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FIXED POINT THEOREMS IN BANACH SPACES WITH NORMAL STRUCTURE

D. K. Ganguly and ***D. Bandyopadhyay***

ABSTRACT: This paper encompasses two theorems on the existence of a unique common fixed point and the approximation of this fixed point by Ishikawa type iterative sequences, for two operators satisfying property – A* in Banach spaces having normal structure. Moreover, certain relationship between normal structure and property – B* has been established.

1. INTRODUCTION: In 1948, Brodskii and Milman [1] introduced the notion of normal structure, an interesting geometrical property, in a Banach space. Since then it has been studying both as a purely geometrical property of a Banach space and a fundamental tool for finding the existence of fixed point of an operator. The definition of normal structure runs as follows:

DEFINITION 1.1: *Let E be a bounded subset of a Banach space X. A point $z \in E$ is a non-diametral point of E if $\sup_{x\in E} \| x - z \| < \delta(E)$ [= diameter of E]. A bounded convex subset K of E is said to have a normal structure if every convex subset C of K containing more than one point has at least one non-diametral point.*

Several authors ([2], [7], [8]) are devoted to the study of normal structure for finding the existence of fixed point of different contraction type operators. According to Kannan [7], an operator T is said to satisfy property-A over a Banach space X if it satisfies

1.1] $\| Tx - Ty \| \leq \frac{1}{2} [\| x - Tx \| + \| y - Ty \|], \forall x, y \in X;$

and also T possesses property – B over $E \subset X$, if for every closed subset K of E containing more

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than one point and mapped into itself by T, then there exists $z \in K$ such that $\| z-Tz \| < \sup_{y \in K} \| y-Ty \|$. Then he proved the following theorem which establishes a relation between normal structure and property-B:

THEOREM A:[6] *Let K be a bounded convex subset of a Banach space X and $T : K \to K$ satisfy 1.1. If K has normal structure then T satisfies property-B over K.*

We shall here introduce the following definitions in a Banach space X which will be of further use in proving our results.

DEFINITION 1.2: *Two mappings $T_1, T_2 : X \to X$ are said to be simultaneous quasi-contraction mappings of order (m , n) , m , n ∈ Z [= set of Natural numbers] if*

1.2] $\| T_1^m x - T_2^n y \| \le q \max \{\|x-y\|, \|x- T_1^m x \|, \|y- T_2^n y \|, \| x- T_2^n y \|, \| y- T_1^m x \|\}, \forall x,y \in X$

$q \in [0,1)$.

If m=n=1 in 1.2] then T_1, T_2 are simultaneous quasi-contraction mappings of order (1,1) and if m=n=1, $T_1 = T_2 = T$ in 1.2], then T is a quasi-contraction mapping, introduced by Cirič [4].

We shall say that T_1, T_2 possess property - A^* over X if T_1, T_2 are simultaneous quasi-contraction mappings of order (1,1) with $q \in [0, \frac{1}{2}]$ instead of $q \in [0,1)$ and T_1, T_2 possess property - A_1^* over X if T_1, T_2 satisfy 1.2] with $q \in [0, \frac{1}{2}]$ instead of $q \in [0, 1]$.

DEFINITION 1.3: *Let E be a bounded subset of a Banach space X and $T_1, T_2 : E \to E$. Then T_1, T_2 are said to possess property- B^* over E if every closed convex subset C of E containing more than one element and mapped into itself by T_1, T_2 both, has at least one point z such that*

$\max\{ \|z - T_1 z\|, \|z - T_2 z \| \} < \max\{\sup_{x, y \in C} \|x - T_1 y\|, \sup_{x, y \in C} \| x - T_2 y \|\}$.

If $T_1 = T_2 = T$, then the above inequality becomes $\|z-Tz \| < \sup_{x, y \in C} \|x - Ty \|$, which clearly includes property-B and hence property-B^* extends property-B of Kannan.

