d(f_i) = 1$

XI₂) $c = (f_1^{\{2\}} f_2^{\{1^2\}}),\ d(f_i) = 1$

XI₃) $c = (f_1^{\{1^2\}} f_2^{\{1^2\}}),\ d(f_i) = 1$

where $f_i \in \mathbf{F}$, i=1,2,3,4.

Two elements T and U of GL(n,q) have the same canonical form, if and only if, there exist a $W \in GL(n,q)$ such that $U=W^{-1}TW$ ([1], p.228). Then two conjugate elements of GL(n,q) have the same canonical form.

Let denote by λ_t, μ_t marks of the $GF(q^t)$ not in the $GF(q^\tau)$, $\tau<t$. For simplicity the subscript unity is omitted from the marks α, β, γ, δ of the GF(q). The types of canonical form of the elements of GL(4,q) are given in Table 1.

Two elements of GL(4,q) which have the same canonical form are conjugate in GL(4,q). But it is not true for SL(4,q). By using the method in [1] we can sec that the elements having the canonical forms of types L II, III, IV_2, V, VI_1, VI_2, VII, $VIII_1$, $VIII_2$, IX_1, IX_2, IX_3, X_2, X_4, X_5, XI_2 and XI_3 are conjugate in SL(4,q), but for the canonical forms of types IV_1, X_1, X_3 and XI_1 the conjugacy classes of GL(4,q) split in SL(4,q). Let d=(4,q-1). The conjugacy classes of the types IV_1, X_3, XI_1 split up two classes in SL(4,q) for d=2 or d=4. The conjugacy classes of the type X_1 split up two and four classes in SL(4,q) respectively for d=2 and d=4.

Type	Canonical forms of the elements.	Number of distinct canonical forms.	Type	Canonical forms of the elements.	Number of distinct canonical forms.
I	$\begin{pmatrix} \lambda_4 & & & \\ & \lambda_4^q & & \\ & & \lambda_4^{q^2} & \\ & & & \lambda_4^{q^3} \end{pmatrix}$	$\frac{1}{4}(q^4-q^2)$	IX_1	$\begin{pmatrix} \alpha & & & \\ & \beta & 1 & \\ & & \beta & 1 \\ & & & \beta \end{pmatrix}$	$(q-1)(q-2)$
II	$\begin{pmatrix} \lambda_3 & & & \\ & \lambda_3^q & & \\ & & \lambda_3^{q^2} & \\ & & & \lambda_1 \end{pmatrix}$	$\frac{1}{3}(q^3-q)(q-1)$	IX_2	$\begin{pmatrix} \alpha & & & \\ & \beta & & \\ & & \beta & 1 \\ & & & \beta \end{pmatrix}$	$(q-1)(q-2)$
III	$\begin{pmatrix} \lambda_2 & & & \\ & \lambda_2^q & & \\ & & \mu_2 & \\ & & & \mu_2^q \end{pmatrix}$	$\frac{1}{8}(q^2-q)(q^2-q-2)$	IX_3	$\begin{pmatrix} \alpha & & & \\ & \beta & & \\ & & \beta & \\ & & & \beta \end{pmatrix}$	$(q-1)(q-2)$
IV_1	$\begin{pmatrix} \lambda_2 & 1 & & \\ & \lambda_2 & & \\ & & \lambda_2^q & 1 \\ & & & \lambda_2^q \end{pmatrix}$	$\frac{1}{2}(q^2-q)$	X_1	$\begin{pmatrix} \alpha & 1 & & \\ & \alpha & 1 & \\ & & \alpha & 1 \\ & & & \alpha \end{pmatrix}$	$(q-1)$

IV_2	$\begin{pmatrix} \lambda_2 & & & \\ & \lambda_2 & & \\ & & \lambda_2^q & \\ & & & \lambda_2^q \end{pmatrix}$	$\frac{1}{2}(q^2 - q)$	X_2	$\begin{pmatrix} \alpha & 1 & & \\ & \alpha & 1 & \\ & & \alpha & \\ & & & \alpha \end{pmatrix}$	$(q-1)$
V	$\begin{pmatrix} \lambda_1 & & & \\ & \mu_1 & & \\ & & \lambda_2 & \\ & & & \lambda_2^q \end{pmatrix}$	$\frac{1}{4}\{(q^2-1)(q-1)(q-2)\}$	X_3	$\begin{pmatrix} \alpha & 1 & & \\ & \alpha & & \\ & & \alpha & 1 \\ & & & \alpha \end{pmatrix}$	$(q-1)$
Vi_1	$\begin{pmatrix} \lambda_1 & 1 & & \\ & \lambda_1 & & \\ & & \lambda_2 & \\ & & & \lambda_2^q \end{pmatrix}$	$\frac{1}{2}\{(q^2-1)(q-1)$	X_4	$\begin{pmatrix} \alpha & 1 & & \\ & \alpha & & \\ & & \alpha & \\ & & & \alpha \end{pmatrix}$	$(q-1)$
VI_2	$\begin{pmatrix} \lambda_1 & & & \\ & \lambda_1 & & \\ & & \lambda_2 & \\ & & & \lambda_2^q \end{pmatrix}$	$\frac{1}{2}(q^2-1)(q-1)$	X_5	$\begin{pmatrix} \alpha & & & \\ & \alpha & & \\ & & \alpha & \\ & & & \alpha \end{pmatrix}$	$(q-1)$
VII	$\begin{pmatrix} \alpha & & & \\ & \beta & & \\ & & \gamma & \\ & & & \delta \end{pmatrix}$	$\frac{1}{24}\{(q-1)(q-2)(q-3)(q-4)\}$	XI_1	$\begin{pmatrix} \alpha & 1 & & \\ & \alpha & & \\ & & \gamma & 1 \\ & & & \gamma \end{pmatrix}$	$\frac{1}{2}(q-1)(q-2)$
$VIII_1$	$\begin{pmatrix} \alpha & & & \\ & \beta & & \\ & & \gamma & 1 \\ & & & \gamma \end{pmatrix}$	$\frac{1}{2}(q-1)(q-2)(q-3)$	XI_2	$\begin{pmatrix} \alpha & 1 & & \\ & \alpha & & \\ & & \gamma & \\ & & & \gamma \end{pmatrix}$	$(q-1)(q-2)$
$VIII_2$	$\begin{pmatrix} \alpha & & & \\ & \beta & & \\ & & \gamma & \\ & & & \gamma \end{pmatrix}$	$\frac{1}{2}(q-1)(q-2)(q-3)$	XI_3	$\begin{pmatrix} \alpha & & & \\ & \alpha & & \\ & & \gamma & \\ & & & \gamma \end{pmatrix}$	$\frac{1}{2}(q-1)(q-2)$

Table 1

1. THE COMPLEX CHARACTERS OF THE GROUP GL(4,q).

Let $GF(q^{n!})^\times = GF(q^{n!})-\{0\} = \langle \varepsilon_* \rangle$ and $\varepsilon_s = \varepsilon_*^{(q^{n!}-1)/(q^s-1)}$ ($i \le s \le n$). Then $GF(q^s)^\times = \langle\varepsilon_s\rangle$. Each non-zero element of $GF(q^s)$ has an expression ε_s^k, and in this, k is uniquely determined mod(q^s-1). The condition for ε_s^k to have the degree s is that all its conjugates $\varepsilon_s^k, \varepsilon_s^{kq}, \ldots, \varepsilon_s^{kq^{s-1}}$ should be distinct, which means that

$$k, kq, \ldots, kq^{s-1} \qquad (1)$$

are distinct residues mod (q^s-1). We shall say that each of the integer (1) is an s-primitive, and that the set (1) is an s-simplex g with k, kq, ..., kq^{s-1} as its roots.

If $\rho = \{1^{r_1}2^{r_2}\ldots\}$ is a partition of n, there is a principal type of class of GL(n,q), represented by

$$c = (f_{11} \ldots f_{1r_1}\ f_{21} \ldots f_{2r_2}\ \ldots)$$

$f_{d1}, \ldots f_{dr_d} \in \mathbf{F}$ being distinct polynomials of degree d (d=1,2,...). We introduce a set X^ρ of variables $x_{di}^\rho = x_{di}$ called ρ- variables. For each positive integer d there are r_d variables $x_{d1}, \ldots, x_{dr_d}$ and each x_{di} said to have degree $d(x_{di})=d$ ($i=1,2,\ldots, r_d$). For each partition $\rho = \{1^{r_1}2^{r_2}\ldots\}$ of n, define the set Y^ρ of "dual ρ-variables" $y_{di}^\rho = y_{di}$ ($i=1,2,\ldots,r_d; d=1,2,\ldots$) and say that y_{di} has degree $d(y_{di})=d$.

For the other definitions and concepts see [3]. By the Theorem 14. in [3], if we know $Q_\rho^\lambda(q)$ polynomials (which are given for n=1, ... 5. in [3]). for ρ, λ partitions of n, by evaluating all modes m of substitution of Y^ρ into c and all modes m′ of substitution of X^ρ into c we obtain all irreducible characters of GL(n,q) and the values of this characters on c. More explicitly (for GL(4,q)):

Type I: They are of type $(g^{\{1\}})$, where g is a 4-simplex, d(g)=4. Let us denote them by χ_1^k where k is a root of g.

In the following, θ is a generating character of the multiplicative group $GF(q^{n!})^\times$ and γ_f is a root of f.

The type of class e	The values of χ_1^k on e
I	$(-1)\{\theta^k(\gamma_f)+\theta^{qk}(\gamma_f)+\theta^{q^2k}(\gamma_f)+\theta^{q^3k}(\gamma_f)\}$
IV_1	$(-1)\{\theta^k(\gamma_f)+\theta^{qk}(\gamma_f)\}$
VI_2	$(q^2-1)\{\theta^k(\gamma_f)+\theta^{qk}(\gamma_f)\}$
X_1	$(-1)\theta^k(\gamma_f)$
X_2	$(q-1)\theta^k(\gamma_f)$
X_3	$(q-1)\theta^k(\gamma_f)$
X_4	$(q-1)(1-q^2)\theta^k(\gamma_f)$
X_5	$(q-1)(1-q^2)(1-q^3)\theta^k(\gamma_f)$

For the other types of classes χ_1^k (e)=0 and,

$$\chi_1^k(1)=(q-1)^3(q+1)(q^2+q+1)$$

is the degree of these characters.

Type II: They are of type $(g_1^{\{1\}}g_2^{\{1\}})$, $d(g_1)=1$, $d(g_2)=3$. Let us denote them by the symbol $\chi_2^{k_1k_2}$ where k_1, k_2 are a roots of g_1, g_2 respectively.

The type of class e	The values of $\chi_2^{k_1k_2}$ on e
II	$\{\theta^{k_1}(\gamma_{f_1})\}\{\theta^{k_2}(\gamma_{f_2})+\theta^{qk_2}(\gamma_{f_2})+\theta^{q^2k_2}(\gamma_{f_2})\}$
IX_1	$\theta^{k_1}(\gamma_{f_1})\theta^{k_2}(\gamma_{f_2})$
IX_2	$(1-q)\theta^{k_1}(\gamma_{f_1})\theta^{k_2}(\gamma_{f_2})$
IX_3	$(1-q)(1-q^2)\theta^{k_1}(\gamma_{f_1})\theta^{k_2}(\gamma_{f_2})$
X_1	$\theta^{k_1}(\gamma_f)\theta^{k_2}(\gamma_f)$
X_2	$\theta^{k_1}(\gamma_f)\theta^{k_2}(\gamma_f)$

X_3	$(1-q^2)\theta^{k_1}(\gamma_f)\theta^{k_2}(\gamma_f)$
X_4	$(1-q^2)\theta^{k_1}(\gamma_f)\theta^{k_2}(\gamma_f)$
X_5	$(1-q^2)(1-q^4)\theta^{k_1}(\gamma_f)\theta^{k_2}(\gamma_f)$

For the other types of classes $\chi_2^{k_1k_2}(c)=0$ and,

$$\chi_2^{k_1k_2}(1)=(1-q^2)(1-q^4)$$

is the degree of these characters.

Type III: They are of type $(g_1^{\{1\}}g_2^{\{1\}})$, $d(g_i)=2$ $(i=1,2)$. Let us denote them by the symbol $\chi_3^{k_1k_2}$ where k_1, k_2 are a roots of g_1, g_2 respectively.

The type of class c	The values of $\chi_3^{k_1k_2}$ on c
III	$\{\theta^{k_1}(\gamma_{f_1})+\theta^{qk_1}(\gamma_{f_1})\}\{\theta^{k_2}(\gamma_{f_2})+\theta^{qk_2}(\gamma_{f_2})\}$
	$+\{\theta^{k_1}(\gamma_{f_2})+\theta^{qk_1}(\gamma_{f_2})\}\{\theta^{k_2}(\gamma_{f_1})+\theta^{qk_2}(\gamma_{f_1})\}$
IV_1	$\{\theta^{k_1}(\gamma_f)+\theta^{qk_1}(\gamma_f)\}\{\theta^{k_2}(\gamma_f)+\theta^{qk_2}(\gamma_f)\}$
IV_2	$(q^2+1)\{\theta^{k_1}(\gamma_f)+\theta^{qk_1}(\gamma_f)\}\{\theta^{k_2}(\gamma_f)+\theta^{qk_2}(\gamma_f)\}$
VI_1	$\theta^{k_1}(\gamma_{f_1})\{\theta^{k_2}(\gamma_{f_2})+\theta^{qk_2}(\gamma_{f_2})\}$
	$+\theta^{k_2}(\gamma_{f_1})\{\theta^{k_1}(\gamma_{f_2})+\theta^{qk_1}(\gamma_{f_2})\}$
VI_2	$(1-q)\{\theta^{k_1}(\gamma_{f_1})\{\theta^{k_2}(\gamma_{f_2})+\theta^{qk_2}(\gamma_{f_2})\}$
	$+\theta^{k_2}(\gamma_{f_1})\{\theta^{k_1}(\gamma_{f_2})+\theta^{qk_1}(\gamma_{f_2})\}\}$
X_1	$\theta^{k_1}(\gamma_f)\theta^{k_2}(\gamma_f)$
X_2	$(1-q)\theta^{k_1}(\gamma_f)\theta^{k_2}(\gamma_f)$
X_3	$(1-q+2q^2)\theta^{k_1}(\gamma_f)\theta^{k_2}(\gamma_f)$
X_4	$(1-q)(1+q^2)\theta^{k_1}(\gamma_f)\theta^{k_2}(\gamma_f)$
X_5	$(1-q)(1+q^2)(1-q^3)\theta^{k_1}(\gamma_f)\theta^{k_2}(\gamma_f)$

XI_1	$\theta^{k_1}(\gamma_{f_1})\theta^{k_2}(\gamma_{f_2})+\theta^{k_1}(\gamma_{f_2})\theta^{k_2}(\gamma_{f_1})$
XI_2	$(1-q)\{\theta^{k_1}(\gamma_{f_1})\theta^{k_2}(\gamma_{f_2})+\theta^{k_1}(\gamma_{f_2})\theta^{k_2}(\gamma_{f_1})\}$
XI_3	$(1-q)^2\{\theta^{k_1}(\gamma_{f_1})\theta^{k_2}(\gamma_{f_2})+\theta^{k_1}(\gamma_{f_2})\theta^{k_2}(\gamma_{f_1})\}$

For the other types of classes $\chi_3^{k_1k_2}(c)=0$ and.

$$\chi_3^{k_1k_2}(1)=(q-1)^2(q^2+1)(q^2+q+1)$$

is the degree of these characters.

Type IV_1: They are of type $(g^{|2|})$, $d(g)=2$. Let us denote them by the symbol $\chi_{4.1}^{k}$, where k is a root of g.

The type of class c	The values of $\chi_{4.1}^{k}$ on c
I	$\theta^{k(q^2+1)}(\gamma_f)+\theta^{kq(q^2+1)}(\gamma_f)$
III	$\{\theta^{k}(\gamma_{f_1})+\theta^{qk}(\gamma_{f_1})\}\{\theta^{k}(\gamma_{f_2})+\theta^{qk}(\gamma_{f_2})\}$
IV_1	$\theta^{k}(\gamma_f^2)+\theta^{qk}(\gamma_f^2)+\theta^{k}(\gamma_f)\theta^{qk}(\gamma_f)$
IV_2	$\theta^{k}(\gamma_f^2)+\theta^{qk}(\gamma_f^2)+(q^2+1)\theta^{k}(\gamma_f)\theta^{qk}(\gamma_f)$
VI_1	$\theta^{k}(\gamma_{f_1})\{\theta^{k}(\gamma_{f_2})+\theta^{qk}(\gamma_{f_2})\}$
VI_2	$(1-q)\theta^{k}(\gamma_{f_1})\{\theta^{k}(\gamma_{f_2})+\theta^{qk}(\gamma_{f_2})\}$
X_1	$\theta^{k}(\gamma_f^2)$
X_2	$(1-q)\,\theta^{k}(\gamma_f^2)$
X_3	$(q^2-q+1)\theta_k(\gamma_f^2)$
X_4	$(1-q)\,\theta^{k}(\gamma_f^2)$
X_5	$(q-1)^2(q^2+q+1)\,\theta^{k}(\gamma_f^2)$
XI_1	$\theta^{k}(\gamma_{f_1})\theta^{k}(\gamma_{f_2})$
XI_2	$(1-q)\theta^{k}(\gamma_{f_1})\theta^{k}(\gamma_{f_2})$
XI_3	$(1-q)^2\theta^{k}(\gamma_{f_1})\theta^{k}(\gamma_{f_2})$

For the other types of classes $\chi_{4,1}^{k}(c)=0$ and,

$$\chi_{4,1}^{k}(1)=(q-1)^2(q^2+q+1)$$

is the degree of these characters.

Type IV$_2$: They are of type $(g\{1^2\})$, $d(g)=2$. Let us denote them by the symbol $\chi_{4,2}^{k}$ where k is a root of g.

The type of class c	The values of $\chi_{4,2}^{k}$ on c
I	$\theta^{k(q^2+1)}(\gamma_f)+\theta^{qk(q^2+1)}(\gamma_f)\}$
III	$\{\theta^k(\gamma_{f_1})+\theta^{qk}(\gamma_{f_1})\}\{\theta^k(\gamma_{f_2})+\theta^{qk}(\gamma_{f_2})\}$
IV$_1$	$\theta^k(\gamma_f)\theta^{qk}(\gamma_f)$
IV$_2$	$q^2\{\theta^k(\gamma_f^2)+\theta^{qk}(\gamma_f^2)\}+(q^2+1)\theta^k(\gamma_f)\theta^{qk}(\gamma_f)$
VI$_1$	$\theta^k(\gamma_{f_1})\{\theta^k(\gamma_{f_2})+\theta^{qk}(\gamma_{f_2})\}$
VI$_2$	$(1-q)\theta^k(\gamma_{f_1})\{\theta^k(\gamma_{f_2})+\theta^{qk}(\gamma_{f_2})\}$
X$_3$	$q^2\theta^k(\gamma_f^2)$
X$_4$	$q^2(1-q)\theta^k(\gamma_f^2)$
X$_5$	$q^2(1-q^2)(q^2+q+1)\theta^k(\gamma_f^2)$
XI$_1$	$\theta^k(\gamma_{f_1})\theta^k(\gamma_{f_2})$
XI$_2$	$(1-q)\theta^k(\gamma_{f_1})\theta^k(\gamma_{f_2})$
XI$_3$	$(1-q)^2\theta^k(\gamma_{f_1})\theta^k(\gamma_{f_2})$

For the other types of classes $\chi_{4,2}^{k}(c)=0$ and

$$\chi_{4,2}^{k}(1)=q^2(1-q)^2(q^2+q+1).$$

Type V: $(g_1^{\{1\}}g_2^{\{1\}}g_3^{\{1\}})$, $d(g_1)=d(g_2)=1, d(g_3)=2$. Let us denote them by $\chi_5^{k_1k_2k_3}$ where k_1, k_2, k_3 are a roots of g_1, g_2, g_3 respectively.

