İstanbul Üniv. Fen Fak. Mat. Dergisi **61-62** (2002-2003), 41-55

PARAMETRIC VIBRATIONS AND RESONANCE

NIGAR M. ILYASOVA

Abstract

System of linear differential equations with periodic coefficients depending on some parameters is examined in this paper.For small changing of parameters the monodromy matrix and its eigenvalues are constructed.We consider the problem of stability of trivial solution of the damped Hill equation and obtain the domains of stability in the space of parameters. Results are used for the problems of dynamic stability of viscoelastic systems, columns and triangle plates.

Key words:monodromy matrix, stability domain,viscoelastic plate, column.
AMS subject classifications: 37C75, 74H55

1.Introduction

Analysis of the parametric vibrations of elastic systems subjected to time varying loadings is one of the well established areas of applied mechanics [1]. Some important results for viscoelastic materials are also obtained [1-6].The problem of dynamic stability of a Kelvin-viscoelastic column is studed in [3].Deterministic as well as random time variations are considered.The effect of viscoelasticity on the instability region is investigated in [4] and it is found that viscoelasticity increases the stability of the plate. The monodromy matrix method is used in [5] for determination of stability regions of Hill equation (and there is small inaccuracy), and such results are obtained in [6] by means of average method.The mathematical properties of a variational second order evolution equation, which includes the equations modelling vibrations of the Euler-Bernoulli and Rayleigh beams with the global or local Kelvin-Voigt damping are studed in [8].Strong asymptotic stability and exponential stability under various conditions on the damping are investigated.Many stability results were obtained by the construction of appropriate Lyapunov functions. Using this method a criterion of asymptotic stability for a linear oscillator with variable parameters is obtained in [9].

2.Statement of problem and general results

Consider the system of linear differential equations

$$\dot{x} = A(t,\tilde{\alpha})x, \quad t \in (a_1, b_1), \quad \tilde{\alpha} \in G_1 \subset R^p, x \in R^n \tag{1}$$

where the matrix A depends on the parameters $\tilde{\alpha} = (\alpha_1, \alpha_2, \ldots, \alpha_p)$ and it is T-periodic,i.e. $A(t+T,\tilde{\alpha}) = A(t,\tilde{\alpha})$ .It is known that if $A \in C^k[(a_1, b_1)\times G_1]$ then $x(t,\tilde{\alpha}) \in C^k(G_1)$ and if A is an analytic matrix-function of all its arguments then the solution is an analytic function of the parameters (Poincare).

Suppose that for the value $\alpha_0 = (\alpha_{10}, \ldots, \alpha_{m0})$ of the first m parameters $(m \le p)$ of $\alpha = (\alpha_1, \ldots, \alpha_m)$, we may solve the system (1).Let us consider $A = A(t,\alpha)$ and expand this function to the series in the neighborhood of α_0

$$A(t,\alpha) = A_0(t) + \sum_{i=1}^{m} A_i(t)(\alpha_i - \alpha_{i0}) + \sum_{i,j=1}^{m} A_{ij}(t)(\alpha_i - \alpha_{i0})(\alpha_j - \alpha_{j0}) + \ldots \tag{2}$$

Evolution matrix $U(t,\alpha)$ of the system (1) satisfies the initial value problem

$$\dot{U} = A(t,\alpha)U, \quad U(0,\alpha) = I \tag{3}$$

If we seek the solution of this problem as

$$U(t,\alpha) = U_0(t) + \sum_{i=1}^{m} U_i(t)(\alpha_i - \alpha_{i0}) + \sum_{i,j=1}^{m} U_{ij}(t)(\alpha_i - \alpha_{i0})(\alpha_j - \alpha_{j0}) + \ldots$$

from (2) and (3) we get the following problems

$$\dot{U}_0 = A_0(t)U_0, U_o(0) = I,$$

$$\dot{U}_i = A_o(t)U_i + A_i(t)U_0, U_i(0) = 0, i = \overline{1,m}, \tag{4}$$

$$\dot{U}_{ij} = A_0(t)U_{ij} + A_{ij}(t)U_0 + A_i(t)U_j + A_j(t)U_i, U_{ij}(0) = 0; i,j = \overline{1,m,\ldots}$$