DEFINITION 1.4: *We consider Reinermann type of infinite matrix [9] defined as $A=(a_{nk}), n,k \in$*

$\mathbb{Z}_0$ [= $\mathbb{Z} \cup \{0\}$], $a_{nk} \in 3$, where 1.3] $a_{nk} = c_k \prod_{r=k+1}^{n}(1-c_r)$, c_n and 0 according as $k<n$, $k=n$ and $k>n$, and the real sequence $\{c_n\}_n$ satisfies (i) $c_0 = 1$, (ii) $0<c_n<1$, $\forall n \in \mathbb{Z}$, and (iii) $\sum_n c_n = \infty$.

Clearly A is a regular matrix [3]. With this matrix we shall define a generalised iteration process of Ishikawa type [6] for two operators as: Let E be a non-empty closed convex subset of a Banach space X and $T_i : X \to X$ $(i = 1,2)$. Then for any $x_0 \in E$,

1.4] $x_{n+1} = (1-c_n)x_n + c_n T_1 y_n$ and $y_n = (1-a_n)x_n + a_n T_2 x_n$, $\forall n \in \mathbb{Z}_0$ where the real sequence $\{a_n\}_n$ satisfies $0<a_n \le c_n<1$ for all $n \in \mathbb{Z}_0$ and the real sequence $\{c_n\}_n$ satisfies (i)-(iii) of 1.3]. We denote this process by $(x_0, A, T_1, T_2, \{a_n\})$ and call it a generalised iteration process for two operators T_1, T_2, determined by an initial point x_0, a matrix A and a sequence $\{a_n\}_n$.

2. MAIN RESULTS

THEOREM-2.1: *Let E be a non-empty bounded convex subset of a Banach space X such that E has normal structure. Let $T_1, T_2 : E \to E$ possess property - A^* over E and satisfy the following condition:*

2.1] $\max\{\|T_1x - z\|, \|T_2x - z\|\} \le \|x - z\|$, $\forall x \in E$, and for every non-diametral point z of E. Then T_1 and T_2 have a unique common fixed point In E.

To prove this theorem, we need the following

LEMMA: *Let X be a Banach space and E, a non-empty bounded convex subset of X having normal structure. Then, corresponding to each non-diametral point z of E, there is a sequence $\{x_n\}_n$ in E such that $\lim_n x_n = z$.*

PROOF: E has normal structure and z is a non-diametral point of E $\Rightarrow \sup_{x \in E} \|x - z\| < \delta(E) =$ d(say) $> 0 \Rightarrow \|x - z\| < d$, $\forall x \in E$. Using this, we shall construct a sequence $\{x_n\}_n$ in E as follows:

Choose $x_1 \in E$ and $r_1 > 1$ such that $\|x_1 - z\| \le d/r_1$.

Now, convexity of E and $x_1, z \in E \Rightarrow \frac{1}{2}(x_1 + z) \in E \Rightarrow \| \frac{1}{2}(x_1 + z) - z\| = \frac{1}{2}\|x_1 - z\| \le d/2r_1$;

writing $x_2 = \frac{1}{2}(x_1 + z)$ we have $\|x_2 - z\| \le d/2r_1$.

Again, writing $x_3 = \frac{1}{2}(x_2 + z) \in E$, we have $\|x_3 - z\| = \| \frac{1}{2}(x_2 + z) - z\| = \frac{1}{2}\|x_2 - z\| \le d/2^2 r_1$.

Similarly, we can write, for each $n \in \mathbb{Z}$, $x_n = \frac{1}{2}(x_{n-1} + z) \in E$ and $\|x_n - z\| \le d/2^{n-1} r_1$.

Now, choose $\varepsilon > 0$ so that for sufficiently large $n \ge N > 0$, $d/2^{N-1} r_1 < \varepsilon$.