The type of class c	The values $\chi_5^{k_1k_2k_3}$ of on c
V	$(-1)\{\theta^{k_1}(\gamma_{f_1})\theta^{k_2}(\gamma_{f_2})+\theta^{k_1}(\gamma_{f_2})\theta^{k_2}(\gamma_{f_1})\}\{\theta^{k_3}(\gamma_{f_3})+\theta^{qk_3}(\gamma_{f_3})\}$
VI_1	$(-1)\{\theta^{k_1}(\gamma_{f_1})\theta^{k_2}(\gamma_{f_1})\}\{\theta^{k_3}(\gamma_{f_2})+\theta^{qk_3}(\gamma_{f_2})\}$
VI_2	$(-1)(q+1)\{\theta^{k_1}(\gamma_{f_1})\theta^{k_2}(\gamma_{f_1})\}\{\theta^{k_3}(\gamma_{f_2})+\theta^{qk_3}(\gamma_{f_2})\}$
$VIII_1$	$(-1)\{\theta^{k_1}(\gamma_{f_1})\theta^{k_2}(\gamma_{f_2})+\theta^{k_1}(\gamma_{f_2})\theta^{k_2}(\gamma_{f_1})\}\theta^{k_3}(\gamma_{f_3})$
$VIII_2$	$(q-1)\{\theta^{k_1}(\gamma_{f_1})\theta^{k_2}(\gamma_{f_2})+\theta^{k_1}(\gamma_{f_2})\theta^{k_2}(\gamma_{f_1})\}\theta^{k_3}(\gamma_{f_2})$
IX_1	$(-1)\{\theta^{k_1}(\gamma_{f_1})\theta^{k_2}(\gamma_{f_2})+\theta^{k_2}(\gamma_{f_1})\theta^{k_1}(\gamma_{f_2})\}\theta^{k_3}(\gamma_{f_3})$
IX_2	$(-1)\{\theta^{k_1}(\gamma_{f_1})\theta^{k_2}(\gamma_{f_2})+\theta^{k_2}(\gamma_{f_1})\theta^{k_1}(\gamma_{f_2})\}\theta^{k_3}(\gamma_{f_2})$
IX_3	$(-1)(1-q^3)\{\theta^{k_1}(\gamma_{f_1})\theta^{k_2}(\gamma_{f_2})+$ $\theta^{k_2}(\gamma_{f_1})\theta^{k_1}(\gamma_{f_2})\}\theta^{k_3}(\gamma_{f_2})$
X_1	$(-1)\theta^{k_1}(\gamma_f)\theta^{k_2}(\gamma_f)\theta^{k_3}(\gamma_f)$
X_2	$(-1)(q+1)\theta^{k_1}(\gamma_f)\theta^{k_2}(\gamma_f)\theta^{k_3}(\gamma_f)$
X_3	$(-1)(q+1)\theta^{k_1}(\gamma_f)\theta^{k_2}(\gamma_f)\theta^{k_3}(\gamma_f)$
X_4	$(-1)(-q^3+q^2+q+1)\theta^{k_1}(\gamma_f)\theta^{k_2}(\gamma_f)\theta^{k_3}(\gamma_f)$
X_5	$(q^4-1)(q^2+q+1)\theta^{k_1}(\gamma_f)\theta^{k_2}(\gamma_f)\theta^{k_3}(\gamma_f)$
XI_1	$(-1)\{\theta^{k_1}(\gamma_{f_1})\theta^{k_2}(\gamma_{f_1})\theta^{k_3}(\gamma_{f_2})+$ $\theta^{k_1}(\gamma_{f_2})\theta^{k_2}(\gamma_{f_2})\theta^{k_3}(\gamma_{f_1})\}$
XI_2	$(q-1)\{\theta^{k_1}(\gamma_{f_1})\theta^{k_2}(\gamma_{f_1})\theta^{k_3}(\gamma_{f_2})\}-$ $(q+1)\{\theta^{k_1}(\gamma_{f_2})\theta^{k_2}(\gamma_{f_2})\theta^{k_3}(\gamma_{f_1})\}$
XI_3	$(q^2-1)\{\theta^{k_1}(\gamma_{f_1})\theta^{k_2}(\gamma_{f_1})\theta^{k_3}(\gamma_{f_1})+$ $\theta^{k_1}(\gamma_{f_2})\theta^{k_2}(\gamma_{f_2})\theta^{k_3}(\gamma_{f_1})\}$

For the other types of classes $\chi_5^{k_1k_2k_3}(c)=0$ and

$$\chi_5^{k_1k_2k_3}(1) = (q^4 - 1)(q^2 + q + 1)$$

is the degree of these characters.

Type VI₁: $(g_1^{\{2\}} g_2^{\{1\}})$, $d(g_1) = 1, d(g_2) = 2$. Let us denote them by the symbol $\chi_{6,1}^{k_1k_2}$ where k_1, k_2 are a roots of g_1, g_2 respectively.

The type of class c	The values $\chi_{6,1}^{k_1k_2}$ of on c
III	$(-1)\{\theta^{k_1(q+1)}(\gamma_{f_1})\{\theta^{k_2}(\gamma_{f_2}) + q^{k_2}(\gamma_{f_2})\} +$
	$\theta^{k_1(q+1)}(\gamma_{f_2})\{\theta^{k_2}(\gamma_{f_1}) + \theta^{qk_2}(\gamma_{f_1})\}\}$
IV₁	$(-1)\{\theta^{k_1(q+1)}(\gamma_f)\{\theta^{k_2}(\gamma_f) + \theta^{qk_2}(\gamma_f)\}\}$
IV₂	$(-1)(q^2+1)\theta^{k_1(q+1)}(\gamma_f)\{\theta^{k_2}(\gamma_f) + \theta^{qk_2}(\gamma_f)\}$
V	$(-1)\theta^{k_1}(\gamma_{f_1})\theta^{k_1}(\gamma_{f_2})\{\theta^{k_2}(\gamma_{f_3}) + \theta^{qk_2}(\gamma_{f_3})\}$
VI₁	$(-1)\{\theta^{k_1}(\gamma_{f_1}^2)\{\theta^{k_2}(\gamma_{f_2}) + \theta^{qk_2}(\gamma_{f_2})\} + \{\theta^{k_1(q+1)}(\gamma_{f_2})\theta^{k_2}(\gamma_{f_1})\}\}$
VI₂	$(-1)\{\theta^{k_1}(\gamma_{f_1}^2)\{\theta^{k_2}(\gamma_{f_2}) + \theta^{qk_2}(\gamma_{f_2})\}$
	$+(q-1)\{\theta^{k_1(q+1)}(\gamma_{f_2})\ \theta^{k_2}(\gamma_{f_1})\}\}$
VIII₁	$(-1)\{\theta^{k_1}(\gamma_{f_1})\theta^{k_1}(\gamma_{f_2}) + \theta^{k_2}(\gamma_{f_3})\}$
VIII₂	$(q-1)\theta^{k_1}(\gamma_{f_1})\theta^{k_1}(\gamma_{f_2})\theta^{k_2}(\gamma_{f_3})$
IX₁	$(-1)\theta^{k_1}(\gamma_{f_1})\theta^{k_1}(\gamma_{f_2})\theta^{k_2}(\gamma_{f_2})$
IX₂	$(-1)\theta^{k_1}(\gamma_{f_1})\theta^{k_1}(\gamma_{f_2})\theta^{k_2}(\gamma_{f_2})$
IX₃	$(q^3-1)\theta^{k_1}(\gamma_{f_1})\theta^{k_1}(\gamma_{f_2})\theta^{k_2}(\gamma_{f_2})$
X₁	$(-1)\theta^{k_1}(\gamma_f^2)\theta^{k_2}(\gamma_f)\}$
X₂	$(-1)\theta^{k_1}(\gamma_f^2)\theta^{k_2}(\gamma_f)$
X₃	$(-1)(1+q^2)\theta^{k_1}(\gamma_f^2)\theta^{k_2}(\gamma_f)$

X_4	$(-1)(-q^3+q^2+1)\theta^{k_1}(\gamma_f^2)\theta^{k_2}(\gamma_f)$
X_5	$(-1)(q^2+q+1)(1+q^2)(1-q)\theta^{k_1}(\gamma_f^2)\theta^{k_2}(\gamma_f)$ }
XI_1	$(-1)\{(\theta^{k_1}(\gamma_{f_1}^2)\theta^{k_2}(\gamma_{f_2})+\theta^{k_1}(\gamma_{f_2}^2)\theta^{k_2}(\gamma_{f_1})\}$
XI_2	$(q-1)\{(\theta^{k_1}(\gamma_{f_1}^2)\theta^{k_2}(\gamma_{f_2})+\theta^{k_1}(\gamma_{f_2}^2)\theta^{k_2}(\gamma_{f_1})\}$
XI_3	$(q-1)\{(\theta^{k_1}(\gamma_{f_1}^2)\theta^{k_2}(\gamma_{f_2})+\theta^{k_1}(\gamma_{f_2}^2)\theta^{k_2}(\gamma_{f_1})\}$

For the other types of classes $\chi_{6,1}^{k_1k_2}(c)=0$ and

$$\chi_{6,1}^{k_1k_2}(1)=(q^2+q+1)(1+q^2)(q-1)$$

is the degree of these characters.

Type VI_2: They are of type $(g_1^{\{1^2\}}g_2^{\{1\}})$, $d(g_1)=1, d(g_2)=2$. Let us denote them by the symbol $\chi_{6,2}^{k_1k_2}$ where k_1, k_2 are a roots of g_1, g_2 respectively.

The type of class c	The values $\chi_{6,2}^{k_1k_2}$ of on e
III	$\theta^{k_1(q+1)}(\gamma_{f_1})\{\theta^{k_2}(\gamma_{f_2})+\theta^{qk_2}(\gamma_{f_2})\}+$
	$\theta^{k_1(q+1)}(\gamma_{f_2})\{\theta^{k_2}(\gamma_{f_1})+\theta^{qk_2}(\gamma_{f_1})\}$
IV_1	$\theta^{k_1(q+1)}(\gamma_f)\{\theta^{k_2}(\gamma_f)+\theta^{qk_2}(\gamma_f)\}$
IV_2	$(q^2+1)\{\theta^{k_1(q+1)}(\gamma_f)\{\theta^{k_2}(\gamma_f)+\theta^{qk_2}(\gamma_f)\}\}$
V	$(-1)\{\theta^{k_1}(\gamma_{f_1})\theta^{k_1}(\gamma_{f_2})\}\{\theta^{k_2}(\gamma_{f_3})+\theta^{qk_2}(\gamma_{f_3})\}$
VI_1	$\theta^{k_1(q+1)}(\gamma_{f_2})\theta^{k_2}(\gamma_{f_1})$
VI_2	$(-1)q\{\theta^{k_1}(\gamma_{f_1}^2)\{\theta^{k_2}(\gamma_{f_2})+\theta^{qk_2}(\gamma_{f_2})\}\}+$
	$(1-q)\{\theta^{k_1(q+1)}(\gamma_{f_2}^2))\theta^{k_2}(\gamma_{f_1})\}$
$VIII_1$	$\theta^{k_1}(\gamma_{f_1})\theta^{k_1}(\gamma_{f_2})\theta^{k_2}(\gamma_{f_3})$
$VIII_2$	$(q-1)\theta^{k_1}(\gamma_{f_1})\theta^{k_1}(\gamma_{f_2})\theta^{k_2}(\gamma_{f_3})$
IX_1	$\theta^{k_1}(\gamma_{f_1})\theta^{k_1}(\gamma_{f_2})\theta^{k_2}(\gamma_{f_2})$

IX_2	$(-1)\theta^{k_1}(\gamma_{f_1})\theta^{k_1}(\gamma_{f_2})\theta^{k_2}(\gamma_{f_2})$
IX_3	$(q^3-1)\theta^{k_1}(\gamma_{f_1})\theta^{k_1}(\gamma_{f_2})\theta^{k_2}(\gamma_{f_2})$
X_2	$(-1)q\theta^{k_1}(\gamma_f^2)\theta^{k_2}(\gamma_f)$
X_3	$(q^2-q)\theta^{k_1}(\gamma_f^2)\theta^{k_2}(\gamma_f)\}$
X_4	$(-1)q\theta^{k_1}(\gamma_f^2)\theta^{k_2}(\gamma_f)$
V	$q(q^2+q+1)(1+q^2)(q-1)\theta^{k_1}(\gamma_f^2)\theta^{k_2}(\gamma_f)$
XI_2	$(-1)q\theta^{k_1}(\gamma_{f_2}^2)\theta^{k_2}(\gamma_{f_1})$
XI_3	$q(q-1)\theta^{k_1}(\gamma_{f_1}^2)\theta^{k_2}(\gamma_{f_2})+\theta^{k_1}(\gamma_{f_1}^2)\theta^{k_2}(\gamma_{f_1})$

For the other type of classes $\chi_{6,2}^{k_1k_2}(c)=0$ and

$$\chi_{6,2}^{k_1k_2}(1)=q(q^2+q+1)(1+q^2)(q-1)$$

Type VII: $(g_1^{\{1\}}g_2^{\{1\}}g_3^{\{1\}}g_4^{\{1\}})$, $d(g_i)=1,(i=1,2,3,4)$. Let us denote them by $\chi_7^{k_1k_2k_3k_4}$.

The type of class c	The values $\chi_7^{k_1k_2k_3k_4}$ of on c
VII	$\sum_{1'2'3'4'}\theta^{k_1}(\gamma_{f_{1'}})\theta^{k_2}(\gamma_{f_{2'}})\theta^{k_3}(\gamma_{f_{3'}})\theta^{k_4}(\gamma_{f_{4'}})$
	Where the summation is over all permutations 1'2'3'4' of 1234.
$VIII_1$	$\frac{1}{2}\{\sum_{1'2'3'4'}\theta^{k_1}(\xi_{11},m')\theta^{k_2}(\xi_{12},m')\theta^{k_3}(\xi_{13},m'))\theta^{k_4}(\xi_{14},m')\}$
$VIII_2$	$\frac{1}{2}(q+1)\{\sum_{1'2'3'4'}\theta^{k_1}(\xi_{11},m')\theta^{k_2}(\xi_{12},m')\theta^{k_3}(\xi_{13},m'))\theta^{k_4}(\xi_{14},m')\}$
	where $m': \xi_{11}\to\gamma_{f_1},\xi_{12}\to\gamma_{f_2},\xi_{13}\to\gamma_{f_3},\xi_{14}\to\gamma_{f_3}$ for the types $VIII_1$ and $VIII_2$.
IX_1	$\frac{1}{6}\{\sum_{1'2'3'4'}\theta^{k_1}(\xi_{11},m')\theta^{k_2}(\xi_{12},m')\theta^{k_3}(\xi_{13},m')\theta^{k_4}(\xi_{14},m')\}$
IX_2	$\frac{1}{6}(2q+1)\{\sum_{1'2'3'4'}\theta^{k_1}(\xi_{11},m')\theta^{k_2}(\xi_{12},m')\theta^{k_3}(\xi_{13},m'))\theta^{k_4}(\xi_{14},m')\}$

IX_3	$\frac{1}{6}(q^2+q+1)(q+1)\{\sum_{1'2'3'4'} \theta^{k_1}(\xi_{1_1},m')\theta^{k_2}(\xi_{1_2},m')\theta^{k_3}(\xi_{1_3},m')\theta^{k_4}(\xi_{1_4},m')\}$

where $m': \xi_{11} \to \gamma_{f_1}, \xi_{12} \to \gamma_{f_2}, \xi_{13} \to \gamma_{f_2}, \xi_{14} \to \gamma_{f_2}$ for the types IX_1, IX_2 and IX_3.

X_1	$\theta^{k_1}(\gamma_f)\theta^{k_2}(\gamma_f)\theta^{k_3}(\gamma_f)\theta^{k_4}(\gamma_f)$
X_2	$(3q+1)\theta^{k_1}(\gamma_f)\theta^{k_2}(\gamma_f)\theta^{k_3}(\gamma_f)\theta^{k_4}(\gamma_f)\}$
X_3	$(2q+1)(q+1)\theta^{k_1}(\gamma_f)\theta^{k_2}(\gamma_f)\theta^{k_3}(\gamma_f)\theta^{k_4}(\gamma_f)$
X_4	$(3q^2+2q+1)(q+1)\theta^{k_1}(\gamma_f)\theta^{k_2}(\gamma_f)\theta^{k_3}(\gamma_f)\theta^{k_4}(\gamma_f)$
X_5	$(q^3+q^2+q+1)(q^2+q+1)(q+1)\theta^{k_1}(\gamma_f)\theta^{k_2}(\gamma_f)\theta^{k_3}(\gamma_f)\theta^{k_4}(\gamma_f)$
XI_1	$\frac{1}{4}\{\sum_{1'2'3'4'} \theta^{k_1}(\xi_{1_1},m')\theta^{k_2}(\xi_{1_2},m')\theta^{k_3}(\xi_{1_3},m')\theta^{k_4}(\xi_{1_4},m')\}$
XI_2	$\frac{1}{4}(q+1)\{\sum_{1'2'3'4'} \theta^{k_1}(\xi_{1_1},m')\theta^{k_2}(\xi_{1_2},m')\theta^{k_3}(\xi_{1_3},m')\theta^{k_4}(\xi_{1_4},m')\}$
XI_3	$\frac{1}{4}(q+1)^2\{\sum_{1'2'3'4'} \theta^{k_1}(\xi_{1_1},m')\theta^{k_2}(\xi_{1_2},m')\theta^{k_3}(\xi_{1_3},m')\theta^{k_4}(\xi_{1_4},m')\}$

where $m': \xi_{11} \to \gamma_{f_1}, \xi_{12} \to \gamma_{f_1}, \xi_{13} \to \gamma_{f_2}, \xi_{14} \to \gamma_{f_2}$ for the types XI_1, XI_2 and XI_3.