For the known $U_0(t)$, solutions of second and third problems in (4) are written as

$$U_i(t)=U_0(t)\int_0^t U_0^{-1}(\tau)A_i(\tau)U_0(\tau)d\tau,$$

$$U_{ij}(t)=U_0(t)\left[\int_0^t U_0^{-1}(\tau)A_{ij}(\tau)U_0(\tau)d\tau+\int_0^t U_0^{-1}(\tau)A_i(\tau)U_0(\tau)\left(\int_0^\tau U_0^{-1}(s)A_j(s)U_0(s)ds\right)d\tau+\right.$$

$$\left.+\int_0^t U_0^{-1}(\tau)A_j(\tau)U_0(\tau)\left(\int_0^\tau U_0^{-1}(s)A_i(s)U_0(s)ds\right)d\tau\right]$$

By substituting $t=T$ in $U(t,\alpha)$, the monodromy matrix

$$U(T,\alpha)=U_0+\sum_{i=1}^m\frac{\partial U}{\partial\alpha_i}(\alpha_i-\alpha_{i0})+\frac{1}{2}\sum_{i,j=1}^m\frac{\partial^2 U}{\partial\alpha_i\partial\alpha_j}(\alpha_i-\alpha_{i0})(\alpha_j-\alpha_{j0})+\dots,$$

is obtained,where $U_0=U_0(T,\alpha_0)$ and all expressions

$$\frac{\partial U}{\partial\alpha_i}=U_0\int_0^T U_0^{-1}(r)\frac{\partial A(r)}{\partial\alpha_i}U_0(r)dr,$$

$$\frac{\partial^2 U}{\partial\alpha_i\partial\alpha_j}=U_0\left[\int_0^T U_0^{-1}\frac{\partial^2 A}{\partial\alpha_i\partial\alpha_j}U_0dr+\int_0^T U_0^{-1}\frac{\partial A}{\partial\alpha_i}U_0dr\left(\int_0^T U_0^{-1}\frac{\partial A}{\partial\alpha_j}U_0ds\right)dr+\right.$$

$$\left.+\int_0^T U_0^{-1}\frac{\partial A}{\partial\alpha_j}U_0\left(\int_0^r U_0^{-1}\frac{\partial A}{\partial\alpha_i}U_0ds\right)dr\right],\quad i,j=\overline{1,m}$$

are written for $\alpha=\alpha_0$.

Eigenvalues (multiplicators) of monodromy matrix are obtained from the equation $\det[U(T)-\rho I]=0$ .Since

$$x(t+nT)=\rho x(t+(n-1)T)=\dots=\rho^n x(t)$$

and the situation $t\to\infty$ is equivalent to $n\to\infty$, then if $|\rho|=1$ the trivial solution $x\equiv 0$ is stable,if $|\rho|<1$ - asymptotic stable,and if $|\rho|>1$ -unstable.If $\rho=1$ solution is $T-$ periodc.

3.Parametric Resonance

Consider the damped Hill equation

$$\ddot{x} + 2\alpha\dot{x} + \left[\omega^2 + \mu\varphi(t)\right]x = 0 \qquad (5)$$

where α, ω *and* μ are constants,and $\varphi(t)$ is $T-$periodic function.We will suppose that the mean value $c = \frac{1}{T}\int_0^T \varphi(t)dt$ of the function $\varphi(t)$ is equal to zero.If it is different from zero,we will do it by using $\omega^2 = \omega_0^{\ 2} + \mu c$ and $\varphi(t) = \varphi_0(t) - c$.

Using $x = x_1$, $\dot{x} = x_2$ we reduce the Eq.(5) to the system of linear differental equations

$$\dot{x}_1 = x_2,\ \dot{x}_2 = -2\alpha x_2 - \left[\omega^2 + \mu\varphi(t)\right]x_1$$

Matrix of this system is represented as

$$A(t) = \begin{pmatrix} 0 & 1 \\ -\omega^2 - \mu\varphi & -2\alpha \end{pmatrix} = A_0 + \mu\varphi A_1,\quad A_0 = \begin{pmatrix} 0 & 1 \\ -\omega^2 & -2\alpha \end{pmatrix},$$

$$A_1 = \begin{pmatrix} 0 & 0 \\ -1 & 0 \end{pmatrix}$$

The solution $U_0(t)$ is obtained as [7]

$$U_0(t) = e^{A_0 t} = e^{-\alpha t}\begin{pmatrix} \cos\beta t + \frac{\alpha}{\beta}\sin\beta t & \frac{1}{\beta}\sin\beta t \\ -\frac{\omega^2}{\beta}\sin\beta t & \cos\beta t - \frac{\alpha}{\beta}\sin\beta t \end{pmatrix}$$

where $\beta = \sqrt{\omega^2 - \alpha^2}$ is the frequency of damped vibrations and α is a damped coefficient.