Thus, for all $n \ge N$, $\|x_n - z\| < \varepsilon \Rightarrow \lim_n \|x_n - z\| = 0$ i.e. $\lim_n x_n = z$.

PROOF OF THEOREM: Normal structure of E and the above lemma $\Rightarrow \exists$ a non-diametral point $z \in E$ and a sequence $\{x_n\}_n \in E \ni \lim_n x_n = z$. From condition-2.1, we have $\|T_1 x_n - z\| \le \|x_n - z\|$ and $\|T_2 x_n - z\| \le \|x_n - z\|$ which yields $\lim_n T_1 x_n = z = \lim_n T_2 x_n$. Now from property- A^*, we have $\|z - T_2 z\| \le \|z - T_1 x_n\| + \|T_1 x_n - T_2 z\| \le \|z - T_1 x_n\| + q \max\{\|x_n - z\|, \|x_n - T_1 x_n\|, \|z - T_2 z\|, \|x_n - T_2 z\|, \|z - T_1 x_n\|\} \Rightarrow \|z - T_2 z\| \le q \max\{0, 0, \|z - T_2 z\|, \|z - T_2 z\|, 0\}$, whenever $n \to \infty$

$\Rightarrow \quad \|z - T_2 z\| \le q \; \|z - T_2 z\| \Rightarrow z = T_2 z$.

Similarly, $z = T_1 z$. If u is another common fixed point of T_1 and T_2 in E, then from property-A^*, it is easy to show that $u = z$. Consequently $z \in E$ is the unique common fixed point of T_1 and T_2.

REMARK: In the above theorem, if T_1, T_2 satisfy property-A_1^* instead of property-A^*, then in the similar way, it can be shown that T_1 and T_2 have a unique common fixed point in E. By means of the following example, it can be easily recognise that T_1, T_2 do not satisfy property-A^* but satisfy property-A_1^*.

EXAMPLE-2.1: Let X = 3 and E = [0,1]. Then clearly E has normal structure. Define $T_1, T_2 : E \to E$ by

$$T_1 x = \begin{cases} 0, & \text{if } x \in \left[0, \frac{1}{2}\right] \\ \frac{1}{2}, & \text{if } x \in \left(\frac{1}{2}, 1\right] \end{cases} \qquad \text{and} \quad T_2 x = \begin{cases} 0, & \text{if } x \in \left[0, \frac{1}{2}\right) \\ \frac{1}{4}, & \text{if } x \in \left[\frac{1}{2}, 1\right) \\ \frac{2}{3}, & \text{if } x = 1 \end{cases}$$

Then $T_1^2 x = 0 \ \forall\, x \in [0, 1]$ and $T_2^3 x = 0 \ \forall\, x \in [0, 1]$. Therefore, $T_1^2, T_2^3 : E \to E$. Also, 0 is the unique common fixed point of T_1, T_2 in E. Further, T_1, T_2 satisfy property-A_1^* for all x, y $\in$ E and for any $q \in [0, \frac{1}{2}]$ but T_1, T_2 fail to satisfy property-A^* with $x = \frac{1}{3}$, y = 1 and $q = \frac{1}{4}$.

3. APPROXIMATION OF FIXED POINT

Consider the class **Q** of all quasi non-expansive mappings in X. If T_1, T_2, satisfying property-A^*, have a common fixed point u in X, then it can be shown easily that $T_1, T_2 \in$ **Q**. Now, we shall prove the following:

THEOREM-3.1: *Let E be a non-empty bounded closed convex subset of a uniformly convex Banach space X, $T_1, T_2 : E \to E$ satisfy property - A^* over E and condition-2.1. Then for any $x_0 \in E$ and a generalised iteration process $(x_0, A, T_1, T_2, \{a_n\}_n)$, the sequences, $\{x_n\}_n$ and $\{y_n\}_n$ obtained by 1.4] with $\{c_n\}_n$ satisfying (i), (ii) of 1.3] and (iv) $\sum_{n\geq 0} \min(c_n, 1 - c_n) = \infty$, if converge, converge to a common fixed point of T_1 and T_2 in E.*