For the other types of classes $\chi_7^{k_1k_2k_3k_4}(c)=0$ and

$$\chi_7^{k_1k_2k_3k_4}(1) = (q^3+q^2+q+1)(q^2+q+1)(q+1)$$

Type $VIII_1$: $(g_1^{\{1\}}g_2^{\{1\}}g_3^{\{2\}})$, $d(g_i)=1\,(i=1,2,3,4)$. Let us denote them by the symbol $\chi_{8,1}^{k_1k_2k_3}$.

The type of class c	The values of $\chi_{8,1}^{k_1k_2k_3}$ on e
V	$\{\theta^{k_1}(\gamma_{f_1})\theta^{k_2}(\gamma_{f_2})+\theta^{k_1}(\gamma_{f_2})\theta^{k_2}(\gamma_{f_1})\}\{\theta^{k_2(q+1)}(\gamma_{f_3})\}$
VI_1	$\theta^{k_1}(\gamma_{f_1})\theta^{k_2}(\gamma_{f_1})\theta^{k_3(q+1)}(\gamma_{f_2})$
VI_2	$(q+1)\theta^{k_1}(\gamma_{f_1})\theta^{k_2}(\gamma_{f_1})+\theta^{k_3(q+1)}(\gamma_{f_2})$

VII
$$\frac{1}{2}\Big\{ \sum_{1'2'3'4'} \theta^{k_1}(\gamma_{f_{1'}})\theta^{k_2}(\gamma_{f_{2'}})\theta^{k_3}(\gamma_{f_{3'}})\theta^{k_3}(\gamma_{f_{4'}}) \Big\}$$

$VIII_1$
$$\theta^{k_1}(\gamma_{f_1})\theta^{k_2}(\gamma_{f_2})\theta^{k_3}(\gamma_{f_3}^2) + \theta^{k_1}(\gamma_{f_2})\theta^{k_2}(\gamma_{f_1})\theta^{k_3}(\gamma_{f_3}^2)$$
$$+\theta^{k_1}(\gamma_{f_3})\theta^{k_2}(\gamma_{f_2})\theta^{k_3}(\gamma_{f_1})\theta^{k_3}(\gamma_{f_3}) + \theta^{k_1}(\gamma_{f_1})\theta^{k_2}(\gamma_{f_3})\theta^{k_3}(\gamma_{f_2})\theta^{k_3}(\gamma_{f_3})$$
$$+\theta^{k_1}(\gamma_{f_3})\theta^{k_2}(\gamma_{f_3})\theta^{k_3}(\gamma_{f_1})\theta^{k_3}(\gamma_{f_2}) + \theta^{k_1}(\gamma_{f_2})\theta^{k_2}(\gamma_{f_3})\theta^{k_3}(\gamma_{f_1})\theta^{k_3}(\gamma_{f_3})$$
$$+\theta^{k_1}(\gamma_{f_3})\theta^{k_2}(\gamma_{f_1})\theta^{k_3}(\gamma_{f_2})\theta^{k_3}(\gamma_{f_3})$$

$VIII_2$
$$\theta^{k_1}(\gamma_{f_1})\theta^{k_2}(\gamma_{f_2})\theta^{k_3}(\gamma_{f_3}^2) + \theta^{k_1}(\gamma_{f_2})\theta^{k_2}(\gamma_{f_1})\theta^{k_3}(\gamma_{f_3}^2)$$
$$+(q+1)\Big\{ \theta^{k_1}(\gamma_{f_3})\theta^{k_2}(\gamma_{f_2})\theta^{k_3}(\gamma_{f_1})\theta^{k_3}(\gamma_{f_3})$$
$$+\theta^{k_1}(\gamma_{f_1})\theta^{k_2}(\gamma_{f_3})\theta^{k_3}(\gamma_{f_2})\theta^{k_3}(\gamma_{f_3}) + \theta^{k_1}(\gamma_{f_3})\theta^{k_2}(\gamma_{f_3})\theta^{k_3}(\gamma_{f_1})\theta^{k_3}(\gamma_{f_2})$$
$$+\theta^{k_1}(\gamma_{f_2})\theta^{k_2}(\gamma_{f_3})\theta^{k_3}(\gamma_{f_1})\theta^{k_3}(\gamma_{f_3}) + \theta^{k_1}(\gamma_{f_3})\theta^{k_2}(\gamma_{f_1})\theta^{k_3}(\gamma_{f_2})\theta^{k_3}(\gamma_{f_3}) \Big\}$$

IX_1
$$\theta^{k_1}(\gamma_{f_1})\theta^{k_2}(\gamma_{f_2})\theta^{k_3}(\gamma_{f_2}^2) + \theta^{k_1}(\gamma_{f_2})\theta^{k_2}(\gamma_{f_1})\theta^{k_3}(\gamma_{f_2}^2)$$
$$+\theta^{k_1}(\gamma_{f_2})\theta^{k_2}(\gamma_{f_2})\theta^{k_3}(\gamma_{f_1})\theta^{k_3}(\gamma_{f_2})$$

IX_2
$$(q+1)\Big\{ \theta^{k_1}(\gamma_{f_1})\theta^{k_2}(\gamma_{f_2})\theta^{k_3}(\gamma_{f_2}^2) + \theta^{k_1}(\gamma_{f_2})\theta^{k_2}(\gamma_{f_1})\theta^{k_3}(\gamma_{f_2}^2) \Big\}$$
$$+(2q+1)\Big\{ \theta^{k_1}(\gamma_{f_2})\theta^{k_2}(\gamma_{f_2})\theta^{k_3}(\gamma_{f_1})\theta^{k_3}(\gamma_{f_2}) \Big\}$$

IX_3
$$(q^2+q+1)\Big\{ \theta^{k_1}(\gamma_{f_1})\theta^{k_2}(\gamma_{f_2})\theta^{k_3}(\gamma_{f_2}^2) + \theta^{k_1}(\gamma_{f_2})\theta^{k_2}(\gamma_{f_1})\theta^{k_3}(\gamma_{f_2}^2) \Big\}$$
$$+(q^2+q+1)(q+1)\Big\{ \theta^{k_1}(\gamma_{f_2})\theta^{k_2}(\gamma_{f_2})\theta^{k_3}(\gamma_{f_1})\theta^{k_3}(\gamma_{f_2}) \Big\}$$

X_1
$$\theta^{k_1}(\gamma_f)\theta^{k_2}(\gamma_f)\theta^{k_3}(\gamma_f^2)$$

X_2
$$(2q+1)\theta^{k_1}(\gamma_f)\theta^{k_2}(\gamma_f)\theta^{k_3}(\gamma_f^2)$$

X_3
$$(q+1)^2\theta^{k_1}(\gamma_f)\theta^{k_2}(\gamma_f)\theta^{k_3}(\gamma_f^2)$$

X_4	$(q^3+3q^2+2q+1)\theta^{k_1}(\gamma_f)\theta^{k_2}(\gamma_f)\theta^{k_3}(\gamma_f^2)$
X_5	$(q^3+q^2+q+1)(q^2+q+1)\theta^{k_1}(\gamma_f)\theta^{k_2}(\gamma_f)\theta^{k_3}(\gamma_f^2)$
XI_1	$\theta^{k_1}(\gamma_{f_1})\theta^{k_2}(\gamma_{f_1})\theta^{k_3}(\gamma_{f_2}^2)+\theta^{k_1}(\gamma_{f_2})\theta^{k_2}(\gamma_{f_2})\theta^{k_3}(\gamma_{f_2}^2)+$ $\theta^{k_1}(\gamma_{f_2})\theta^{k_2}(\gamma_{f_1})\theta^{k_3}(\gamma_{f_1})\theta^{k_3}(\gamma_{f_2})+\theta^{k_1}(\gamma_{f_1})\theta^{k_2}(\gamma_{f_2})\theta^{k_3}(\gamma_{f_1})\theta^{k_3}(\gamma_{f_2})$
XI_2	$\theta^{k_1}(\gamma_{f_1})\theta^{k_2}(\gamma_{f_1})\theta^{k_3}(\gamma_{f_2}^2)+(q+1)\{\ \theta^{k_1}(\gamma_{f_2})\theta^{k_2}(\gamma_{f_2})\theta^{k_3}(\gamma_{f_1}^2)+$ $\theta^{k_1}(\gamma_{f_2})\theta^{k_2}(\gamma_{f_1})\theta^{k_3}(\gamma_{f_1})\theta^{k_3}(\gamma_{f_2})+\theta^{k_1}(\gamma_{f_1})\theta^{k_2}(\gamma_{f_2})\theta^{k_3}(\gamma_{f_1})\theta^{k_3}(\gamma_{f_2})\ \}$
XI_3	$(q+1)\{\ \theta^{k_1}(\gamma_{f_1})\theta^{k_2}(\gamma_{f_1})\theta^{k_3}(\gamma_{f_2}^2)+\theta^{k_1}(\gamma_{f_2})\theta^{k_2}(\gamma_{f_2})\theta^{k_3}(\gamma_{f_1}^2)\ \}+$ $+(q+1)^2\{\ \theta^{k_1}(\gamma_{f_2})\theta^{k_2}(\gamma_{f_1})\theta^{k_3}(\gamma_{f_1})\theta^{k_3}(\gamma_{f_2})$ $+\theta^{k_1}(\gamma_{f_1})\theta^{k_2}(\gamma_{f_2})\theta^{k_3}(\gamma_{f_1})\theta^{k_3}(\gamma_{f_2})\ \}$

For the other types of classes $\chi_{8,1}^{k_1k_2k_3}(c)=0$ and

$$\chi_{8,1}^{k_1k_2k_3}(1)=(q+1)(q^2+1)(q^2+q+1)$$

Type VIII₂: $(g_1^{\{1\}}g_2^{\{1\}}g_3^{\{1^2\}})$, $d(g_i)=1\ (i=1,2,3)$. Let us denote them by $\chi_{8,2}^{k_1k_2k_3}$

The type of class c	The values of $\chi_{8,2}^{k_1k_2k_3}$ on c
V	$(-1)\{\ \theta^{k_1}(\gamma_{f_1})\theta^{k_2}(\gamma_{f_2})+\theta^{k_1}(\gamma_{f_2})\theta^{k_2}(\gamma_{f_1})\ \}\theta^{k_3(q+1)}(\gamma_{f_3})$
VI_1	$(-1)\{\ \theta^{k_1}(\gamma_{f_1})\theta^{k_2}(\gamma_{f_1})\theta^{k_3(q+1)}(\gamma_{f_2})\ \}$
VI_2	$(-1)(q+1)\{\ \theta^{k_1}(\gamma_{f_1})\theta^{k_2}(\gamma_{f_1})\theta^{k_3(q+1)}(\gamma_{f_2})\ \}$
VII	$\frac{1}{2}\{\ \sum_{1'2'3'4'}\theta^{k_1}(\gamma_{f_{1'}})\theta^{k_2}(\gamma_{f_{2'}})\theta^{k_3}(\gamma_{f_{3'}})\theta^{k_3}(\gamma_{f_{4'}})\ \}$
$VIII_1$	$\theta^{k_1}(\gamma_{f_3})\theta^{k_2}(\gamma_{f_2})\theta^{k_3}(\gamma_{f_1})\theta^{k_3}(\gamma_{f_3})$

$$+\theta^{k_1}(\gamma_{f_1})\theta^{k_2}(\gamma_{f_3})\theta^{k_3}(\gamma_{f_2})\theta^{k_3}(\gamma_{f_3})$$

$$+\theta^{k_1}(\gamma_{f_3})\theta^{k_2}(\gamma_{f_3})\theta^{k_3}(\gamma_{f_1})\theta^{k_3}(\gamma_{f_2})$$

$$+\theta^{k_1}(\gamma_{f_2})\theta^{k_2}(\gamma_{f_3})\theta^{k_3}(\gamma_{f_1})\theta^{k_3}(\gamma_{f_3})$$

$$+\theta^{k_1}(\gamma_{f_3})\theta^{k_2}(\gamma_{f_1})\theta^{k_3}(\gamma_{f_2})\theta^{k_3}(\gamma_{f_3})$$

VIII$_2$

$$q\{\ \theta^{k_1}(\gamma_{f_1})\theta^{k_2}(\gamma_{f_2})\theta^{k_3}(\gamma_{f_3}^2)+\theta^{k_1}(\gamma_{f_2})\theta^{k_2}(\gamma_{f_1})\theta^{k_3}(\gamma_{f_3}^2)\ \}$$

$$+(q+1)\{\ \theta^{k_1}(\gamma_{f_3})\theta^{k_2}(\gamma_{f_2})\theta^{k_3}(\gamma_{f_1})\theta^{k_3}(\gamma_{f_3})$$

$$+\theta^{k_1}(\gamma_{f_1})\theta^{k_2}(\gamma_{f_3})\theta^{k_3}(\gamma_{f_2})\theta^{k_3}(\gamma_{f_3})$$

$$+\theta^{k_1}(\gamma_{f_3})\theta^{k_2}(\gamma_{f_3})\theta^{k_3}(\gamma_{f_1})\theta^{k_3}(\gamma_{f_2})$$

$$+\theta^{k_1}(\gamma_{f_2})\theta^{k_2}(\gamma_{f_3})\theta^{k_3}(\gamma_{f_1})\theta^{k_3}(\gamma_{f_3})$$

$$+\theta^{k_1}(\gamma_{f_3})\theta^{k_2}(\gamma_{f_1})\theta^{k_3}(\gamma_{f_2})\theta^{k_3}(\gamma_{f_3})\ \}$$

IX$_1$

$$\theta^{k_1}(\gamma_{f_2})\theta^{k_2}(\gamma_{f_2})\theta^{k_3}(\gamma_{f_1})\theta^{k_3}(\gamma_{f_2})$$

IX$_2$

$$(2q+1)\{\ \theta^{k_1}(\gamma_{f_2})\theta^{k_2}(\gamma_{f_2})\theta^{k_3}(\gamma_{f_1})\theta^{k_3}(\gamma_{f_2})\ \}+$$

$$q\{\ \theta^{k_1}(\gamma_{f_1})\theta^{k_2}(\gamma_{f_2})\theta^{k_3}(\gamma_{f_2}^2)+\theta^{k_1}(\gamma_{f_2})\theta^{k_2}(\gamma_{f_1})\theta^{k_3}(\gamma_{f_2}^2)\ \}$$

IX$_3$

$$q(q^2+q+1)\{\ \theta^{k_1}(\gamma_{f_1})\theta^{k_2}(\gamma_{f_2})\theta^{k_3}(\gamma_{f_2}^2)$$

$$+\theta^{k_1}(\gamma_{f_2})\theta^{k_2}(\gamma_{f_1})\theta^{k_3}(\gamma_{f_2}^2)\ \}$$

$$(q^2+q+1)(q+1)\{\ \theta^{k_1}(\gamma_{f_2})\theta^{k_2}(\gamma_{f_2})\theta^{k_3}(\gamma_{f_1})\theta^{k_3}(\gamma_{f_2})\ \}$$

X$_2$

$$q\theta^{k_1}(\gamma_f)\theta^{k_2}(\gamma_f)\theta^{k_3}(\gamma_f^2)$$

X$_3$

$$q(q+1)\theta^{k_1}(\gamma_f)\theta^{k_2}(\gamma_f)\theta^{k_3}(\gamma_f^2)$$

X$_4$

$$(2q^3+2q^2+q)\theta^{k_1}(\gamma_f)\theta^{k_2}(\gamma_f)\theta^{k_3}(\gamma_f^2)$$

X$_5$

$$(q^2+q+1)(q^4+q^3+q^2+q)\theta^{k_1}(\gamma_f)\theta^{k_2}(\gamma_f)\theta^{k_3}(\gamma_f^2)$$

XI$_1$

$$\{\ \theta^{k_1}(\gamma_{f_2})\theta^{k_2}(\gamma_{f_1})\theta^{k_3}(\gamma_{f_1})\theta^{k_3}(\gamma_{f_2})+\theta^{k_1}(\gamma_{f_1})\theta^{k_2}(\gamma_{f_2})\theta^{k_3}(\gamma_{f_1})\theta^{k_3}(\gamma_{f_2})\ \}$$

XI$_2$

$$\{\ \theta^{k_1}(\gamma_{f_1})\theta^{k_2}(\gamma_{f_1})\theta^{k_3}(\gamma_{f_2}^2)\ \}+(q+1)\{\ \theta^{k_1}(\gamma_{f_2})+\theta^{k_2}(\gamma_{f_1})\theta^{k_3}(\gamma_{f_1})\theta^{k_3}(\gamma_{f_2})$$

$$+\theta^{k_1}(\gamma_{f_1})\theta^{k_2}(\gamma_{f_2})\theta^{k_3}(\gamma_{f_1})\theta^{k_3}(\gamma_{f_2})\ \}$$

XI_3	$q(q+1)\{\ \theta^{k_1}(\gamma_{f_1})\theta^{k_2}(\gamma_{f_1})\theta^{k_3}(\gamma_{f_2}^2)+\theta^{k_1}(\gamma_{f_2})\theta^{k_2}(\gamma_{f_2})\theta^{k_3}(\gamma_{f_1}^2)\ \}+$
	$+(q+1)^2\{\ \theta^{k_1}(\gamma_{f_2})\theta^{k_2}(\gamma_{f_1})\theta^{k_3}(\gamma_{f_1})\theta^{k_3}(\gamma_{f_2})$
	$+\theta^{k_1}(\gamma_{f_1})\theta^{k_2}(\gamma_{f_2})\theta^{k_3}(\gamma_{f_1})\theta^{k_3}(\gamma_{f_2})\ \}$

For the other types of classes $\chi_{8,2}^{k_1k_2k_3}(c)=0$ and

$$\chi_{8,2}^{k_1k_2k_3}(1)=q(q+1)(q^2+1)(q^2+q+1)$$

Type IX$_1$: $(g_1^{\{1\}}g_2^{\{3\}})$, $d(g_i)=1\ (i=1,2)$. Let us denote them by $\chi_{9,1}^{k_1k_2}$.