Let $U(t) = U_0(t)V(t)$,then the problem (3) is reduced to

$$\dot{V} = \mu\varphi U_0^{-1}A_1U_0V, V(0) = I \qquad (6)$$

Assuming μ suffisiently small, by the method of successive approximation we found the following solution of the problem (6)

$$V(t) = I + \mu\int_0^t \varphi U_0^{-1}A_1U_0 d\tau + \mu^2\int_0^t \varphi U_0^{-1}A_1U_0\int_0^\tau \varphi U_0^{-1}A_1U_0 ds d\tau + \ldots$$

Taking into account $U_0(t)$ and $V(t)$ the monodromy matrix

$$U(T) = U_0(T) + \mu U_0(T)\int_0^T \varphi U_0^{-1} A_1 U_0 d\tau + \mu^2 U_0(T)\int_0^T \varphi U_0^{-1} A_1 U_0 \int_0^\tau \varphi U_0^{-1} A_1 U_0 ds d\tau + ...$$

is obtained.The multiplicators of this matrix are obtained from the equation

$$\rho^2 - \rho Tr U(T) + \det U(T) = 0$$

Since

$$\det U(T) = \exp\left(\int_0^T Tr A d\tau\right) = \exp(-2\alpha T),$$

we use monodromy matrix to found its trace $TrU(T)$.For this aim we represent $U(T)$ in the form

$$U(T) = U_0(T)\left[I + \mu T\begin{pmatrix} a - \alpha b/\beta & -b/\beta \\ \dfrac{\alpha^2 - \beta^2}{\beta} b - 2a\alpha & -a + \alpha b/\beta \end{pmatrix} + \mu^2\begin{pmatrix} y_{11} & y_{12} \\ y_{21} & y_{22} \end{pmatrix} + ...\right]$$

where

$$a = \frac{1}{2\beta T}\int_0^T \varphi(\tau)\sin 2\beta\tau d\tau, b = \frac{1}{2\beta T}\int_0^T \varphi(\tau)\cos 2\beta\tau d\tau .$$

Since $\det U(T) = \det U_0(T)$,determinant of the matrix in the square brackets is equal to one. Using this, we found

$$y_{11} + y_{22} = T^2(a^2 + b^2).$$

Up to μ^2 the equation for multiplicators is written as

$$\rho^2 - \rho e^{-\alpha T}\left[2 + \mu^2 T^2\left(a^2 + b^2\right)\right]\cos\beta T + e^{-2\alpha T} = 0. \qquad (7)$$

For $\mu = 0$ we get

$$\rho_{1,2} = e^{-\alpha T}\left(\cos\beta_0 T \pm i\sin\beta_0 T\right)$$

where $\beta_0 = \sqrt{\omega_0^2 - \alpha^2}$. For the values $\beta_0 T \neq k\pi, k = 1,2,...$ multiplicators are complex-conjugate and $|\rho_1| = |\rho_2| = \exp(-\alpha T)$.For $\alpha > 0$ we see that $|\rho_i < 1|$,therefore the trivial solution of the Eq.(5) is asymtotic stable,if $\alpha = 0$ it is stable and if $\alpha < 0$ it is unstable.For the values

$\beta T = k\pi, k = 1,2,...$ multiplicators are real and one of them may be greather than one.Consider this cace.From (7) up to μ^2 we have found

$$\rho_{1,2} = \left[(-1)^k \pm T \sqrt{r_k^{\ 2} \mu^2 - \left(\beta_0 - \frac{k\pi}{T} + \mu c_k \right)^2} \right] e^{-\alpha T}$$

where $r_k^{\ 2} = a_k^{\ 2} + b_k^{\ 2}, \beta_0 = \sqrt{\omega_0^{\ 2} - \alpha^2}$,

$$a_k = \frac{1}{2k\pi} \int_0^T \varphi(r) \sin \frac{2k\pi r}{T} dr, \quad b_k = \frac{1}{2k\pi} \int_0^T \varphi(r) \cos \frac{2k\pi r}{T} dr, \quad c_k = \frac{1}{2k\pi} \int_0^T \varphi(r) dr$$

In order to be $|\rho| \geq 1$ the inequality

$$r_k^{\ 2} \mu^2 \geq \frac{1}{T^2} \left(e^{\alpha T} - 1 \right)^2 + \left(\beta_0 - \frac{k\pi}{T} + \mu c_k \right)^2 \tag{8}$$

must be satisfyes.