To prove this theorem, we need the following:

LEMMA: *Let X be a uniformly convex Banach space with modulus of convexity δ. Then, for u, v $\in X$, with $\|u\| \leq 1$, $\|v\| \leq 1$ and for all $c \in (0, 1]$, $\| cu + (1-c)v \| \leq 1-2 \min(c, 1-c)\, \delta(\|u-v\|)$.*

PROOF OF LEMMA: Obviously, we may assume that $c \leq \frac{1}{2}$. Since X is uniformly convex, therefore we have, $\delta(\|u-v\|) \leq 1- \frac{1}{2}\|u + v\|$. It, now, suffices to prove that $2c\,(1- \frac{1}{2}\|u+v\|) \leq 1 - \|cu + (1-c)v\|$ i.e., $2c + \|cu + (1-c)v\| \leq 1 + c\|u+v\|$. Now, $2c + \|cu+(1-c)v\| = 2c + \|c(u + v) + (1-2c)v\| \leq 2c + c\|u + v\|+(1-2c)\|v\| \leq 1+ c\|u +v\|$ (as $\|v\| \leq 1$) $\Rightarrow 2c\,(1- \frac{1}{2}\|u+v\|) \leq 1 - \| cu + (1-c)v\| \Rightarrow 2c\, \delta(\|u-v\|) \leq 1-\| cu + (1-c)v \| \Rightarrow \|cu + (1-c)v\| \leq 1- 2c\, \delta(\|u-v\|)$.

Hence, in other words, we have, $\| cu + (1-c) v \| \leq 1- 2 \min(c, 1-c)\, \delta(\|u-v\|)$.

PROOF OF THEOREM: Since uniform convexity of X implies the normal structure of E. therefore from theorem-2.1. T_1 and T_2 have a unique common fixed point u, say, in E. Then. $T_1, T_2 \in \mathbf{Q}$. For any $x_0 \in E$, and for all $n \in \angle_0$.

3.1] $\| y_n - u \| = \| (1- a_n)x_n + a_n T_2x_n -u \| \leq (1-a_n) \| x_n - u \| + a_n \| T_2 x_n -u \| \leq \| x_n -u \|$, and

3.2] $\| x_{n+1} -u \| = \| (1-c_n)x_n + c_n T_1y_n -u\| \leq (1-c_n) \| x_n - u \| + c_n \| T_1 y_n -u \| \leq \| x_n -u \|$[using (3.1)].

Thus, we have, for all $n \in \angle_0$, $\| x_n -u \| \leq \| x_{n-1} - u \| \leq .... \leq \|x_0 - u \| = K$ (say).

This reveals that $\{\|x_n - u \|\}_n$ is a non-increasing sequence in [0, K]: hence it must converge to a limit. We shall show that this limit is zero. If not, let it be $a > 0$. Now, we shall first show that $\lim_n\|x_n-T_1y_n\| = 0$. Suppose not. Then, either (I) $\exists$ an $\varepsilon > 0 \ni \| x_n - T_1 y_n \| \geq \varepsilon$, $\forall n \in \angle_0$, or (II) $\lim_n \inf \|x_n - T_1 y_n \| = 0$.

For case - I], we use modulus of convexity δ (>0) of X. Now, for all $n \in \angle_0$,

3.3] $\| x_{n+1} - u \| = \| (1-c_n)x_n + c_n T_1 y_n -u \| = \|(1-c_n) (x_n -u) + c_n (T_1 y_n -u) \|$

$$= \|x_n -u \| \left\| (1-c_n)\frac{x_n - u}{\|x_n - u\|} + c_n \frac{T_1 y_n - u}{\|x_n - u\|} \right\| = \|x_n - u \| \, \| (1-c_n)x + c_n y\|, \text{ where}$$

3.4] $x = (x_n -u) / \|x_n -u\|$, $y = (T_1 y_n -u) / \| x_n - u \|$; then

3.5] $\| x \| = 1$, $\| y \| \leq 1$, and $\| x -y \| = (\|x_n - T_1 y_n \| / \|x_n -u \|) \geq \varepsilon/K$, $\forall n \in \angle_0$.