The type of class c	The values of $\chi_{9,1}^{k_1k_2}$ on c
II	$\theta^{k_1}(\gamma_{f_1})\theta^{k_2(q^2+q+1)}(\gamma_{f_2})$
V	$\{\ \theta^{k_1}(\gamma_{f_1})\theta^{k_2}(\gamma_{f_2})+\theta^{k_1}(\gamma_{f_2})\theta^{k_2}(\gamma_{f_1})\ \}\{\ \theta^{k_2(q+1)}(\gamma_{f_3})\ \}$
VI$_1$	$\theta^{k_1}(\gamma_{f_1})\theta^{k_2}(\gamma_{f_1})\theta^{k_2(q+1)}(\gamma_{f_2})$
VI$_2$	$(q+1)\theta^{k_1}(\gamma_{f_1})\theta^{k_2}(\gamma_{f_1})\theta^{k_2(q+1)}(\gamma_{f_2})\ \}$
VII	$\frac{1}{6}\{\ \sum_{1'2'3'4'}\theta^{k_1}(\gamma_{f_{1'}})\theta^{k_2}(\gamma_{f_{2'}})\theta^{k_2}(\gamma_{f_{3'}})\theta^{k_2}(\gamma_{f_{4'}})\ \}$
VIII$_1$	$\{\ \theta^{k_1}(\gamma_{f_1})\theta^{k_2}(\gamma_{f_2})\theta^{k_2}(\gamma_{f_3}^2)+\theta^{k_1}(\gamma_{f_2})\theta^{k_2}(\gamma_{f_1})\theta^{k_2}(\gamma_{f_3}^2)$
	$+\theta^{k_1}(\gamma_{f_3})\theta^{k_2}(\gamma_{f_1})\theta^{k_2}(\gamma_{f_2})\theta^{k_2}(\gamma_{f_3})\ \}$
VIII$_2$	$\{\ \theta^{k_1}(\gamma_{f_1})\theta^{k_2}(\gamma_{f_2})\theta^{k_2}(\gamma_{f_3}^2)+\theta^{k_1}(\gamma_{f_2})\theta^{k_2}(\gamma_{f_1})\theta^{k_2}(\gamma_{f_3}^2)$
	$+(q+1)\{\ \theta^{k_1}(\gamma_{f_3})\theta^{k_2}(\gamma_{f_1})\theta^{k_2}(\gamma_{f_2})\theta^{k_2}(\gamma_{f_3})\ \}$
IX$_1$	$\{\ \theta^{k_1}(\gamma_{f_1})\theta^{k_2}(\gamma_{f_2}^3)+\theta^{k_1}(\gamma_{f_2})\theta^{k_2}(\gamma_{f_1})\theta^{k_2}(\gamma_{f_2}^2)\ \}$
IX$_2$	$\{\ \theta^{k_1}(\gamma_{f_1})\theta^{k_2}(\gamma_{f_2}^3)\ \}+(q+1)\{\ \theta^{k_1}(\gamma_{f_2})\theta^{k_2}(\gamma_{f_1})\theta^{k_2}(\gamma_{f_2}^2)\ \}$
IX$_3$	$\{\ \theta^{k_1}(\gamma_{f_1})\theta^{k_2}(\gamma_{f_2}^3)\ \}+(q^2+q+1)\{\ \theta^{k_1}(\gamma_{f_2})\theta^{k_2}(\gamma_{f_1})\theta^{k_2}(\gamma_{f_2}^2)\ \}$

X_1	$\theta^{k_1}(\gamma_f)\theta^{k_2}(\gamma_f^3)$
X_2	$(q+1)\theta^{k_1}(\gamma_f)\theta^{k_2}(\gamma_f^3)\ \}$
X_3	$(q+1)\{\ \theta^{k_1}(\gamma_f)\theta^{k_2}(\gamma_f^3)\ \}$
X_4	$(q^2+q+1)\{\ \theta^{k_1}(\gamma_f)\theta^{k_2}(\gamma_f^3)\ \}$
X_5	$(q^3+q^2+q+1)\{\ \theta^{k_1}(\gamma_f)\theta^{k_2}(\gamma_f^3)\ \}$
XI_1	$\{\ \theta^{k_1}(\gamma_{f_1})\theta^{k_2}(\gamma_{f_1})\theta^{k_2}(\gamma_{f_2}^2)+\theta^{k_1}(\gamma_{f_2})\theta^{k_2}(\gamma_{f_2})\theta^{k_2}(\gamma_{f_1}^2)\ \}$
XI_2	$\{\ \theta^{k_1}(\gamma_{f_1})\theta^{k_2}(\gamma_{f_1})\theta^{k_2}(\gamma_{f_2}^2)\ \}+(q+1)\{\ \theta^{k_1}(\gamma_{f_2})\theta^{k_2}(\gamma_{f_2})\theta^{k_2}(\gamma_{f_1}^2)\ \}$
XI_3	$(q+1)\{\ \theta^{k_1}(\gamma_{f_1})\theta^{k_2}(\gamma_{f_1})\theta^{k_2}(\gamma_{f_2}^2)+\theta^{k_1}(\gamma_{f_2})\theta^{k_2}(\gamma_{f_2})\theta^{k_2}(\gamma_{f_1}^2)\ \}$

For the other types of classes $\chi_{9,1}^{k_1k_2}(c)=0$ and

$$\chi_{9,1}^{k_1k_2}(1)=(q+1)(q^2+1)$$

Type IX_2: $(g_1^{\{1\}}g_2^{\{2\}})$, $d(g_i)=1\ (i=1,2)$. Let us denote them by $\chi_{9,2}^{k_1k_2}$.

The type of class c	**The values of $\chi_{9,2}^{k_1k_2}$ on c**
II	$(-1)\{\ \theta^{k_1}(\gamma_{f_1})\theta^{k_2(q^2+q+1)}(\gamma_{f_2})\ \}$
VII	$\frac{1}{3}\{\ \sum_{1'2'3'4'}\theta^{k_1}(\gamma_{f_{1'}})\theta^{k_2}(\gamma_{f_{2'}})\theta^{k_2}(\gamma_{f_{3'}})\theta^{k_2}(\gamma_{f_{4'}})\ \}$
$VIII_1$	$\theta^{k_1}(\gamma_{f_1})\theta^{k_2}(\gamma_{f_2})\theta^{k_2}(\gamma_{f_3}^2)+\theta^{k_1}(\gamma_{f_2})\theta^{k_2}(\gamma_{f_1})\theta^{k_2}(\gamma_{f_3}^2)$
	$+2\{\ \theta^{k_1}(\gamma_{f_3})\theta^{k_2}(\gamma_{f_1})\theta^{k_2}(\gamma_{f_2})\theta^{k_2}(\gamma_{f_3})\ \}$
$VIII_2$	$(q+1)\{\ \theta^{k_1}(\gamma_{f_1})\theta^{k_2}(\gamma_{f_2})\theta^{k_2}(\gamma_{f_3}^2)+\theta^{k_1}(\gamma_{f_2})\theta^{k_2}(\gamma_{f_1})\theta^{k_2}(\gamma_{f_3}^2)\ \}$
	$+2\{\ \theta^{k_1}(\gamma_{f_3})\theta^{k_2}(\gamma_{f_1})\theta^{k_2}(\gamma_{f_2})\theta^{k_2}(\gamma_{f_3})\ \}$
IX_1	$\theta^{k_1}(\gamma_{f_2}).\theta^{k_2}(\gamma_{f_1})\theta^{k_2}(\gamma_{f_2}^2)$
IX_2	$q\{\ \theta^{k_1}(\gamma_{f_1})\theta^{k_2}(\gamma_{f_2}^3)\ \}+(2q+1)\{\ \theta^{k_1}(\gamma_{f_2})+\theta^{k_2}(\gamma_{f_1})\theta^{k_2}(\gamma_{f_2}^2)\ \}$
IX_3	$q(q+1)\{\ \theta^{k_1}(\gamma_{f_1})\theta^{k_2}(\gamma_{f_2}^3)\ \}+$

	$(q^2+q+1)(q+1)\{\theta^{k_1}(\gamma_{f_2})+\theta^{k_2}(\gamma_{f_1})\theta^{k_2}(\gamma^2_{f_2})\}$
X_2	$q\{\theta^{k_1}(\gamma_f)\theta^{k_2}(\gamma_f^3)\}$
X_3	$q(q+1)\{\theta^{k_1}(\gamma_f)\theta^{k_2}(\gamma_f^3)\}$
X_4	$q(q+1)^2\{\theta^{k_1}(\gamma_f)\theta^{k_2}(\gamma_f^3)\}$
X_5	$q(q+1)^2(q^2+1)\{\theta^{k_1}(\gamma_f)\theta^{k_2}(\gamma_f^3)\}$
XI_1	$\theta^{k_1}(\gamma_{f_1})\theta^{k_2}(\gamma_{f_1})\theta^{k_2}(\gamma^2_{f_2})+\theta^{k_1}(\gamma_{f_2})\theta^{k_2}(\gamma_{f_2})\theta^{k_2}(\gamma^2_{f_1})$
XI_2	$\theta^{k_1}(\gamma_{f_1})\theta^{k_2}(\gamma_{f_1})\theta^{k_2}(\gamma^2_{f_2})+\theta^{k_1}(\gamma_{f_2})\theta^{k_2}(\gamma_{f_2})\theta^{k_2}(\gamma^2_{f_1})$
XI_3	$(q+1)^2\{\theta^{k_1}(\gamma_{f_1})\theta^{k_2}(\gamma_{f_1})\theta^{k_2}(\gamma^2_{f_2})+\theta^{k_1}(\gamma_{f_2})\theta^{k_2}(\gamma_{f_2})\theta^{k_2}(\gamma^2_{f_1})\}$

For the other types of classes $\chi_{9,2}^{k_1k_2}(c)=0$ and

$$\chi_{9,2}^{k_1k_2}(1)=q(q+1)^2(q^2+1)$$

Type IX_3: $(g_1^{\{1\}}g_2^{\{1^3\}})$, $d(g_i)=1\ (i=1,2)$. Let us denote them by $\chi_{9,3}^{k_1k_2}$

The type of Class c	The values of $\chi_{9,3}^{k_1k_2}$ on c
II	$\{\theta^{k_1}(\gamma_{f_1})\theta^{k_2}(\gamma_{f_2}^{(q^2+q+1)})\}$
V	$(-1)\{\theta^{k_1}(\gamma_{f_1})\theta^{k_2}(\gamma_{f_2})+\theta^{k_1}(\gamma_{f_2})\theta^{k_2}(\gamma_{f_1})\}\{\theta^{k_2(q+1)}(\gamma_{f_3})\}$
VI_1	$(-1)\{\theta^{k_1}(\gamma_{f_1})\theta^{k_2}(\gamma_{f_1})+\theta^{k_2(q+1)}(\gamma_{f_2})\}$
VI_2	$(-1)(q+1)\{\theta^{k_1}(\gamma_{f_1})\theta^{k_2}(\gamma_{f_1})\theta^{k_2(q+1)}(\gamma_{f_2})\}$
VII	$\frac{1}{6}\{\sum_{1'2'3'4'}\theta^{k_1}(\gamma_{f_{1'}})\theta^{k_2}(\gamma_{f_{2'}})\theta^{k_2}(\gamma_{f_{3'}})\theta^{k_2}(\gamma_{f_{4'}})\}$
$VIII_1$	$\{\theta^{k_1}(\gamma_{f_3})\theta^{k_2}(\gamma_{f_1})\theta^{k_2}(\gamma_{f_2})\theta^{k_2}(\gamma_{f_3})\}$
$VIII_2$	$q\{\theta^{k_1}(\gamma_{f_1})\theta^{k_2}(\gamma_{f_2})\theta^{k_2}(\gamma^2_{f_3})+\theta^{k_1}(\gamma_{f_2})\theta^{k_2}(\gamma_{f_1})\theta^{k_2}(\gamma^2_{f_3})\}$ $+(q+1)\{\theta^{k_1}(\gamma_{f_3})\theta^{k_2}(\gamma_{f_1})\theta^{k_2}(\gamma_{f_2})\theta^{k_2}(\gamma_{f_3})\}$
IX_2	$q\{\theta^{k_1}(\gamma_{f_2})\theta^{k_2}(\gamma_{f_1})\theta^{k_2}(\gamma^2_{f_2})\}$
IX_3	$q^3\{\theta^{k_1}(\gamma_{f_1})\theta^{k_2}(\gamma^3_{f_2})\}+q(q^2+q+1)\{\theta^{k_1}(\gamma_{f_2})\theta^{k_2}(\gamma_{f_1})\theta^{k_2}(\gamma^2_{f_2})\}$

X_4	$q^3\{\theta^{k_1}(\gamma_f)\theta^{k_2}(\gamma_f^3)\}$
X_5	$q^3(q+1)(q^2+1)\{\theta^{k_1}(\gamma_f)\theta^{k_2}(\gamma_f^3)\}$
XI_2	$q\{\theta^{k_1}(\gamma_{f_1})\theta^{k_2}(\gamma_{f_2})\theta^{k_2}(\gamma_{f_2}^2)+\theta^{k_1}(\gamma_{f_2})\theta^{k_2}(\gamma_{f_2})\theta^{k_2}(\gamma_{f_1}^2)\}$
XI_3	$q(q+1)\{\theta^{k_1}(\gamma_{f_1})\theta^{k_2}(\gamma_{f_2})\theta^{k_2}(\gamma_{f_2}^2)+\theta^{k_1}(\gamma_{f_2})\theta^{k_2}(\gamma_{f_2})\theta^{k_2}(\gamma_{f_1}^2)\}$

For the other types of classes $\chi_{9,3}^{k_1k_2}(c)=0$ and

$$\chi_{9,3}^{k_1k_2}(1)=q^3(q+1)(q^2+1)$$

Type X_1: $(g^{\{4\}}):=\chi_{10,1}^{k}, d(g_i)=1$.

The type of class c	The values of $\chi_{10,1}^{k}$ on c	The type of class c	The values of $\chi_{10,1}^{k}$ on c
I	$\theta^k(\gamma_f^{(q^3+q^2+q+1)})$	IX_1	$\theta^k(\gamma_{f_1})\theta^k(\gamma_{f_2}^3)$
II	$\theta^k(\gamma_{f_1})\theta^k(\gamma_{f_2}^{(q^2+q+1)})$	IX_2	$\theta^k(\gamma_{f_1})\theta^k(\gamma_{f_2}^3)$
III	$\theta^k(\gamma_{f_1}^{(q+1)})\theta^k(\gamma_{f_2}^{(q+1)})$	IX_3	$\theta^k(\gamma_{f_1})\theta^k(\gamma_{f_2}^3)$
IV_1	$\theta^k(\gamma_f^{2(q+1)})$	X_1	$\theta^k(\gamma_f^4)$
IV_2	$\theta^k(\gamma_f^{2(q+1)})$	X_2	$\theta^k(\gamma_f^4)$
V	$\theta^k(\gamma_{f_1})\theta^k(\gamma_{f_2})\theta^k(\gamma_{f_3}^{(q+1)})$	X_3	$\theta^k(\gamma_f^4)$
VI_1	$\theta^k(\gamma_{f_1}^2)\theta^k(\gamma_{f_2}^{(q+1)})$	X_4	$\theta^k(\gamma_f^4)$
VI_2	$\theta^k(\gamma_{f_1}^2)\theta^k(\gamma_{f_2}^{(q+1)})$	X_5	$\theta^k(\gamma_f^4)$
VII	$\theta^k(\gamma_{f_1})\theta^k(\gamma_{f_2})\theta^k(\gamma_{f_3})\theta^k(\gamma_{f_4})$	XI_1	$\theta^k(\gamma_{f_1}^2)\theta^k(\gamma_{f_2}^2)$
$VIII_1$	$\theta^k(\gamma_{f_1})\theta^k(\gamma_{f_2})\theta^k(\gamma_{f_3}^2)$	XI_2	$\theta^k(\gamma_{f_1}^2)\theta^k(\gamma_{f_2}^2)$
$VIII_2$	$\theta^k(\gamma_{f_1})\theta^k(\gamma_{f_2})\theta^k(\gamma_{f_3}^2)$	XI_3	$\theta^k(\gamma_{f_1}^2)\theta^k(\gamma_{f_2}^2)$

$$\chi_{10,1}^{k}(1)=1$$

Type X_2: $(g^{\{13\}}):=\chi_{10,2}^{k}, d(g)=1$

The type of class c	The values of $\chi^k_{10,2}$ on c	The type of class c	The values of $\chi^k_{10,2}$ on c
I	$(-1)\{\theta^k(\gamma_f^{(q^3+q^2+q+1)})\}$	IX_2	$(q+1)\{\theta^k(\gamma_{f_1})\theta^k(\gamma_{f_2}^3)\}$
III	$(-1)\{\theta^k(\gamma_{f_1}^{(q+1)})\theta^k(\gamma_{f_2}^{(q+1)})\}$	IX_3	$(q^2+q+1)\{\theta^k(\gamma_{f_1})\theta^k(\gamma_{f_2}^3)\}$
IV_1	$(-1)\{\theta^k(\gamma_f^{2(q+1)})\}$	X_2	$q\{\theta^k(\gamma_f^4)\}$
IV_2	$(-1)\{\theta^k(\gamma_f^{2(q+1)})\}$	X_3	$q\{\theta^k(\gamma_f^4)\}$
V	$\theta^k(\gamma_{f_1})\theta^k(\gamma_{f_2})\theta^k(\gamma_{f_3}^{(q+1)})$	X_4	$q(q+1)\{\theta^k(\gamma_f^4)\}$
VI_2	$q\{\theta^k(\gamma_{f_1}^2)\theta^k(\gamma_{f_2}^{(q+1)})\}$	X_5	$q(q^2+q+1)\{\theta^k(\gamma_f^4)\}$
VII	$3\{\theta^k(\gamma_{f_1})\theta^k(\gamma_{f_2})\theta^k(\gamma_{f_3})\theta^k(\gamma_{f_4})\}$	XI_1	$\theta^k(\gamma_{f_1}^2)\theta^k(\gamma_{f_2}^2)$
$VIII_1$	$2\{\theta^k(\gamma_{f_1})\theta^k(\gamma_{f_2})\theta^k(\gamma_{f_3}^2)\}$	XI_2	$(1+q)\{\theta^k(\gamma_{f_1}^2)\theta^k(\gamma_{f_2}^2)\}$
$VIII_2$	$(2+q)\{\theta^k(\gamma_{f_1})\theta^k(\gamma_{f_2})\theta^k(\gamma_{f_3}^2)\}$	XI_3	$(2q+1)\{\theta^k(\gamma_{f_1}^2)\theta^k(\gamma_{f_2}^2)\}$
IX_1	$\theta^k(\gamma_{f_1})\theta^k(\gamma_{f_2}^3)$		

For the other types of classes $\chi^k_{10,2}(c)=0$ and

$$\chi^k_{10,2}(1)=q(q^2+q+1)$$

Type X_3: $(g^{(2^2)}) := \chi^k_{10,3}, d(g)=1$

The type of class c	The values of $\chi^k_{10,3}$ on c	The type of class c	The values of $\chi^k_{10,3}$ on c
II	$(-1)\{\theta^k(\gamma_{f_1})\theta^k(\gamma_{f_2}^{(q^2+q+1)})\}$	VI_2	$(1-q)\{\theta^k(\gamma_{f_1}^2)\theta^k(\gamma_{f_2}^{(q+1)})\}$
III	$2\{\theta^k(\gamma_{f_1}^{(q+1)})\theta^k(\gamma_{f_2}^{(q+1)})\}$	VII	$2\{\theta^k(\gamma_{f_1})\theta^k(\gamma_{f_2})\theta^k(\gamma_{f_3})\theta^k(\gamma_{f_4})$
IV_1	$\theta^k(\gamma_f^{2(q+1)})$	$VIII_1$	$\theta^k(\gamma_{f_1})\theta^k(\gamma_{f_2})\theta^k(\gamma_{f_3}^2)$
IV_2	$(q^2+1)\theta^k(\gamma_f^{2(q+1)})$	$VIII_2$	$(1+q)\theta^k(\gamma_{f_1})\theta^k(\gamma_{f_2})\theta^k(\gamma_{f_3}^2)$
VI_1	$\theta^k(\gamma_{f_1}^2)\theta^k(\gamma_{f_2}^{(q+1)})\}$	IX_2	$q\{\theta^k(\gamma_{f_1})\theta^k(\gamma_{f_2}^3)\}$

IX_3	$q(q+1)\{\theta^k(\gamma_{f_1})\theta^k(\gamma_{f_2}^3)\}$	XI_1	$\theta^k(\gamma_{f_1}^2)\theta^k(\gamma_{f_1}^2)$
X_3	$q^2\{\theta^k(\gamma_f^4)\}$	XI_2	$\theta^k(\gamma_{f_1}^2)\theta^k(\gamma_{f_2}^2)$
X_4	$q^2\{\theta^k(\gamma_f^4)\}$	XI_3	$(q^2+1)\{\theta^k(\gamma_{f_1}^2)\theta^k(\gamma_{f_2}^2)\}$
X_5	$q^2(q^2+1)\{\theta^k(\gamma_f^4)\}$		$\chi_{10,3}^k(1)=q^2(q^2+1)$

For the other types of classes $\chi_{10,3}^k(c)=0$.