If T-periodic function $\varphi(t)$ in the domain $0 < t < T$ is given as Dirac delta $\varphi(t) = \delta\left(t - \frac{T}{2}\right)$, we found $a_k = 0, \quad b_k = \frac{(-1)^k}{2k\pi}, \quad c_k = r_k = \frac{1}{2k\pi}$.The inequality (8) becomes as

$$\mu^2 \geq \left(\frac{2k\pi}{T} \right)^2 \left[\left(e^{\alpha T} - 1 \right)^2 + \left(\beta_0 T - k\pi + \frac{\mu T}{2k\pi} \right)^2 \right]$$

As we see,in order that the parametric resonance take place, the amplitude of external actions must be greate than certain positive number.This limit increases whenever the number of resonance domain k increase.

Up to now we put only limitation on α,that is $\alpha^2 < \omega_0^{\ 2}$.As we know [7], for the cases $\alpha^2 \geq \omega_0^{\ 2}$ the vibrations is not take place. If $\alpha << 1$,then the condition (8) may be written as

$$\alpha^2 + \left(\beta_0 - \frac{k\pi}{T} + \mu c_k \right)^2 \leq \mu^2 r_k^{\ 2} \tag{9}$$

In the space of parameters (α, ω_0, μ) the condition (9) is defined the half conical domains with the axis $\omega_0 = \frac{k\pi}{T} + \mu c_k$ and the generators

$\omega_0 = \frac{k\pi}{T} + \mu c_k \pm \mu r_k$ on the plane $\alpha = 0$. Since r_k decreases for increasing k ,the angle between this lines is narrow down.

The intersections of conies (9) with the plane $\alpha = const$ are the following hyperbolas

$$\frac{r_k^2}{\alpha^2}\mu^2 - \left(\frac{\omega_0 - \frac{k\pi}{T} + \mu c_k}{\alpha} \right)^2 \geq 1$$

Here we found the lower bound $\mu \geq \frac{\alpha}{r_k}$ for the amplitude of external perturbation.For increasing k the fraction $\frac{\alpha}{r_k}$ also increases,therefore the corresponding domains of instability be far away from the ω axis.For the case $\alpha = 0$ this domains are intersects of the ω_0 axis [1]. If $c_k = 0$, from (9) it follows that

$$-\mu r_k < \omega_0 - \frac{k\pi}{T} < r\gamma_k, \quad \alpha = 0.$$

As we see, the resonance domains begin from the points $\omega_0 = \frac{k\pi}{T}$ .The numbers μc_k are differences of the resonance frequences and the numbers $k\pi/T$ .For decreasing k the numbers c_k *and* r_k are increasing ,therefore the axises of resonance conies becomes perpendicular to the ϖ_0 axis.

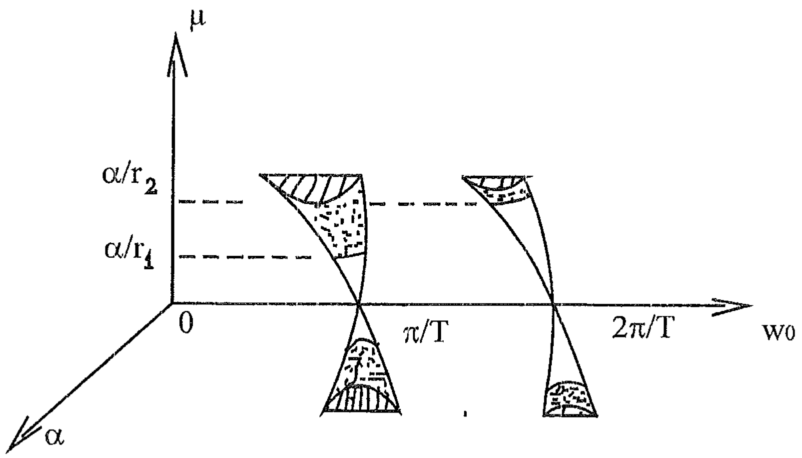

Fig.The domains of instablity.

The intesection of the domains (9) with the plane $\mu = const$ are the ellipses with the half-axises $\left|\omega_0 - \frac{k\pi}{T} + \mu c_k\right| = \mu r_k$ *and* $\alpha = \mu r_k$,and the center is on the $\omega_0 = \frac{k\pi}{T} - \mu c_k$ axis.For increasing α the instablity domains on ω_0 decreases and vanishes for $\alpha > r_k \mu$.