Since δ is non-decreasing in (0, 2], therefore from (3.5) we have 3.6] $\delta(\|x - y \|) \geq \delta(\varepsilon/K)$.

Now, using lemma and (3.6), we can write from (3.3) that, for all $n \in \angle_0$,

$$a \leq \| x_{n+1} -u\| \leq \| x_n -u \| \, \| (1-c_n) x + c_n y \| \leq \|x_n -u \| \{1-2 \min (c_n , 1-c_n) \delta(\|x-y\|)\}$$

$$\leq \| x_n -u \| \{1-2\delta(\varepsilon/K) \min (c_n , 1- c_n)\}$$

$$= \| x_n -u \|\{1-b \min (c_n ,1-c_n)\}, \text{ where } b = 2 \delta(\varepsilon/K) > 0$$

$$= \| x_n -u \| - b \min (c_n , 1-c_n) \| x_n - u \|$$

$$\leq \|x_{n-1} - u \| - b \min (c_{n-1} , 1-c_{n-1}) \| x_{n-1} -u\| - b \min (c_n , 1-c_n) \| x_n -u \|$$

$$\leq K - b.\, a \sum_{i=0}^{n} \min(c_i, 1 - c_i)$$

$\Rightarrow \sum_n \min(c_n, 1-c_n) \leq (K-a)/(ba)$ which contradicts (iv), as r.h.s. is finite. Hence case-I is not true.

For case - II]. $\exists \{x_{n_i}\}_i \subset \{x_n\}_n$, $\{y_{n_i}\}_i \subset \{y_n\}_n$ such that 3.7] $\lim_i \|x_{n_i} - T_1 y_{n_i}\| = 0$.

We shall show that $\lim_i \|x_{n_i} - T_2 x_{n_i}\| = 0$. For this it suffices to show with (3.7) that $\lim_i \|T_1 y_{n_i} - T_2 x_{n_i}\| = 0$.

Now, for any $n \in Z_0$, we have

3.8] $\| T_1 y_n - T_2 x_n \| \leq q \max \{\| x_n - y_n \|, \| y_n - T_1 y_n \|, \|x_n - T_2 x_n \|, \| y_n - T_2 x_n \|, \|x_n - T_1 y_n \|\} = qD$

where $D = \max \{\| x_n - y_n \|, \| y_n - T_1 y_n \|, \|x_n - T_2 x_n \|, \| y_n - T_2 x_n \|, \|x_n - T_1 y_n \| \}$.

Now, if $D = \| x_n - y_n \|$, then from (3.8) we have on using (1.4)

$\|T_1 y_n - T_2 x_n \| \leq q \| x_n - y_n \| = qa_n \|x_n - T_2 x_n \| \leq qa_n \| x_n - T_1 y_n \| + qa_n \|T_1 y_n - T_2 x_n \|$

$\Rightarrow$ 3.9] $$\| T_1 y_n - T_2 x_n \| \leq \frac{qa_n}{1 - qa_n} \|x_n - T_1 y_n \|.$$

If $D = \| y_n - T_1 y_n \|$, then using (1.4), we have from (3.8)

$\| T_1 y_n - T_2 x_n \| \leq q \| y_n - T_1 y_n \| = q \| (1 - a_n) x_n + a_n T_2 x_n - T_1 y_n \| \leq q(1 - a_n) \| x_n - T_1 y_n \| + qa_n \| T_1 y_n - T_2 x_n \|$