Type X_4: $(g^{\{1^2 2\}}) := \chi_{10,4}^k,\ d(g)=1.$

The type of class c	The values of $\chi_{10,4}^k$ on c	The type of class c	The values of $\chi_{10,4}^k$ on c
I	$\theta^k(\gamma_f^{(q^3+q^2+q+1)})$	$VIII_2$	$(2q+1)\{\theta^k(\gamma_{f_1})\theta^k(\gamma_{f_2})\theta^k(\gamma_{f_3}^2)\}$
III	$(-1)\{\theta^k(\gamma_{f_1}^{(q+1)})\theta^k(\gamma_{f_2}^{(q+1)})\}$	IX_2	$q\{\theta^k(\gamma_{f_1})\theta^k(\gamma_{f_2}^3)\}$
IV_2	$(-1)q^2\{\theta^k(\gamma_f^{2(q+1)})\}$	IX_3	$q(q^2+q+1)\{\theta^k(\gamma_{f_1})\theta^k(\gamma_{f_2}^3)\}$
V	$(-1)\{\theta^k(\gamma_{f_1})\theta^k(\gamma_{f_2})\theta^k(\gamma_{f_3}^{(q+1)})\}$	X_4	$q^3\{\theta^k(\gamma_f^4)\}$
VI_1	$(-1)\{\theta^k(\gamma_{f_1}^2)\theta^k(\gamma_{f_2}^{(q+1)})\}$	X_5	$q^3(q^2+q+1)\{\theta^k(\gamma_f^4)\}$
VI_2	$(-1)\{\theta^k(\gamma_{f_1}^2)\theta^k(\gamma_{f_2}^{(q+1)})\}$	XI_2	$q\{\theta^k(\gamma_f^2)\theta^k(\gamma_{f_2}^2)\}$
VII	$3\{\theta^k(\gamma_{f_1})\theta^k(\gamma_{f_2})\theta^k(\gamma_{f_3})\theta^k(\gamma_{f_4})\}$	XI_3	$q(q+2)\{\theta^k(\gamma_{f_1}^2)\theta^k(\gamma_{f_2}^2)\}$
$VIII_1$	$\theta^k(\gamma_{f_1})\theta^k(\gamma_{f_2})\theta^k(\gamma_{f_3}^2)\}$		$\chi_{10,4}^k(1)=q^3(q^2+q+1)$

For the other type of classes $\chi_{10,4}^k(c)=0$.

Type X_5: $(g^{\{1^4\}}) := \chi_{10,5}^k,\ d(g)=1.$

The type of class c	The values of $\chi_{10,5}^k$ oo c	The type of class c	The values of $\chi_{10,5}^k$ on c
I	$(-1)\{\theta^k(\gamma_f^{(q^3+q^2+q+1)})\}$	IV_2	$q^2\{\theta^k(\gamma_{f_1}^{2(q+1)})\}$
II	$\theta^k(\gamma_{f_1})\theta^k(\gamma_{f_2}^{(q^2+q+1)})$	V	$(-1\{\theta^k(\gamma_{f_1})\theta^k(\gamma_{f_2})\theta^k(\gamma_{f_3}^{(q+1)})\}$
III	$\theta^k(\gamma_{f_1}^{(q+1)})\theta^k(\gamma_f^{(q+1)})$	VI_2	$(-1)q\{\theta^k(\gamma_{f_1}^2)\theta^k(\gamma_{f_2}^{(q+1)})\}$

VII	$\theta^{k}(\gamma_{f_1})\theta^{k}(\gamma_{f_2})\theta^{k}(\gamma_{f_3})\theta^{k}(\gamma$	X_5	$q^{6}\{\theta^{k}(\gamma_{f}^{4})\}$
$VIII_2$	$q\{\theta^{k}(\gamma_{f_1})\theta^{k}(\gamma_{f_2})\theta^{k}(\gamma_{f_3}^{2})\}$	XI_5	$q^{2}\{\theta^{k}(\gamma_{f_1}^{2})\theta^{k}(\gamma_{f_2}^{2})\}$
IX_3	$q^{3}\{\theta^{k}(\gamma_{f_1})\theta^{k}(\gamma_{f_2}^{3})\}$		$\chi_{10,5}^{k}(1)=q^{6}$

For the other types of classes $\chi_{10,5}^{k}(c)=0$.

Type XI_1: $(g_1^{\{2\}}g_2^{\{2\}}):=\chi_{11,1}^{k_1k_2}$, $d(g_i)=1\,(i=1,2)$.

The type of class c	The values of $\chi_{11,1}^{k_1k_2}$ on c
III	$\theta^{k_1}(\gamma_{f_1}^{(q+1)})\theta^{k_2}(\gamma_{f_2}^{(q+1)})+\theta^{k_1}(\gamma_{f_2}^{(q+1)})\theta^{k_2}(\gamma_{f_1}^{(q+1)})$
IV_1	$\theta^{k_1}(\gamma_{f}^{(q+1)})\theta^{k_2}(\gamma_{f}^{(q+1)})$
IV_2	$(q^{2}+1)\theta^{k_1}(\gamma_{f}^{(q+1)})\theta^{k_2}(\gamma_{f}^{(q+1)})$
V	$\theta^{k_1}(\gamma_{f_1})\theta^{k_1}(\gamma_{f_2})\theta^{k_2(q+1)}(\gamma_{f_3})+\theta^{k_2}(\gamma_{f_1})\theta^{k_2}(\gamma_{f_2})\theta^{k_1(q+1)}(\gamma_{f_3})\}$
VI_1	$\theta^{k_1}(\gamma_{f_1}^{2})\theta^{k_2(q+1)}(\gamma_{f_2})+\theta^{k_2}(\gamma_{f_2}^{2})\theta^{k_2(q+1)}(\gamma_{f_2})$
VI_2	$\theta^{k_1}(\gamma_{f_1}^{2})\theta^{k_2(q+1)}(\gamma_{f_2})+\theta^{k_2}(\gamma_{f_1}^{2})\theta^{k_1}(\gamma_{f_2}^{(q+1)})$
VII	$\frac{1}{4}\{\sum_{1'2'3'4'}\theta^{k_1}(\gamma_{f_{1'}})\theta^{k_1}(\gamma_{f_{2'}})\theta^{k_2}(\gamma_{f_{3'}})\theta^{k_2}(\gamma_{f_{4'}})\}$
$VIII_1$	$\{\theta^{k_1}(\gamma_{f_1})\theta^{k_1}(\gamma_{f_2})\theta^{k_2}(\gamma_{f_3}^{2})+\theta^{k_1}(\gamma_{f_2})\theta^{k_1}(\gamma_{f_3})\theta^{k_2}(\gamma_{f_1})\theta^{k_2}(\gamma_{f_3})$ $+\theta^{k_1}(\gamma_{f_1})\theta^{k_1}(\gamma_{f_3})\theta^{k_2}(\gamma_{f_2})\theta^{k_2}(\gamma_{f_3})+\theta^{k_1}(\gamma_{f_3}^{2})\theta^{k_2}(\gamma_{f_1})\theta^{k_2}(\gamma_{f_2})\}$
$VIII_2$	$\{\theta^{k_1}(\gamma_{f_1})\theta^{k_1}(\gamma_{f_2})\theta^{k_2}(\gamma_{f_3}^{2})+\theta^{k_1}(\gamma_{f_3}^{2})\theta^{k_2}(\gamma_{f_1})\theta^{k_2}(\gamma_{f_2})\}$ $+(q+1)\{\theta^{k_1}(\gamma_{f_2})\theta^{k_1}(\gamma_{f_3})\theta^{k_2}(\gamma_{f_1})\theta^{k_2}(\gamma_{f_3})$ $+\theta^{k_1}(\gamma_{f_1})\theta^{k_1}(\gamma_{f_3})\theta^{k_2}(\gamma_{f_2})\theta^{k_2}(\gamma_{f_3})\}$
IX_1	$\{\theta^{k_1}(\gamma_{f_1})\theta^{k_1}(\gamma_{f_2})\theta^{k_2}(\gamma_{f_2}^{2})+\theta^{k_2}(\gamma_{f_1})\theta^{k_2}(\gamma_{f_2})\theta^{k_1}(\gamma_{f_2}^{2})\}$
IX_2	$(q+1)\{\theta^{k_1}(\gamma_{f_1})\theta^{k_1}(\gamma_{f_2})\theta^{k_2}(\gamma_{f_2}^{2})+\theta^{k_2}(\gamma_{f_1})\theta^{k_2}(\gamma_{f_2})\theta^{k_1}(\gamma_{f_2}^{2})\}$

IX_3	$(q^2+q+1)\{\theta^{k_1}(\gamma_{f_1})\theta^{k_1}(\gamma_{f_2})\theta^{k_2}(\gamma_{f_2}^2)+\theta^{k_2}(\gamma_{f_1})\theta^{k_2}(\gamma_{f_2})\theta^{k_1}(\gamma_{f_2}^2)\}$
X_1	$\theta^{k_1}(\gamma_f^2)\theta^{k_2}(\gamma_f^2)$
X_2	$(q+1)\{\theta^{k_1}(\gamma_f^2)\theta^{k_2}(\gamma_f^2)\}$
X_3	$(q^2+q+1)\{\theta^{k_1}(\gamma_f^2)\theta^{k_2}(\gamma_f^2)\}$
X_4	$(2q^2+q+1)\{\theta^{k_1}(\gamma_f^2)\theta^{k_2}(\gamma_f^2)\}$
X_5	$(q^2+1)(q^2+q+1)\{\theta^{k_1}(\gamma_f^2)\theta^{k_2}(\gamma_f^2)\}$
IX_1	$\{\theta^{k_1}(\gamma_{f_1}^2)\theta^{k_2}(\gamma_{f_2}^2)+\theta^{k_1}(\gamma_{f_2}^2)\theta^{k_2}(\gamma_{f_1}^2)$ $+\theta^{k_1}(\gamma_{f_1})\theta^{k_1}(\gamma_{f_2})\theta^{k_2}(\gamma_{f_1})\theta^{k_2}(\gamma_{f_2})\}$
IX_2	$\{\theta^{k_1}(\gamma_{f_1}^2)\theta^{k_2}(\gamma_{f_2}^2)+\theta^{k_1}(\gamma_{f_2}^2)\theta^{k_2}(\gamma_{f_1}^2)\}+$ $+(q+1)\{\theta^{k_1}(\gamma_{f_1})\theta^{k_1}(\gamma_{f_2})\theta^{k_2}(\gamma_{f_1})\theta^{k_2}(\gamma_{f_2})\}$
IX_3	$\{\theta^{k_1}(\gamma_{f_1}^2)\theta^{k_2}(\gamma_{f_2}^2)+\theta^{k_2}(\gamma_{f_1}^2)\theta^{k_1}(\gamma_{f_2}^2)\}+$ $+(q+1)^2\{\theta^{k_1}(\gamma_{f_1})\theta^{k_1}(\gamma_{f_2})\theta^{k_2}(\gamma_{f_1})\theta^{k_2}(\gamma_{f_2})\}$

For the other types of classes $\chi_{11,1}^{k_1k_2}(c)=0$ and $\chi_{11,1}^{k_1k_2}(1)=(q^2+1)(q^2+q+1)$.

Type XI_2: $(g_1^{\{2\}}g_2^{\{1^2\}}) := \chi_{11,2}^{k_1k_2}$, $d(g_i)=1\ (i=1,2)$.

The type of class c	**The values of $\chi_{11,2}^{k_1k_2}$ on c**
III	$(-1)\{\theta^{k_1}(\gamma_{f_1}^{(q+1)})\theta^{k_2}(\gamma_{f_2}^{(q+1)})+\theta^{k_1}(\gamma_{f_2}^{(q+1)})\theta^{k_2}(\gamma_{f_1}^{(q+1)})\}$
IV_1	$(-1)\{\theta^{k_1}(\gamma_f^{(q+1)})\theta^{k_2}(\gamma_f^{(q+1)})\}$
IV_2	$(-1)(q^2+1)\{\theta^{k_1}(\gamma_f^{(q+1)})\theta^{k_2}(\gamma_f^{(q+1)})\}$
V	$\{\theta^{k_2}(\gamma_{f_1})\theta^{k_2}(\gamma_{f_2})\theta^{k_1}(\gamma_{f_3}^{(q+1)})-\theta^{k_1}(\gamma_{f_1})\theta^{k_1}(\gamma_{f_2})\theta^{k_2}(\gamma_{f_3}^{(q+1)})\}$
VI_1	$(-1)\{\theta^{k_1}(\gamma_{f_1}^2)\theta^{k_2}(\gamma_{f_2}^{(q+1)})\}$
VI_2	$(-1)\{\theta^{k_1}(\gamma_{f_1}^2)\theta^{k_2}(\gamma_{f_2}^{(q+1)})+q\theta^{k_2}(\gamma_{f_1}^2)\theta^{k_1}(\gamma_{f_2}^{(q+1)})\}$
VII	$\frac{1}{4}\{\sum_{1'2'3'4'}\theta^{k_1}(\gamma_{f_{1'}})\theta^{k_1}(\gamma_{f_{2'}})\theta^{k_2}(\gamma_{f_{3'}})\theta^{k_2}(\gamma_{f_{4'}})\}$

$VIII_1$	$\{\theta^{k_1}(\gamma_{f_2})\theta^{k_1}(\gamma_{f_3})\theta^{k_2}(\gamma_{f_1})\theta^{k_2}(\gamma_{f_3}) + \theta^{k_1}(\gamma_{f_1})\theta^{k_1}(\gamma_{f_3})\theta^{k_2}(\gamma_{f_2})\theta^{k_2}(\gamma_{f_3}) + \theta^{k_1}(\gamma_{f_3}^2)\theta^{k_2}(\gamma_{f_1})\theta^{k_2}(\gamma_{f_2})\}$
$VIII_2$	$q\{\theta^{k_1}(\gamma_{f_1})\theta^{k_1}(\gamma_{f_2})\theta^{k_2}(\gamma_{f_3}^2)\} + (q+1)\{\theta^{k_1}(\gamma_{f_2})\theta^{k_1}(\gamma_{f_3})\theta^{k_2}(\gamma_{f_1})\theta^{k_2}(\gamma_{f_3}) + \theta^{k_1}(\gamma_{f_1})\theta^{k_1}(\gamma_{f_3})\theta^{k_2}(\gamma_{f_2})\theta^{k_2}(\gamma_{f_3})\} + \theta^{k_1}(\gamma_{f_3}^2)\theta^{k_2}(\gamma_{f_1})\theta^{k_2}(\gamma_{f_2})$
IX_1	$\theta^{k_1}(\gamma_{f_2}^2)\theta^{k_2}(\gamma_{f_1})\theta^{k_2}(\gamma_{f_2})$
IX_2	$q\{\theta^{k_1}(\gamma_{f_1})\theta^{k_1}(\gamma_{f_2})\theta^{k_2}(\gamma_{f_2}^2)\} + (q+1)\{\theta^{k_1}(\gamma_{f_2}^2)\theta^{k_2}(\gamma_{f_1})\theta^{k_2}(\gamma_{f_2})\}$
IX_3	$q(q^2+q+1)\{\theta^{k_1}(\gamma_{f_1})\theta^{k_1}(\gamma_{f_2})\theta^{k_2}(\gamma_{f_2}^2)\} + (q^2+q+1)\{\theta^{k_1}(\gamma_{f_2}^2)\theta^{k_2}(\gamma_{f_1})\theta^{k_2}(\gamma_{f_2})\}$
X_2	$q\{\theta^{k_1}(\gamma_f^2)\theta^{k_2}(\gamma_f^2)\}$
X_3	$q\{\theta^{k_1}(\gamma_f^2)\theta^{k_2}(\gamma_f^2)\}$
X_4	$q(q^2+q+1)\{\theta^{k_1}(\gamma_f^2)\theta^{k_2}(\gamma_f^2)\}$
X_5	$q(q^2+1)(q^2+q+1)\{\theta^{k_1}(\gamma_f^2)\theta^{k_2}(\gamma_f^2)\}$
XI_1	$\theta^{k_1}(\gamma_{f_1})\theta^{k_1}(\gamma_{f_2})\theta^{k_2}(\gamma_{f_1})\theta^{k_2}(\gamma_{f_2})$
XI_2	$(q+1)\{\theta^{k_1}(\gamma_{f_1})\theta^{k_1}(\gamma_{f_2})\theta^{k_2}(\gamma_{f_1})\theta^{k_2}(\gamma_{f_2})\}$
XI_3	$q\{\theta^{k_1}(\gamma_{f_1}^2)\theta^{k_2}(\gamma_{f_2}^2) + \theta^{k_1}(\gamma_{f_2}^2)\theta^{k_2}(\gamma_{f_1}^2)\} + (q+1)^2\{\theta^{k_1}(\gamma_{f_1})\theta^{k_1}(\gamma_{f_2})\theta^{k_2}(\gamma_{f_1})\theta^{k_2}(\gamma_{f_2})\}$

For the other types of classes $\chi_{11,2}^{k_1k_2}(c)=0$ and

$$\chi_{11,2}^{k_1k_2}(1)=q(q^2+1)(q^2+q+1).$$

Type XI_3: $(g_1^{\{f\}}g_2^{\{f\}}) := \chi_{11,3}^{k_1k_2}$, $d(g_i)=1$.