$\Rightarrow$ 3.10] $$\| T_1 y_n - T_2 x_n \| \leq \frac{q(1 - a_n)}{1 - qa_n} \|x_n - T_1 y_n \|.$$

If $D = \| x_n - T_2 x_n \|$, then from (3.8) we have

$\| T_1 y_n - T_2 x_n \| \leq q \| x_n - T_2 x_n \| \leq q \| x_n - T_1 y_n \| + q \|T_1 y_n - T_2 x_n \|$

$\Rightarrow$ 3.11] $$\| T_1 y_n - T_2 x_n \| \leq \frac{q}{1 - q} \| x_n - T_1 y_n \|.$$

If $D = \| y_n - T_2 x_n \|$, then using (1.4), we have from (3.8)

$\| T_1 y_n - T_2 x_n \| \leq q \| y_n - T_2 x_n \| = q \| (1-a_n) x_n + a_n T_2 x_n - T_2 x_n \| = q (1-a_n) \|x_n - T_2 x_n \|$

$\leq q(1-a_n)\|x_n - T_1 y_n\| + q(1-a_n)\|T_1 y_n - T_2 x_n\|$

$\Rightarrow$ 3.12] $$\|T_1 y_n - T_2 x_n\| \leq \frac{q(1-a_n)}{1-q(1-a_n)} \|x_n - T_1 y_n\|.$$

If $D = \|x_n - T_1 y_n\|$, then from (3.8), we have 3.13] $\|T_1 y_n - T_2 x_n\| \leq q\|x_n - T_1 y_n\|$.

Thus, using (3.9) - (3.13) in (3.8) we have $\forall n \in \angle_0$, $\|T_1 y_n - T_2 x_n\| \leq \max\{r_n, r\}\|x_n - T_1 y_n\|$

where $r = \dfrac{q}{1-q} \in (0, 1]$, $r_n = \max\left\{\dfrac{qa_n}{1-qa_n}, \dfrac{q(1-a_n)}{1-qa_n}, \dfrac{q(1-a_n)}{1-q(1-a_n)}\right\} \in (0, 1]$ $\forall n \in \angle_0$ and

$\lim_n r_n > 0$. Therefore, $\|T_1 y_{n_i} - T_2 x_{n_i}\| \leq \max\{r_{n_i}, r\}\|x_{n_i} - T_1 y_{n_i}\| \to 0$ as $i \to \infty$, using (3.7). Hence,

$\lim_i \|x_{n_i} - T_2 x_{n_i}\| = 0$.

Consequently, $\lim_i \|x_{n_i} - y_{n_i}\| = 0$ and $\lim_i \|y_{n_i} - T_1 y_{n_i}\| = 0$. Further, for any $i, j \in \angle$,

3.14] $$\|T_2 x_{n_i} - T_2 y_{n_j}\| \leq \|T_2 x_{n_i} - T_1 y_{n_j}\| + \|T_1 y_{n_j} - T_2 x_{n_j}\|.$$

Now, 3.15] $\|T_2 x_{n_i} - T_1 y_{n_j}\| \leq q \max\{\|x_{n_i} - y_{n_j}\|, \|x_{n_i} - T_2 x_{n_i}\|, \|y_{n_j} - T_1 y_{n_j}\|, \|x_{n_i} - T_1 y_{n_j}\|, \|y_{n_j} - T_2 x_{n_i}\|\}$.

Let; $\lim_{i,j} \|T_2 x_{n_i} - T_1 y_{n_j}\| = t$. Then, we have

$$\|x_{n_i} - y_{n_j}\| \leq \|x_{n_i} - T_2 x_{n_i}\| + \|T_2 x_{n_i} - T_1 y_{n_j}\| + \|y_{n_j} - T_1 y_{n_j}\| \Rightarrow \lim_{i,j} \|x_{n_i} - y_{n_j}\| \leq t;$$

$$\|x_{n_i} - T_1 y_{n_j}\| \leq \|x_{n_i} - y_{n_j}\| + \|y_{n_j} - T_1 y_{n_j}\| \Rightarrow \lim_{i,j} \|x_{n_i} - T_1 y_{n_j}\| \leq t;$$

$$\|y_{n_j} - T_2 x_{n_i}\| \leq \|x_{n_i} - y_{n_j}\| + \|x_{n_i} - T_2 x_{n_i}\| \Rightarrow \lim_{i,j} \|y_{n_j} - T_2 x_{n_i}\| \leq t.$$

In view of these inequalities, (3.15) reduces to

$\lim_{i,j} \|T_2 x_{n_i} - T_1 y_{n_j}\| \leq qt \Rightarrow t \leq qt \Rightarrow t = 0$ i.e. $\lim_{i,j} \|T_2 x_{n_i} - T_1 y_{n_j}\| = 0$.

Using this in (3.14) we have, $\lim_{i,j} \|T_2 x_{n_i} - T_2 y_{n_j}\| = 0 \Rightarrow \{T_2 x_{n_i}\}_i$ is a Cauchy sequence in $E \Rightarrow \exists$

$v \in E \ni \lim_i T_2 x_{n_i} = v$. Consequently, $\lim_i x_{n_i} = v = \lim_i T_2 x_{n_i} = \lim_i T_1 y_{n_i} = \lim_i y_{n_i}$.

Again, $\|T_2 x_{n_i} - T_1 v\| \leq q \max\{\|x_{n_i} - v\|, \|x_{n_i} - T_2 x_{n_i}\|, \|v - T_1 v\|, \|x_{n_i} - T_1 v\|, \|v - T_2 x_{n_i}\|\}$.

Now taking limit as $i \to \infty$ on both sides we have

$\| v - T_1v \| \le q \max \{ 0, 0, \| v-T_1v \| . \|v - T_1v \|, 0 \} = q \| v - T_1v \| \Rightarrow \| v - T_1v \| = 0$ i.e., $v = T_1v$. From property-A^*, it is easy to show that $v = T_2v$. Therefore, v is a common fixed point of T_1 and T_2 in E. Since u is the unique common fixed point of T_1 and T_2, hence $u = v$. Thus, $\lim_i x_{n_i} = u$ and $\{\|x_n - u\|\}_n$ monotone non-increasing, together would imply $\lim_n x_n = u$. Also, form (3.1), $\lim_n y_n = u$. Hence, the iterative sequences $\{x_n\}_n$ and $\{y_n\}_n$ both converge to a unique common fixed point of T_1 and T_2 in E .

4. RELATION BETWEEN NORMAL STRUCTURE AND PROPERTY - B^*:

In this section, we shall establish a relationship between normal structure and property-B^* which extends theorem-A of Kannan.

THEOREM-4.1: *Let E be a bounded convex subset of a Banach space X and $T_1, T_2 : E \to E$ satisfy property-A^* over E. If E has normal structure, then T_1 and T_2 satisfy property - B^* over E.*

PROOF: Let us suppose that E has normal structure. We further assume that the contrapositive statement hold. Then there exists a closed convex subset C of E such that

4.1] $\max \{\|x-T_1x\|, \|x-T_2x\|\} = \max \{\sup_{x, y \in C} \|x-T_1y\|, \sup_{x, y \in C} \|x-T_2y\|\} = t\ (\neq 0)$ (say), $\forall\ x \in C$.

Then, clearly the image sets $T_i\ (C)$ (i=1, 2) of C contain more than one element ; otherwise, for some $u \in C$, $T_i\ (C) = \{u\} \Rightarrow T_1\ u = u = T_2\ u$ and therefore, $\max \{\|u-T_1u \| , \| u-T_2u \|\} = 0$ which contradicts (4.1). Let $S = T_1\ (C) \cup T_2\ (C) \subset C$ and $D = co\ [S]$, the convex hull of S.