The type of class e	The values of $\chi_{11,3}^{k_1k_2}$ on e
III	$\{\theta^{k_1}(\gamma_{f_1}^{(q+1)})\theta^{k_2}(\gamma_{f_2}^{(q+1)})\} + \{\theta^{k_1}(\gamma_{f_2}^{(q+1)})\theta^{k_2}(\gamma_{f_1}^{(q+1)})\}$
IV_1	$\theta^{k_1}(\gamma_f^{(q+1)})\theta^{k_2}(\gamma_f^{(q+1)})$

IV_2	$(q^2+1)\{\theta^{k_1}(\gamma_f^{(q+1)})\theta^{k_2}(\gamma_f^{(q+1)})\}$
V	$(-1)\{\theta^{k_1}(\gamma_{f_1})\theta^{k_1}(\gamma_{f_2})\theta^{k_2}(\gamma_{f_3}^{(q+1)})$
	$+\theta^{k_2}(\gamma_{f_1})\theta^{k_2}(\gamma_{f_2})\theta^{k_1}(\gamma_{f_3}^{(q+1)})\}$
VI_2	$(-q)\{\theta^{k_1}(\gamma_{f_1}^2)\theta^{k_2}(\gamma_{f_2}^{(q+1)})+\theta^{k_2}(\gamma_{f_1}^2)\theta^{k_1}(\gamma_{f_2}^{(q+1)})\}$
VII	$\frac{1}{4}\{\sum_{1'2'3'4'}\theta^{k_1}(\gamma_{f_{1'}})\theta^{k_1}(\gamma_{f_{2'}})\theta^{k_2}(\gamma_{f_{3'}})\theta^{k_2}(\gamma_{f_{4'}})\}$
$VIII_1$	$\theta^{k_1}(\gamma_{f_1})\theta^{k_1}(\gamma_{f_3})\theta^{k_2}(\gamma_{f_2})\theta^{k_2}(\gamma_{f_3})+$
	$\theta^{k_1}(\gamma_{f_2})\theta^{k_1}(\gamma_{f_3})\theta^{k_2}(\gamma_{f_1})\theta^{k_2}(\gamma_{f_3})$
$VIII_2$	$(-q)\{\theta^{k_1}(\gamma_{f_1})\theta^{k_1}(\gamma_{f_2})\theta^{k_2}(\gamma_{f_3}^2)+\theta^{k_1}(\gamma_{f_3}^2)\theta^{k_2}(\gamma_{f_1})\theta^{k_2}(\gamma_{f_2})\}+$
	$(-1)(q+1)\{\theta^{k_1}(\gamma_{f_2})\theta^{k_1}(\gamma_{f_3})\theta^{k_2}(\gamma_{f_1})\theta^{k_2}(\gamma_{f_3})$
	$+\theta^{k_1}(\gamma_{f_1})\theta^{k_1}(\gamma_{f_3})\,^{k_2}(\gamma_{f_2})\,^{k_2}(\gamma_{f_3})\}$
IX_2	$q\{\theta^{k_1}(\gamma_{f_1})\theta^{k_1}(\gamma_{f_2})\theta^{k_2}(\gamma_{f_2}^2)+\theta^{k_1}(\gamma_{f_2}^2)\theta^{k_2}(\gamma_{f_1})\theta^{k_2}(\gamma_{f_2})\}$
IX_3	$q(q^2+q+1)\{\theta^{k_1}(\gamma_{f_1})\theta^{k_1}(\gamma_{f_2})\theta^{k_2}(\gamma_{f_2}^2)+\theta^{k_1}(\gamma_{f_2}^2)\theta^{k_2}(\gamma_{f_1})\theta^{k_2}(\gamma_{f_2})\}$
X_3	$q^2\{\theta^{k_1}(\gamma_f^2)\theta^{k_2}(\gamma_f^2)\}$
X_4	$q^2(q+1)\{\theta^{k_1}(\gamma_f^2)\theta^{k_2}(\gamma_f^2)\}$
X_5	$q^2(q^2+1)(q^2+q+1)\{\theta^{k_1}(\gamma_f^2)\theta^{k_2}(\gamma_f^2)\}$
XI_1	$\{\theta^{k_1}(\gamma_{f_1})\theta^{k_1}(\gamma_{f_2})\theta^{k_2}(\gamma_{f_1})\theta^{k_2}(\gamma_{f_2})\}$
XI_2	$q\{\theta^{k_1}(\gamma_{f_1}^2)\theta^{k_2}(\gamma_{f_2}^2)+\theta^{k_1}(\gamma_{f_2}^2)\theta^{k_2}(\gamma_{f_1}^2)\}$
	$+(q+1)\{\theta^{k_1}(\gamma_{f_1})\theta^{k_1}(\gamma_{f_2})\theta^{k_2}(\gamma_{f_1})\theta^{k_2}(\gamma_{f_2})\}$
IX_3	$q^2\{\theta^{k_1}(\gamma_{f_1}^2)\theta^{k_2}(\gamma_{f_2}^2)+\theta^{k_2}(\gamma_{f_1}^2)\theta^{k_1}(\gamma_{f_2}^2)\}$
	$+(q+1)^2\{\theta^{k_1}(\gamma_{f_1})\theta^{k_1}(\gamma_{f_2})\theta^{k_2}(\gamma_{f_1})\theta^{k_2}(\gamma_{f_2})\}$

For the other types of classes $\chi_{11,3}^{k_1k_2}(c)=0$ and

$$\chi_{11,3}^{k_1k_2}(1)=q^2(q^2+1)(q^2+q+1).$$

2. ON THE COMPLEX CHARACTERS OF SL(4,q)

In this chapter we'll restrict the complex characters of G=GL(4,q) down to S=SL(4,q) and determine which restricted characters are irreducible, which of them split and how many parts they split up into irreducible components.

Theorem 2.1. Let $H \lhd G$ with G/H cyclic, $\chi \in Irr(G)$, $\theta \in Irr(H)$, $(\chi_H,\theta) \neq 0$. Then

$$\chi_H = \sum_{i=1}^{t} \theta_i \quad (\theta_i = \theta)$$

where $t = |G : I_G(\theta)|$. Let $T = I_G(\theta) = \{g \in G |\ \theta^g = \theta\}$ then

$$\chi_T = \sum_{i=1}^{t} \Psi_i$$

where $\Psi_1,\dots,\Psi_t$ are the distinct irreducible characters of T and $(\Psi_i)_H = \theta_i$, $\Psi_i^G = \chi$ (i=1,..., t).

Proof. Since G/H is cyclic then $T \lhd G$. By Clifford Theorem and [4,Th.6.11] we obtain

$$\chi_T = \sum_{i=1}^{t} \Psi_i$$

where $\Psi_i \in Irr(T)$, $\Psi_i^G = \chi$, $(\Psi_i)_H = e\theta_i$ (i=1, ..., t). Since $H \lhd T$, T/H is cyclic, $\theta \in Irr(H)$ is invariant on T, then by [4,Cor.11.22] there exist an $\eta \in Irr(T)$ such that $\eta_H = \theta$. Since $H \lhd T$ and $\eta \in Irr(T)$, $\eta_H = \theta$, by [4,Cor.6.17] we have

$$\theta^T = \sum \beta_i \eta$$

where $\beta_i \in Irr(T/H)$, $(\beta_i \eta)_H = \eta_H = \theta$. Since $\chi_T = \sum_{i=1}^{t} \Psi_i \ (\Psi_i = \Psi)$, then $e = (\Psi_H,\theta) = (\psi,\theta^T) \neq 0$. By [4,Cor,6.17], $\Psi = \beta_i \eta$ for some i, then we obtain

$$e = (\Psi_H,\theta) = (\theta,\theta) = 1.$$

Let G=GL(4,q), S=SL(4,q). Since $S \lhd G$ and G/S is cyclic then by Theorem 2.1

$$\chi_S = \sum_{i=1}^{t} \theta_i (\theta = \theta_i)$$

for $\chi \in Irr(G)$, $\theta \in Irr(S)$, $(\chi_{S}, \theta) \neq 0$, where $t = |G : I_G(\theta)|$, $\theta = \theta_1, \ldots, \theta_t$ are distinct conjugates of θ in G. So

$$\chi_S = \sum_{i=1}^{t} \Psi_i$$

where $T = I_G(\theta)$, $\Psi_1, \ldots, \Psi_t$ are distinct irreducible characters of T and $(\Psi_i)_S = \theta_i$, $\Psi_i^G = \chi$ (i=1, ..., t).

Lemma 2.1. With t defined above for G=GL(4,q) and S=SL(4,q), then t|d, where d = (4,q-1) (See [6], Th.4.7).

Definition. Let

$$M(d)= \{X \in G | \det X = \varepsilon_1^{dk},\ k=1, \ldots, (q-1)/d\}$$

Where d = (4,q-1) and $<\varepsilon_1> = GF(q)^{\times} = GF(q)-\{0\}$.

If d>1 then $S < M(d) \leq M(2) < G$.

If d=1 then χ_S is irreducible.

Lemma 2.2. Let $\chi \in Irr(G)$ and d>1. Then χ_S is irreducible if and only if there exists an $X \in G - M(2)$ such that $\chi(X) \neq 0$.

Proof. 1. Let χ_S be irreducible; since $S<M(2)$ then $\chi_{M(2)}$ is irreducible. Suppose that $\chi(X) = 0$ for each $X \in G-M(2)$, then it follows that

$$|G| = \sum_{X \in G} \chi(X)\overline{\chi(X)} = \sum_{X \in M(2)} \chi(X)\overline{\chi(X)} = M(2),$$

and this contradicts M(2)<G. So there exits an $X \in G-M(2)$ such that $\chi(X) \neq 0$.

2. Assume that there exists an $X \in G-M(2)$ such that $\chi(X) \neq 0$. Suppose that χ_S is not irreducible, then $t = |G:T| = 2$ or 4. Thus $T \leq M(2)$. By Theorem 2.1 there exists a $\Psi \in Irr(T)$ such that $\Psi^G = \chi$. Then $\chi(Y)=0$ each $Y \in G-T$, which contradicts our assumption.

Corollary 2.1. Let $\chi \in Irr(G)$. If $\chi(X)=0$ for each $\chi \in G-M(2)$ then χ_S is reducible.

Lemma 2.3. Let $\chi \in Irr(G)$ and d = 4. Then χ_S is sum of four distinct irreducible characters of S, i.e.

$$\chi_S = \sum_{i=1}^{4} \theta_i$$

if and only if, $\chi(X) = 0$ for each $\chi \in G\text{-}M(d)$.

Proof.

If $\chi_S = \sum_{i=1}^{4} \theta_i$ then T = M(d) and by Theorem 2.1 there exists a $\Psi \in Irr(T)$ such that $\Psi^G = \chi$. If follows that $\chi(X)=0$ for each $X \in G\text{-}M(d)$. If $\chi(X) = 0$ for each $X \in G\text{-}M(d)$ then

$$|G| = \sum_{X \in G} \chi(X)\overline{\chi(X)} = \sum_{X \in M(d)} \chi(X)\overline{\chi(X)} = u.|M(d)|$$

then $u=|G:M(d)| = 4$, i.e. $\chi_{M(d)}$ split up into four irreducible components. Since $S<M(d)$ then $t = 4$.

Corollary 2.2. Let $\chi \in Irr(G)$, then:

(i) In the case d=2: If $\chi(X) = 0$ for each $X \in G\text{-}M(2)$, then χ_S is sum of two irreducible conjugate characters of S.

(ii) In the case d=4: **a)** If $\chi(X) = 0$ for all $X \in G\text{-}M(4)$, then χ_S is sum of four irreducible conjugate characters of S.

b) If $\chi(X) = 0$ for each $X \in G\text{-}M(2)$ and if there exists a $Y \in M(2)\text{-}M(4)$ such that $\chi(Y) \neq 0$ then χ_S is sum of two irreducible conjugate characters of S.

Now consider the subgroup H: = M(2) of G. Since H consists of elements of G whose determinants are square, then the elements of G having the canonical forms of types IV_i (i=1,2), X_i (i=1,2,3,4,5) and XI_i (i=1,2,3) belong to H. The following table denotes the canonical forms of elements belonging to G-H.

1. The case d = (4, q-1) = 2

Type I. The types of classes on which the character χ_1^k takes non-zero values are I, IV_i (i=1,2) and X_i (i=1, ..., 5). Within these types only the classes of the type-I don't contain elements belonging to H. From the type I we consider a special class c whose elements have the canonical form

Type	Canonical Forms	Properties	Type	Canonical Forms	Properties
I	$\begin{pmatrix} \varepsilon_4^{t} & & & \\ & \varepsilon_4^{qt} & & \\ & & \varepsilon_4^{q^2 t} & \\ & & & \varepsilon_4^{q^3 t} \end{pmatrix}$	$t=2\ell-1$, $\ell \in IN$ $t < q^4-1$, $\varepsilon_4^t \notin GF(q^2)^x$	VII	$\begin{pmatrix} \varepsilon_1^{r} & & & \\ & \varepsilon_1^{s} & & \\ & & \varepsilon_1^{t} & \\ & & & \varepsilon_1^{u} \end{pmatrix}$	$r+s+t+u=2\ell+1$, $\ell \in IN$ $r,s,t,u \le q-1$
II	$\begin{pmatrix} \varepsilon_3^{t} & & & \\ & \varepsilon_3^{qt} & & \\ & & \varepsilon_3^{q^2 t} & \\ & & & \varepsilon_1^{u} \end{pmatrix}$	$t+u=2\ell+1$, $\ell \in IN$ $t < q^3-1$, $u \le q-1$ $\varepsilon_3^t \notin GF(q)^x$.	VIII₁	$\begin{pmatrix} \varepsilon_1^{s} & & & \\ & \varepsilon_1^{t} & & \\ & & \varepsilon_1^{u} & 1 \\ & & & \varepsilon_1^{u} \end{pmatrix}$	$u \in IN \cup \{0\}$ $s+l=2\ell-1$, $\ell \in IN$ $s,t,u \le q-1$
III	$\begin{pmatrix} \varepsilon_2^{t} & & & \\ & \varepsilon_2^{qt} & & \\ & & \varepsilon_2^{u} & \\ & & & \varepsilon_2^{qu} \end{pmatrix}$	$t+u=2\ell+1$, $\ell \in IN$ $t,u < q^2-1$ $\varepsilon_2^t, \varepsilon_2^u \notin GF(q)^x$.	VIII₂	$\begin{pmatrix} \varepsilon_1^{s} & & & \\ & \varepsilon_1^{t} & & \\ & & \varepsilon_1^{u} & \\ & & & \varepsilon_1^{u} \end{pmatrix}$	"
V	$\begin{pmatrix} \varepsilon_1^{s} & & & \\ & \varepsilon_1^{t} & & \\ & & \varepsilon_2^{u} & \\ & & & \varepsilon_2^{qu} \end{pmatrix}$	$s+t+u=2\ell+1$, $\ell \in IN$ $s,t \le q-1$, $u < q^2-1$, $\varepsilon_2^u \notin GF(q)^x$.	IX₁	$\begin{pmatrix} \varepsilon_1^{t} & & & \\ & \varepsilon_1^{u} & 1 & \\ & & \varepsilon_1^{u} & 1 \\ & & & \varepsilon_1^{u} \end{pmatrix}$	$u+t=2\ell-1$, $\ell \in IN$ $t,u \le q-1$
VI₁	$\begin{pmatrix} \varepsilon_1^{t} & 1 & & \\ & \varepsilon_1^{t} & & \\ & & \varepsilon_2^{u} & \\ & & & \varepsilon_2^{qu} \end{pmatrix}$	$t \in IN \cup \{0\}$ $u=2\ell-1$, $\ell \in IN$ $t \le q-1$, $u < q^2-1$ $\varepsilon_2^u \notin GF(q)^x$	IX₂	$\begin{pmatrix} \varepsilon_1^{t} & & & \\ & \varepsilon_1^{u} & & \\ & & \varepsilon_1^{u} & 1 \\ & & & \varepsilon_1^{u} \end{pmatrix}$	"
VI₂	$\begin{pmatrix} \varepsilon_1^{t} & & & \\ & \varepsilon_1^{t} & & \\ & & \varepsilon_2^{u} & \\ & & & \varepsilon_2^{qu} \end{pmatrix}$	"	IX₃	$\begin{pmatrix} \varepsilon_1^{t} & & & \\ & \varepsilon_1^{u} & & \\ & & \varepsilon_1^{u} & \\ & & & \varepsilon_1^{u} \end{pmatrix}$	"

Where $\varepsilon_d^x \neq \varepsilon_d^y$ for $x \neq y$ with d=1,2,3,4.

Table 2

$$\begin{pmatrix} \varepsilon_4 & & & \\ & \varepsilon_4^{q} & & \\ & & \varepsilon_4^{q^2} & \\ & & & \varepsilon_4^{q^3} \end{pmatrix}$$

According to Table 2 the elements of this class don't belong to H. On the other hand, we have

$$\chi_1^k(c) = (-1)\left\{\varepsilon^k + \varepsilon^{qk} + \varepsilon^{q^2k} + \varepsilon^{q^3k}\right\}$$

where ε is a primitive (q^4-1)-th root of unity.

Assertion. $\left\{\varepsilon^k + \varepsilon^{qk} + \varepsilon^{q^2k} + \varepsilon^{q^3k}\right\}$ is different from zero except the case $k \equiv (q^2+1)/2(\mathrm{mod}(q^2+1))$.

Proof. Since k is a root of a simplex of fourth order, the values $\varepsilon^k, \varepsilon^{qk}, \varepsilon^{q^2k}, \varepsilon^{q^3k}$ are all different. The points of the plane corresponding to these numbers, say A_1, A_2, A_3 and A_4 lie on the unit circle (see Figure 1). Let $\vec{OA_1} + \vec{OA_2} = \vec{OB_1}$, $\vec{OA_3} + \vec{OA_4} = \vec{OB_2}$. If $\varepsilon^k + \varepsilon^{qk} + \varepsilon^{q^2k} + \varepsilon^{q^3k} = 0$ then $\vec{OB_1} = -\vec{OB_2}$, since $\vec{OA_1} + \vec{OA_2} + \vec{OA_3} + \vec{OA_4} = 0$. On the other hand, the quadrangles $[OA_1B_1A_2]$ and $[OA_3B_2A_4]$ are two rhombuses such that the length of sides are equal and the points B_1, O, B_2 are on the same line. Consequently

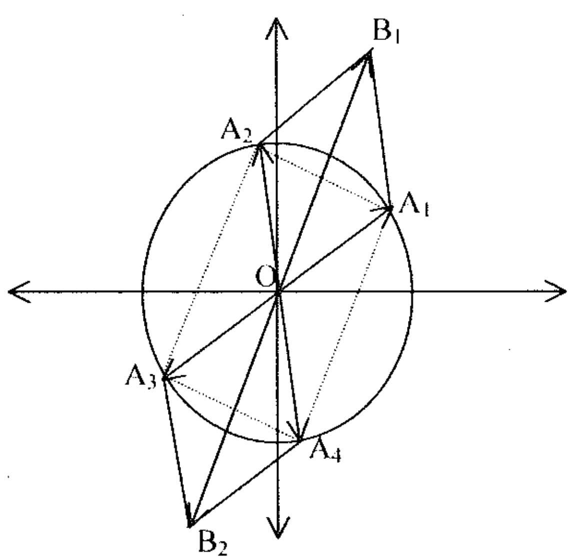

Figure 1

$[A_1A_2A_3A_4]$ is a rectangle, so the sum of the numbers corresponding to A_1 and A_3 is zero,

If

$$\varepsilon^{k}+\varepsilon^{qk}+\varepsilon^{q^2k}+\varepsilon^{q^3k}=0$$

then the following three cases can arise:

$1^0)\ \varepsilon^{k}+\varepsilon^{qk}=0,\qquad 2^0)\ \varepsilon^{k}+\varepsilon^{q^2k}=0\qquad 3^0)\ \varepsilon^{k}+\varepsilon^{q^3k}=0$

It can be easily seen that the cases 1^0) and 3^0) are impossible and the case 2^0 is possible only for

$$k\equiv(q^2+1)/2(\mathrm{mod}(q^2+1)).$$

So, $\varepsilon^{k}+\varepsilon^{qk}+\varepsilon^{q^2k}+\varepsilon^{q^3k}=0$ if and only if $k\equiv(q^2+1)/2(\mathrm{mod}(q^2+1))$.

By using simple operations it can be easily seen that: If

$$k\equiv(q^2+1)/2(\mathrm{mod}(q^2+1))$$

then $\chi_1^k(X)=0$ for all $X\in G\text{-}H$.

Corollary. In the case d = 2

1) if $k\not\equiv(q^2+1)/2(\mathrm{mod}(q^2+1))$ then restriction of χ_1^k *ea* down to S is an irreducible character of S,

2) if $k\equiv(q^2+1)/2(\mathrm{mod}(q^2+1))$ then restriction of χ_1^k down to S is the sum of two irreducible conjugate characters of S of degree

$$\frac{1}{2}\left\{(q-1)^3(q+1)(q^2+q+1)\right\}.$$

Type II. The character $\chi_2^{k_1k_2}$ take the value

$$\chi_2^{k_1k_2}(c)=(1-q)(1-q^2)\left\{\theta^{k_1}(\gamma_{f_1})\theta^{k_2}(\gamma_{f_2})\right\}$$

for the type of class IX_3. If we choose c in the special form

$$\begin{pmatrix} \varepsilon_1 & & & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{pmatrix}$$

then $X \notin H$ and

$$\chi_2^{k_1 k_2}(c) = (1-q)(1-q^2)\varepsilon^{k_1} \neq 0$$

where ε is a (q-1)-th primitive root of unity. By Lemma 2.2 the restriction of the character $\chi_2^{k_1 k_2}$ down to S gives an irreducible character of S.

Type III. The types of classes on which the character $\chi_3^{k_1 k_2}$ takes non zero values are the types III, IV_i, VI_i (i=1,2), X_i (i=1, ..., 5) and XI_i (i=1,2,3). Among them the types of classes including elements not belonging to H are III and VI_i (i=1,2). If we choose from the classes of the type VI_1 a special class c with the canonical form

$$\begin{pmatrix} 1 & 1 & & \\ & 1 & & \\ & & \varepsilon_2 & \\ & & & \varepsilon_2^q \end{pmatrix}$$

then, according to Table 2, the elements of c can not belong to H and

$$\chi_3^{k_1 k_2}(c) = \varepsilon^{k_1} + \varepsilon^{qk_1} + \varepsilon^{k_2} + \varepsilon^{qk_2}$$

where ε is a primitive (q^2-1)-th root of unity. By using simple operation we have.

1. It holds

$$\varepsilon^{k_1} + \varepsilon^{qk_1} + \varepsilon^{k_2} + \varepsilon^{qk_2} \neq 0$$

except the cases

i) $k_i \equiv (q+1)/2 \pmod{q+1}$, i=1,2

ii) $k_1 - k_2 \equiv (q^2-1)/2(\mathrm{mod}(q^2-1))$

2. If

i) $k_i \equiv (q+1)/2\ (\mathrm{mod}(q+1))$, $i=1,2$

or

ii) $k_1 - k_2 \equiv (q^2-1)/2(\mathrm{mod}(q^2-1))$

then $\chi_3^{k_1k_2}(X) = 0$ for each $X \in G\text{-}H$.

Corollary. In the case $d = 2$ if

i) $k_i \equiv (q+1)/2(\mathrm{mod}(q+1))$, $i=1,2$

or

ii) $k_1 - k_2 \equiv (q^2-1)/2(\mathrm{mod}(q^2-1))$

then the restriction of the character $\chi_3^{k_1k_2}$ down to S is the sum of two irreducible conjugate characters of S of degrees $\frac{1}{2}\{(q-1)^2(q^2+1)(q^2+q+1)\}$. Out of these cases the restriction of $\chi_3^{k_1k_2}$ down to S is an irreducible character of S.

Type IV₁. The types of classes on which the character $\chi_{1,1}^{k}$ takes nonzero values are I, III, IV_i, VI_i $(i=1,2)$, X_i $(i=1, \ldots, 5)$ and XI_i $(i=1,2,3)$. The elements of class c from the type I with the canonical form

$$\begin{pmatrix} \varepsilon_4 & & & \\ & \varepsilon_4^{q} & & \\ & & \varepsilon_4^{q^2} & \\ & & & \varepsilon_4^{q^3} \end{pmatrix}$$

can not belong to H. On the other hand

$$\chi_{1,1}^{k}(c) = \left\{\varepsilon^{k} + \varepsilon^{qk}\right\}$$

where ε is a primitive (q^2-1)-th root of unity.

It can be seen that:

1. It holds $\varepsilon^k + \varepsilon^{qk} \neq 0$ except $k \equiv (q+1)/2 (\mathrm{mod}(q+1))$
2. If $k \equiv (q+1)/2(\mathrm{mod}(q+1))$ then $\chi_{4,1}^{k}(X) = 0$ for each $X \in G\text{-}H$.

Corollary. In the case $d = 2$:

1. If $k \not\equiv (q+1)/2(\mathrm{mod}(q+1))$ then the restriction of the character $\chi_{4,1}^{k}$ down to S is an irreducible character of S.

2. If $k \equiv (q+1)/2(\mathrm{mod}(q+1))$ then the restriction of $\chi_{4,1}^{k}$ down to S is the sum of two irreducible conjugate characters of S of degrees $1/2\{(q-1)^2(q^2+q+1)\}$.

Type IV$_2$. If we consider the values of the character $\chi_{4,2}^{k}$ on the types of classes, then we can see that all results which we have obtained above for the character $\chi_{4,1}^{k}$ are valid exactly for the character $\chi_{4,2}^{k}$ but the degrees of two irreducible constituents of $\chi_{4,2}^{k}$ are $\frac{1}{2}\left\{q^2(q-1)^2(q^2+q+1)\right\}$

Type V. The types of classes on which the character $\chi_5^{k_1k_2k_3}$ takes nonzero values are V, VI$_i$, VIII$_i$ (i=1,2), IX$_i$ (i=1,2,3), X$_i$ (i=1, ..., 5) and XI$_i$ (i=1,2,3). The elements not belonging to H are in the classes V, VI$_i$, VIII$_i$ (i=1,2) and IX$_i$ (i=1,2,3). If we choose from the type IX$_1$ a special class c with the canonical form

$$\begin{pmatrix} \varepsilon_1 & & & \\ & 1 & 1 & \\ & & 1 & 1 \\ & & & 1 \end{pmatrix}$$

then the elements of this class can not belong to H and

$$\chi_5^{k_1k_2k_3}(c) = (-1)\left\{\omega^{k_1} + \omega^{k_3}\right\}$$

where ω is a primitive (q-1)-th root of unity.

One can see that:

1. $\{\omega^{k_1}+\omega^{k_2}\}\neq 0$ except $k_1-k_2 \equiv (q-1)/2(\mathrm{mod}(q-1))$.

On the other hand, if we choose from the type of classes VI_1 a special class c with the canonical form

$$\begin{pmatrix} 1 & 1 & & \\ & 1 & & \\ & & \varepsilon_2 & \\ & & & \varepsilon_2^q \end{pmatrix}$$

we can say that the elements of this class not belonging to H and it holds

$$\chi_5^{k_1k_2k_3}(c)=(-1)\{\varepsilon^{k_3}+\varepsilon^{qk_3}\}$$

where ε is a primitive (q^2-1)-th root of unity.

2. $\{\varepsilon^{k_3}+\varepsilon^{qk_3}\}\neq 0$ except $k_3 \equiv (q+1)/2(\mathrm{mod}(q+1))$.

3. If $k_1 - k_2 \equiv (q-1)/2(\mathrm{mod}(q-1))$ and $k_3 \equiv (q+1)/2(\mathrm{mod}(q+1))$ then it holds $\chi_5^{k_1k_2k_3}(X)=0$ for each $X\in G-H$.

Corollary. 1. Except the cases $k_1 - k_2 \equiv (q-1)/2(\mathrm{mod}(q-1))$ and $k_3 \equiv (q+1)/2$ $(\mathrm{mod}(q+1))$ the restriction of the character $\chi_5^{k_1k_2k_3}$ down to S is an irreducible character of S.

2. If $k_1 - k_2 \equiv (q-1)/2(\mathrm{mod}(q-1))$ and $k_3 \equiv (q+1)/2(\mathrm{mod}(q+1))$ then the restriction of the character $\chi_5^{k_1k_2k_3}$ down to S is the sum of two irreducible cojugate characters of S of degree $\frac{1}{2}\{(q^2+q+1)(q^2+1)(q^2-1)\}$.

Type VI_1. On a class c from the type IX_1 having the canonical form

$$\begin{pmatrix} \varepsilon_1 & & & \\ & 1 & 1 & \\ & & 1 & 1 \\ & & & 1 \end{pmatrix}$$

the character $\chi_{6,1}^{k_1k_2}$ takes the value $(-1)\varepsilon^{k_1} \neq 0$, where ε is a primitive (q-1)-th root of unity. On the other hand, since the elements of the class c don't belong to H, the restriction of the character $\chi_{6,1}^{k_1k_2}$ down to S remains irreducible by Lemma 2.2.

Type VI_2. Similarly, the restriction of the character $\chi_{6,2}^{k_1k_2}$ down to S remains irreducible.

Type VII. Under the condition q>3 the types of classes on which the character $\chi_7^{k_1k_2k_3}$ takes nonzero values are VII, $VIII_i$ (i=1,2), IX_i (i=1,2,3), X_i (i=1, ..., 5) and XI_i (i=1,2,3). Among them the types including elements not belonging to H are VII, $VIII_i$ (i=1,2) and IX_i (i=1,2,3). If we consider a special class c from the type of classes IX_1 having the canonical form

$$\begin{pmatrix} \varepsilon_1 & & & \\ & 1 & 1 & \\ & & 1 & 1 \\ & & & 1 \end{pmatrix}$$

then we obtain

$$\chi_7^{k_1k_2k_3k_4}(c) = \varepsilon^{k_1} + \varepsilon^{k_2} + \varepsilon^{k_3} + \varepsilon^{k_4}$$

where ε is a primitive (q-1)-th root of unity.

It can be seen that

1. Except the cases

i) $k_1 - k_3 \equiv (q-1)/2(\mathrm{mod}(q-1))$ and $k_2 - k_4 \equiv (q-1)/2(\mathrm{mod}(q-1)$

Ii) $k_1 - k_2 \equiv (q-1)/2(\mathrm{mod}(q-1))$ and $k_3 - k_4 \equiv (q-1)/2(\mathrm{mod}(q-1))$

iii) $k_1 - k_4 \equiv (q-1)/2(\mathrm{mod}(q-1))$ and $k_2 - k_3 \equiv (q-1)/2(\mathrm{mod}(q-1))$

it holds

$$\left\{\varepsilon^{k_1} + \varepsilon^{k_2} + \varepsilon^{k_3} + \varepsilon^{k_4}\right\} \neq 0$$

2. In each of the cases i), ii), iii) above it holds $\chi_7^{k_1 k_2 k_3 k_4}(X) = 0$ for each $X \in G\text{-}H$.

Corollary. If in the case d = 2

i) $k_1 - k_3 \equiv (q-1)/2(\mathrm{mod}(q-1)$ and $k_2 - k_4 \equiv (q-1)/2(\mathrm{mod}(q-1))$

ii) $k_1 - k_2 \equiv (q-1)/2(\mathrm{mod}(q-1)$ and $k_3 - k_4 \equiv (q-1)/2(\mathrm{mod}(q-1))$

iii) $k_1 - k_4 \equiv (q-1)/2(\mathrm{mod}(q-1)$ and $k_2 - k_3 \equiv (q-1)/2(\mathrm{mod}(q-1))$

then the restriction of the character $\chi_7^{k_1 k_2 k_3 k_4}$ down to S is the sum of two irreducible conjugate characters of S of degrees $\frac{1}{2}\{(q^3+q^2+q+1)(q^2+q+1)(q+1)\}$.

Except these three cases the restriction of the character $\chi_7^{k_1 k_2 k_3 k_4}$ down to S remains irreducible.

Type VIII$_1$. Under the condition q>3 the character $\chi_{8,1}^{k_1 k_2 k_3}$ takes on a class c from the type VI$_1$ with the canonical form

$$\begin{pmatrix} \varepsilon_1 & 1 & & \\ & \varepsilon_1 & & \\ & & \varepsilon_2 & \\ & & & \varepsilon_2^q \end{pmatrix}$$

the value

$$\chi_{8,1}^{k_1 k_2 k_3}(c) = \varepsilon^{k_1}.\varepsilon^{k_2}.\varepsilon^{k_3}$$

where ε is a primitive (q-1)-th root of unity. But this value is always different from zero. On the other hand, since the elements of class c don't belong to H, the restriction of the character $\chi_{8,1}^{k_1k_2k_3}$ down to S remains irreducible by Lemma 2.2.

Type VIII$_2$. Similarly, the restriction of the character $\chi_{8,1}^{k_1k_2k_3}$ down to S remains irreducible.

Type IX$_1$. On a class c from the type II having the canonical form

$$\begin{pmatrix} \varepsilon_3 & & & \\ & \varepsilon_3^q & & \\ & & \varepsilon_3^{q^2} & \\ & & & \varepsilon_1^2 \end{pmatrix}$$

the character $\chi_{9,1}^{k_1k_2}$ takes the value

$$\chi_{9,1}^{k_1k_2}(c) = \varepsilon^{2k_1}\varepsilon^{k_2} \neq 0$$

where ε is a primitive (q-1)-th root of unity. On the other hand, since the elements of the class c don't belong to H, the restriction of the character $\chi_{9,1}^{k_1k_2}$ down to S remains irreducible by Lemma 2.2.

Types IX$_2$ and IX$_3$. Similarly, the restrictions of the characters of these types down to S remain irreducible.

Type X$_1$. On a class c from the type IX$_3$ having the canonical form

$$\begin{pmatrix} \varepsilon_1 & & & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{pmatrix}$$

the character $\chi_{10,1}^{k}$ takes the value

$$\chi_{10,1}^{k}(c) = \varepsilon^{k_1}$$

where ε is a primitive (q-1)-th root of unity. This value is always nonzero. On the other hand, since the elements of class c don't belong to H, the restriction of character $\chi_{10,1}^{k}$ down to S is an irreducible character of S by Lemma 2.2.

Similarly, the characters of the types X_i (i=2, ..., 5) take nonzero values for the elements of class c mentioned above. So the restrictions of these characters down to S are irreducible characters of S.

Type XI_1. The value of the character $\chi_{11,1}^{k_1 k_2}$ on a class c from the type IX_1 having the canonical form

$$\begin{pmatrix} \varepsilon_1 & & & \\ & 1 & 1 & \\ & & 1 & 1 \\ & & & 1 \end{pmatrix}$$

is $\chi_{11,1}^{k_1 k_2}(c) = \varepsilon^{k_1} + \varepsilon^{k_2}$, where ε is a primitive (q-1)-th root of unity.

It can be seen that:

1. $\varepsilon^{k_1} + \varepsilon^{k_2} \neq 0$ except $k_1 - k_2 \equiv (q-1)/2(\mathrm{mod}(q-1))$.
2. If $k_1 - k_2 \equiv (q-1)/2(\mathrm{mod}(q-1))$ then it holds $\chi_{11,1}^{k_1 k_2}(X) = 0$ for each $X \in G\text{-}H$.

Corollary. If $k_1 - k_2 \equiv (q-1)/2(\mathrm{mod}(q-1))$ then the restriction of the character $\chi_{11,1}^{k_1 k_2}$ down to S is the sum of two irreducible conjugate characters of S of degrees $\frac{1}{2}\{(q^2+1)(q^2+q+1)\}$.

If $k_1 - k_2 \not\equiv (q-1)/2(\mathrm{mod}(q-1))$ then the restriction of $\chi_{11,1}^{k_1 k_2}$ down to S is an irreducible character of S.

Type XI_2. The value of the character $\chi_{11,2}^{k_1 k_2}$ on a class c from the type IX_1 having the canonical form

$$\begin{pmatrix} \varepsilon_1 & & & \\ & 1 & 1 & \\ & & 1 & 1 \\ & & & 1 \end{pmatrix}$$

is $\chi_{11,2}^{k_1 k_2}(c) = \varepsilon^{k_2} \neq 0$, where ε is a primitive (q-1)-th root of unity. On the other hand, since the elements of this class don't belong to H, the restriction of $\chi_{11,2}^{k_1 k_2}$ down to S is an irreducible character of S by Lemma 2.2.

Type XI₃. The value of the character $\chi_{11,3}^{k_1 k_2}$ on a class c from the type IX_2 having the canonical form

$$\begin{pmatrix} \varepsilon_1 & & & \\ & 1 & & \\ & & 1 & 1 \\ & & & 1 \end{pmatrix}$$

is $\chi_{11,3}^{k_1 k_2}(c) = q\left\{\varepsilon^{k_1} + \varepsilon^{k_2}\right\}$, where ε is a primitive (q-1)-th root of unity.

One can see that:

1. $\varepsilon^{k_1} + \varepsilon^{k_2} \neq 0$ except $k_1 - k_2 \equiv (q-1)/2(\mathrm{mod}(q-1))$.
2. If $k_1 - k_2 \equiv (q-1)/2(\mathrm{mod}(q-1))$ then it holds $\chi_{11,3}^{k_1 k_2}(X) = 0$ for each $X \in G-H$.

Corollary. In the case d = 2: 1. If $k_1 - k_2 \equiv (q-1)/2(\mathrm{mod}(q-1))$ then the restriction of the character $\chi_{11,3}^{k_1 k_2}$ down to S is the sum of two irreducible conjugate characters of S degrees $\frac{1}{2}\{q^2(q^2+1)(q^2+q+1)\}$.

2. If $k_1 - k_2 \not\equiv (q-1)/2(\mathrm{mod}(q-1))$ then the restriction of this character is an irreducible character of S.

2. The case d = (4,q-1) = 4

In this case q is of the from 4 m+1, and

$$M(d) = \{X \in G | \det X = \varepsilon_1^{4t}, t=1, \ldots, (q-1)/4\}.$$

Type I. As we have shown above (in the case d=2), if

$$k \equiv (q^2+1)/2(\mathrm{mod}(q^2+1))$$

then it holds $\chi_1^k(X) = 0$ for each $X \in G-H$.