Now for any $x, y \in D$, any one of the following holds:

(i) $\exists\ x^/ , y^/ \in C$, $\ni x = T_1\ x^/ , y = T_2\ y^/$;

(ii) $\exists\ x^/, y_i^/ \in C$ (i=1,2 .....,n) $\ni x = T_1\ x^/$, $y = \sum_{i=1}^{n} a_i\ T_2\ y_i^/$, $\sum_{i=1}^{n} a_i = 1$;

(iii) $\exists\ x_i^/ , y_j^/ \in C$ (i=1,2, ...m, j=1,2,.......,n) $\ni x = \sum_{i=1}^{m} a_i\ T_1\ x_i^/$, $y = \sum_{j=1}^{n} b_j\ T_2\ y_j^/$, $\sum_{i=1}^{m} a_i = 1 = \sum_{j=1}^{n} b_j$.

Now, we shall first show that 4.2] $\|\ T_1\ x- T_2\ y\ \| \le t$, $\forall\ x, y \in C$; in fact, since $\mathbf{T_1}$, $\mathbf{T_2}$ satisfy property - A^* over E and $C \subset E$, therefore, for all $x, y \in C$,

$\| T_1 x - T_2 y \| \leq q \max \{\|x-y\|, \|x-T_1 x\|, \| y- T_2 y \|, \| x - T_2 y \|, \| y - T_1 x \| \}$

$\leq q \max \{ (\| x - T_1 x \| + \| T_1 x - y \|). t, t, t, t \}$ [using (4.1)]

$\leq q \max \{ 2t, t, t, t, t \} = 2qt \leq t$, as $q \leq 1/2$.

Again, it can be easily shown for each of the above cases (i-iii) that $\|x-y\| \leq t$. So $\delta(D) \leq t$.

The same conclusion can be obtained on interchanging T_1 and T_2 in (i-iii).

Now, for any $x \in D$, [$x = T_j x'$ or $\sum_{i=1}^{n} a_i T_j x_i'$, $x', x_i' \in C$ ($i= 1, 2, ..., n, j = 1,2$), $\sum_{i=1}^{n} a_i = 1$], $T_j x \in D$ for $j=1, 2$, as $T_j : C \to C$ and $D \subseteq C$. Further, from (4.1), $\max \{\|x-T_1x\|, \|x -T_2x\|\} = t, \forall x \in C$. Thus, for any $y \in D$, there exists $x \in C$ such that either $y = T_1 x$ or $y = T_2 x$; and therefore, for any $y \in D$, $\sup_{x \in D} \| x-y \| = t \geq \delta (D)$ which contradicts the assumption of normal structure of E.

Hence the theorem follows.

REMARK: The converse of the above theorem may not always hold good which can be revealed form the following

EXAMPLE - 4.1: Let m be the space of bounded sequences of real numbers with the supremum norm [5] and let $C = \{x \in m \mid \| x \| \leq 2\}$. Evidently, C is a bounded convex subset of m. Now, let $K= \{x_i \mid i \in \mathbb{Z}\} \subset C$ where $x_i = \{0, 0, ..., 1, 0, 0, ...\}$ (1 in the i^{th} place). Then, $\delta(K) = 1$. Also, $\sup_{y \in C} \|x -y\| = 1$ for every $x \in K$. Hence C does not have normal structure. Now, define $T_1, T_2 : C \to C$ by $T_1x = 0$ and $T_2x = 1$, for all $x \in C$. Then T_1, T_2 possess property - A^* over C for some $q \in (0, \frac{1}{2}]$ and also have property - B^* over C.

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DEPARTMENT OF PURE MATHEMATICS,
UNIVERSITY OF CALCUTTA,
35, BALLYGUNGE CIRCULAR ROAD,
CALCUTTA-700 019, INDIA.
E-mail: msngr@cal.vsnl.net.in
[code: 31/DHK/0007(H)]