Now if we choose from the type of classes I a special class c having the canonical form

$$\begin{pmatrix} \varepsilon_4^2 & & & \\ & \varepsilon_4^{2q} & & \\ & & \varepsilon_4^{2q^2} & \\ & & & \varepsilon_4^{2q^3} \end{pmatrix}$$

then we obtain

$$\chi_1^k(c) = (-1)\{\varepsilon^{2k} + \varepsilon^{2qk} + \varepsilon^{2q^2k} + \varepsilon^{2q^3k}\}$$

where ε is a primitive (q^4-1)-th root of unity. If $k \equiv (q^2+1)/2(\mathrm{mod}(q^2+1))$ then it follows from $\varepsilon^{2k} = \varepsilon^{2q^2k}$ and $\varepsilon^{2qk} = \varepsilon^{2q^3k}$ that

$$\chi_1^k(c) = (-1)\,2\{\varepsilon^{2k} + \varepsilon^{2qk}\}$$

For this type it can be seen that:

1. $\varepsilon^{2k} + \varepsilon^{2qk} \neq 0$ except $2k \equiv (q+1)(q^2+1)/2(\mathrm{mod}(q+1)(q^2+1))$.

2. If k is of the form $k = (2z+1)(q+1)(q^2+1)/4$ where $z \in \mathbf{Z}$ (the set of intg.) then it holds $\chi_1^k(X) = 0$ for each $X \in H-M(d)$.

Corollary. In the case $d = 4$ if $k \equiv (q^2+1)/2(\mathrm{mod}(q^2+1))$ then the restriction of the character χ_1^k down to S splits up and the following results can be obtained by Corollary 2.2:

1. If $k \neq (2z+1)(q+1)(q^2+1)/4$, $z \in \mathbf{Z}$, then the restriction of the character χ_1^k down to S is the sum of two irreducible conjugate characters of S of degree $\frac{1}{2}\{(q-1)^3(q+1)(q^2+q+1)\}$.

2. If $k = (2z+1)(q+1)(q^2+1)/4$, $z \in \mathbf{Z}$, then the restriction of χ_1^k down to S is the sum of four irreducible conjugate characters of S of degrees $\frac{1}{4}\{(q-1)^3(q+1)(q^2+q+1)\}$.

3. If $k \not\equiv (q^2+1)/2(\mathrm{mod}(q^2+1))$ then the restriction of χ_1^k down to S is an irreducible character of S by Lemma 2.2.

Type II. As we have proved above (in the case $d=2$), the restriction of the character $\chi_2^{k_1k_2}$ down to S is an irreducible character of S.

Type III. As we have shown above (in the case $d=2$), if

i) $k_i \equiv (q+1)/2(\mathrm{mod}(q+1))$, $i=1,2$ or

ii) $k_1 - k_2 \equiv (q^2-1)/2(\mathrm{mod}(q^2-1))$

then it holds $\chi_3^{k_1k_2}(X) = 0$ for each $X \in G\text{-}H$.

The case i) We consider an element X of G from the type of classes VI_1 having the canonical form

$$\begin{pmatrix} 1 & 1 & & \\ & 1 & & \\ & & \varepsilon_2^2 & \\ & & & \varepsilon_2^{2q} \end{pmatrix}.$$

Since $\det X = \varepsilon_1^2$, then $X \in H\text{-}M(d)$. On the other hand, it holds

$$\chi_3^{k_1k_2}(X) = \{ (\varepsilon^{k_2})^2 + (\varepsilon^{qk_2})^2 + (\varepsilon^{k_1})^2 + (\varepsilon^{qk_1})^2 \}$$

where ε is a primitive (q^2-1)–th root of unity. Since $\varepsilon^{qk_i} = -\varepsilon^{k_i}$ (i=1,2), it follows $\chi_3^{k_1k_2}(X) = 2\{ (\varepsilon^{2k_2}\varepsilon^{2k_1} \}$.

In this case it can be seen that:

1. $\varepsilon^{2k_2} + \varepsilon^{2k_1} \neq 0$ except $k_1 - k_2 \equiv (q^2-1)/4(\mathrm{mod}(q^2-1)/2)$.
2. If $k_1 - k_2 \equiv (q^2-1)/4(\mathrm{mod}(q^2-1)/2)$ then it holds $\chi_3^{k_1k_2}(X) = 0$ for each $X \in$ II-M(d).

The case ii) We consider an element X of G from the type of classes XI_1 having the canonical form

$$\begin{pmatrix} 1 & 1 & & \\ & 1 & & \\ & & \varepsilon_1 & 1 \\ & & & \varepsilon_1 \end{pmatrix}$$

Since $\det X = \varepsilon_1^2$, then $X \in$ II-M(d). On the other hand, it holds

$$\chi_3^{k_1k_2}(X) = \varepsilon^{k_1} + \varepsilon^{k_2},$$

where is a primitive (q-1)-th root of unity. Since $\varepsilon^{k_1} = \varepsilon^{k_2}$, we obtain

$$\chi_3^{k_1k_2}(X) = 2\{ \varepsilon^{k_1} \} \neq 0.$$

Corollary 1. In the case d = 4: If $k_i \equiv (q+1)/2(\mathrm{mod}(q+1))$, i=1,2 then the restriction of the character $\chi_3^{k_1k_2}$ down to S splits up and by Corollary 2.2 we obtain the following:

1°) If $k_1 - k_2 \not\equiv (q^2-1)/4(\mathrm{mod}(q^2-1)/2)$ then the restriction of character $\chi_3^{k_1k_2}$ down to S is the sum of two irreducible conjugate characters of S degrees $\frac{1}{2}\{(1-q)(1+q^2)(1-q^3)\}$.

2°) If $k_1 - k_2 \equiv (q^2-1)/4(\mathrm{mod}(q^2-1)/2)$ then the restriction of $\chi_3^{k_1k_2}$ down to S is the sum of four reducible conjugate characters of S degrees $\frac{1}{4}\{(1-q)(1+q^2)(1-q^3)\}$.

Corollary 2. In the case d = 4: If $k_1 - k_2 \equiv (q^2-1)/2(\mathrm{mod}(q^2-1))$ then by Corollary 2.2 the restriction of character $\chi_3^{k_1k_2}$ down to S is the sum of two irreducible conjugate characters of S.

Corollary 3. In the case d=4: Except the cases $k_i \equiv (q+1)/2(\mathrm{mod}(q+1))$ (i=1,2) and $k_1 - k_2 \equiv (q^2-1)/2(\mathrm{mod}(q^2-1))$ the restriction of the character $\chi_3^{k_1k_2}$ down to S remains irreducible.

Type IV$_1$. As we have shown above (in the case d=2), if $k \equiv (q+1)/2$ $(\mathrm{mod}(q+1))$ then it holds $\chi_{4,1}^{k}(X) = 0$ for each $X \in G\text{-}H$. On the other hand, for an element X of G from the type of classes XI_1 having the canonical form

$$\begin{pmatrix} 1 & 1 & & \\ & 1 & & \\ & & \varepsilon_1 & 1 \\ & & & \varepsilon_1 \end{pmatrix},$$

we have $\chi_{4,1}^{k}(X) = \varepsilon^k \neq 0$, where ε is a primitive (q-1)-th root of unity. Since $\det X = \varepsilon_1^2$, then $X \in H\text{-}M(d)$.

Corollary. If $k \equiv (q+1)/2(\mathrm{mod}(q+1))$ then the restriction of the character $\chi_{4,1}^{k}$ down to S is the sum of two irreducible conjugate characters of S of degrees $\frac{1}{2}\{(q-1)^2(q^2+q+1)\}$ and with the exception of this case the restriction of $\chi_{4,1}^{k}$ down to S remains irreducible.

Type IV$_2$. Similarly, if $k \equiv (q+1)/2(\mathrm{mod}(q+1))$ then the restriction of the character $\chi_{4,2}^{k}$ down to S is the sum of two irreducible conjugate characters of S of degrees $\frac{1}{2}\{(q^2(1-q)(q^2+q+1)\}$. Except this case the restriction of $\chi_{4,2}^{k}$ down to S remains irreducible.

Type V. As we have shown above (in the case d=2), $k_1 - k_2 \equiv (q-1)/2 \pmod{q+1}$ and $k_3 \equiv (q+1)/2 \pmod{q+1}$ then it holds $\chi_5^{k_1k_2k_3}(X) = 0$ for each $X \in G-H$. On the other hand, for an element X of G from the type of classes IX_3 having the canonical form

$$\begin{pmatrix} \varepsilon_1^2 & & & \\ & 1 & & \\ & & 1 & 1 \\ & & & 1 \end{pmatrix}$$

we have $\chi_5^{k_1k_2k_3}(X) = (-1)(1-q^3)\{\varepsilon^{2k_1} + \varepsilon^{2k_2}\}$, where ε is a primitive (q-1)–th root of unity. Since in this case, $\varepsilon^{k_2} = -\varepsilon^{k_1}$ then we obtain

$$\chi_5^{k_1k_2k_3}(X) = (q^3-1)\{2\ \varepsilon^{2k}1\} \neq 0.$$

Since $\det X = \varepsilon_1^2$, then $X \in H\text{-}M(d)$.

Corollary. In the case d = 4: 1. Except the cases $k_1 - k_2 \equiv (q-1)/2 \pmod{q-1}$ and $k_3 \equiv (q+1)/2 \pmod{q+1}$ the restriction of the character $\chi_5^{k_1k_2k_3}$ down to S remains irreducible.

2. If $k_1 - k_2 \equiv (q-1)/2 \pmod{q-1}$ and $k_3 \equiv (q+1)/2 \pmod{q+1}$ than the restriction of $\chi_5^{k_1k_2k_3}$ down to S is the sum of two irreducible conjugate characters of S of degrees $1/2\ \{(q^2+q+1)(q^2+1)(q^2-1)\}$.

Types VI_i (i=1,2). In the case d = 4 the restriction of the character $\chi_{6,i}^{k_1k_2}$ down to S remains irreducible just as in the case d = 2.

Type VII. As we have shown above (in the case d=2), in the cases

i) $k_1 - k_3 \equiv (q-1)/2 \pmod{q-1}$ and $k_2 - k_4 \equiv (q-1)/2 \pmod{q-1}$

ii) $k_1 - k_2 \equiv (q-1)/2 \pmod{q-1}$ and $k_3 - k_4 \equiv (q-1)/2 \pmod{q-1}$

iii) $k_1 - k_4 \equiv (q-1)/2 \pmod{q-1}$ and $k_2 - k_3 \equiv (q-1)/2 \pmod{q-1}$

it holds $\chi_7^{k_1k_2k_3k_4}(X)=0$ for each $X \in G-H$. On the other hand, for an element X of G from the type of classes IX_1 having the canonical form

$$\begin{pmatrix} \varepsilon_1^2 & & & \\ & 1 & 1 & \\ & & 1 & 1 \\ & & & 1 \end{pmatrix}$$

we have $\chi_7^{k_1k_2k_3k_4}(x) = \varepsilon^{2k_1} + \varepsilon^{2k_2} + \varepsilon^{2k_3} + \varepsilon^{2k_4}$,

where ε is a primitive (q-1)–th root of unity.

The case i) Since $\varepsilon^{k_1} = -\varepsilon^{k_3}$ and $\varepsilon^{k_2} = -\varepsilon^{k_4}$, it follows that

$$\chi_7^{k_1k_2k_3k_4}(X) = 2(\varepsilon^{2k_3} + \varepsilon^{2k_4}).$$

Here it can be seen that:

1. $\varepsilon^{2k_3} + \varepsilon^{2k_4} \neq 0$ except $k_3 - k_4 \equiv (q-1)/4(\mathrm{mod}(q-i)/2)$.

2. If $k_3 - k_4 \equiv (q-1)/4(\mathrm{mod}(q-1)/2)$ under the conditions

   $k_1 - k_3 \equiv (q-1)/2(\mathrm{mod}(q-1))$ and $k_2 - k_4 \equiv (q-1)/2(\mathrm{mod}(q-1))$,

   then it holds $\chi_7^{k_1k_2k_3k_4}(X)=0$ for each $X \in$ H-M(d)

Since k_1, k_2, k_3, k_4 play symmetric roles in the values of the characters $\chi_7^{k_1k_2k_3k_4}$ on the classes, the result obtained for the case i) can be symmetrically obtained for the cases ii) and iii).

So we can summarize our results as follows:

i) Under the conditions

$k_1 - k_3 \equiv (q-1)/2(\mathrm{mod}(q-1))$ and $k_2 - k_4 \equiv (q-1)/2\ (\mathrm{mod}(q-1))$,

1. if $k_3 - k_4 \not\equiv (q-1)/4(\mathrm{mod}(q-1)/2)$ then the restriction of the character $\chi_7^{k_1k_2k_3k_4}$ down to S is the sum of two irreducible conjugate characters of S of degrees $1/2\{(q^3+q^2+q+1)(q^2+q+1)(q+1)\}$.

2. if $k_3 - k_4 \equiv (q-1)/4(\mathrm{mod}(q-1)/2)$ then the restriction of the character $X_7^{k_1k_2k_3k_4}$ down to S is the sum of two irreducible conjugate characters of S of degrees $\frac{1}{4}\{q^3+q^2+q+1)(q^2+q+1)(q+1)\}$.

ii) Under the conditions

$k_1 - k_2 \equiv (q-1)/2(\mathrm{mod}(q-1))$ and $k_3 - k_4 \equiv (q-1)/2(\mathrm{mod}(q-1))$,

1. if $k_2 - k_4 \not\equiv (q-1)/4(\mathrm{mod}(q-1)/2)$ then the restriction of the character $\chi_7^{k_1k_2k_3k_4}$ down to S is the sum of two irreducible conjugate characters of S of degrees $\frac{1}{2}\{q^3+q^2+q+1)(q^2+q+1)(q+1)\}$.

2. if $k_2 - k_4 \equiv (q-1)/4(\mathrm{mod}(q-1)/2)$ then the restriction of the character $\chi_7^{k_1k_2k_3k_4}$ down to S is the sum of four irreducible conjugate characters of S of degrees

$\frac{1}{4}\{q^3+q^2+q+i)(q^2+q+1)(q+i)\}$.

iii) Under the conditions

$k_1 - k_4 \equiv (q-1)/2(\mathrm{mod}(q-1))$ and $k_2 - k_3 \equiv (q-1)/2(\mathrm{mod}(q-1))$

1. if $k_4 - k_3 \not\equiv (q-1)/4(\mathrm{mod}(q-1)/2)$ then the restriction of the character $\chi_7^{k_1k_2k_3k_4}$ down to S is the sum of two irreducible conjugate characters of S of degrees $\frac{1}{2}(q^3+q^2+q+1)(q^2+q+1)(q+1)$.

2. if $k_4 - k_3 \equiv (q-1)/4(\mathrm{mod}(q-1)/2)$ then the restriction of the character $\chi_7^{k_1k_2k_3k_4}$ down to S is the sum of four irreducible conjugate characters of S of degrees $\frac{1}{4}(q^3+q^2+q+1)(q^2+q+1)(q+1)$

Except for the cases listed above the restriction of the character $\chi_7^{k_1k_2k_3k_4}$ down to S is an irreducible character of S.

Types VIII$_i$ (i=1,2), **IX$_j$** (j=1,2,3), **X$_t$** (t=1, ..., 5). All results obtained in the case d = 2 are also valid for the case d = 4. So the restrictions of the characters $\chi_{8,i}^{k_1k_2k_3}$, $\chi_{9,j}^{k_1k_2}$ and $\chi_{10,t}^{k}$ remain irreducible.

Type XI$_1$.As we have said in the case d=2, if $k_1 - k_2 \equiv (q-1)/2$ (mod(q-1)) then it holds $\chi_{11,1}^{k_1k_2}(X) = 0$ for each **X**$\in$**G-H**. On the other hand, for an element X of G from the type of classes **IV**$_1$ having the canonical form

$$\begin{pmatrix} \varepsilon_2 & 1 & & \\ & \varepsilon_2 & & \\ & & \varepsilon_2^q & 1 \\ & & & \varepsilon_2^q \end{pmatrix}$$

we have $\chi_{11,1}^{k_1k_2}(X) = \varepsilon^{k_1+k_2} \neq 0$

where ε is a primitive (q-1)–th root of unity. Since $\det X = \varepsilon_1^2$, then $X \in H$-$M(d)$.

Corollary. In the case d = 4 if $k_1 - k_2 \equiv (q-1)/2(\bmod(q-1))$ then the restriction of the character $\chi_{11,1}^{k_1k_2}$ down to S is the sum of two irreducible conjugate characters of S of degrees $\frac{1}{2}\{(q^2+q+1)(q^2+1)\}$ and with the exception of this case the restriction of $\chi_{11,1}^{k_1k_2}$ is an irreducible character of S.

Type XI$_2$. All result obtained in the case d = 2 are also valid for the case d=4. So the restriction of the character $\chi_{11,2}^{k_1k_2}$ down to S is an irreducible character for this case.

Type XI$_3$. As we have proved in the case d = 2, if

$$k_1 - k_2 \equiv (q-1)/2(\bmod(q-1))$$

then it holds $\chi_{11,3}^{k_1k_2}(X) = 0$ for each $X \in G$-H. On the other hand, an element X from the type of classes IV_1 having the canonical form

$$\begin{pmatrix} \varepsilon_2 & 1 & & \\ & \varepsilon_2 & & \\ & & \varepsilon_2^q & 1 \\ & & & \varepsilon_2^q \end{pmatrix}$$

belong to H-M(d), since $\det X = \varepsilon_1^2$ and $\chi_{11,3}^{k_1k_2}(X) = \varepsilon^{k_1+k_2} \neq 0$.

Corollary. In the case d = 4, if $k_1 - k_2 \equiv (q-1)/2(\mathrm{mod}(q-1))$ then the restriction of the character $X_{11,3}^{k_1k_2}$ down to S is the sum of two irreducible conjugate characters of S of degrees $\frac{1}{2}\{(q^2(q^2+1)(q^2+q+1)\}$ and with the exception of this cases the restriction of $\chi_{11,3}^{k_1k_2}$ down to S is an irreducible character of S.

REFERENCES

[1] Dickson, L.E.; Linear Groups. Dover, New York, 1958.

[2] Dornhoff, L; Group Representation Theory (Part A), Dekker, New York, 1971.

[3] Green, J.A.; The Characters of The Finite General Linear Groups. Trans. Amer. Math.Soc. 80 (1955), 402-477.

[4] Isaacs, I.M.; Character Theory of Finite Groups, Academic Press, New York-London, 1976.

[5] Littlewood, D.E.; The Theory of Group Characters and Matrix Representations of Groups, Oxford, 1940.

[6] Simpson W.A. and Frame, J.S.; The Character Tables for SL(3,q), SU(3,q^2), PSL(3,q), PSU(3,q^2), Can.J Math. vol 25, No. 3 (1973), 486-494.

[7] Steinberg, R.; The Representation of GL(3,q), GL(4,q), PGL(3,q) and PGL(4,q), Can. J. Math. 3 (1951), 225-235.

Address:

Department of Mathematics, Faculty of Science, University of Istanbul, 34459 Vezneciler-Istanbul, TURKEY.