For the function $\varphi(t) = \cos\theta t$ we obtain $a_1 = c_1 = 0, b_1 = r_1 = \frac{1}{2\theta}$ and $\theta = \frac{2\omega_0}{k}$. Since $\theta = 2\omega_0$ for $k = 1$, that is the parametric resonance take place when the frequence θ of the external action is equal to the double eigenvalues of the system.From (9) it follows that $\mu^2 \geq \frac{\alpha^2}{r_1^2} = 4\theta^2\alpha^2 = 16\omega^2\alpha^2$.

Generally,the resonance domain in the space(α, ω_0, μ) consist of the union of half space $\alpha < 0$ with the half horn domains on $\alpha \geq 0$ satisfies the conditions $\mu^2 \geq \frac{\alpha^2}{r_k^2}$.

4.Kelvin – viscoelastic column

The equation,governing the lateral displacement of a column is given by [3]

$$EI\frac{\partial^4 u}{\partial x^4}+\vartheta I\frac{\partial^5 u}{\partial x^4 \partial t}+[P_0+P(t)]\frac{\partial^2 u}{\partial x^2}+\rho\frac{\partial^2 u}{\partial t^2}=0$$

where $u(x,t)$ is lateral displacement, I is the second moment of area, ρ is the mass per unit length of the column, P_0 is the static load, $P(t)$ is the time varying load and E and η are the elastic modulus of the spring and the damping coefficient of the damper in the Kelvin . viscoelastic model,respectively.

Assuming that the normal modes of vibration of the column are given by $\phi_n(x)$,the general series solution is given by

$$U(x,t)=\sum_n A_n(t)\phi_n(x)$$

For a simply supported column $\phi_n=\sin\frac{n\pi x}{\ell}$ and for a fixed-fixed column, ϕ_n is approximately given by $\phi_n=1-\cos\frac{n\pi x}{\ell}$, in which l is the length of the column.

Employing the series solution in the equation of motion yields

$$A_n''+2\xi_n\Omega_n A_n'+\Omega_n^{\,2}[1-f_n(t)]A_n=0 \tag{10}$$

in which

$$\Omega_n=\varpi_n\left(1-\frac{P_0}{P_n}\right)^{\frac{1}{2}} \quad are \quad \xi_n=\eta_n\left(1-\frac{P_0}{P_n}\right)^{\frac{1}{2}}$$

are the natural frequency and damping coefficient,respectively,of the corresponding column subjected to an axial dead load P_0 whwre $P_n=EI\lambda_n^{\,2}l^{-2}$ is the nth Euler bucking load,

$$f_n(t)=\frac{P(t)}{P_n-P_0},\quad \varpi_n=\frac{\lambda_n^{\,2}}{\ell^2}\left(\frac{EI}{\rho}\right)^{\frac{1}{2}},\quad \eta_n=\frac{\lambda_n^{\,2}}{2\ell^2}\left(\frac{I}{\rho E}\right)^{\frac{1}{2}}\eta,$$

and $\lambda_n = n\pi$ for simply supported and $\lambda_n = 2n\pi$ for a fixed-fixed column.Comparison with the Eq. (5) gives

$$\alpha = \xi_n\Omega_n, \quad \omega = \Omega_n, \quad \mu\varphi(t) = -\Omega_n^{\,2} f_n(t)$$

The resonance domain

$$\xi_n^{\,2}\Omega_n^{\,2} + \left[\Omega_n\sqrt{1-\xi_n^{\,2}} - \frac{k\pi}{T} - \frac{\Omega_n^{\,2}}{2k\pi}\int_0^T f_n(r)dr\right]^2 \leq$$

$$\leq \left(\frac{\Omega_n^{\,2}}{2k\pi}\right)^2 \left[\left(\int_0^T f_n(r)\sin 2\beta r dr\right)^2 + \left(\int_0^T f_n(r)\cos 2\beta r dr\right)^2\right]$$

is obtained from (9),where $\beta = \Omega_n\sqrt{1-\xi_n^{\,2}}$ .For the function $P(t) = q(P_n - P_0)\cos\theta t$ in the case $k = 1, \theta = 2\Omega_n$ *and* $\xi_n << 1$, the condition $q^2 \geq 16\xi_n^{\,2}$ is obtained.The right hand side of this inequality is proportional to η, ω_n and E^{-1}.

In [3] the unstable domains for damping vibrations are unfortunately not obtained.

5.Viscoelastic plates, shells and columns

Many processes in mechanics, physics, biology, economy, ecology, automatic regulation etc. can be modelled by hereditary equations. By means of various methods (separation of variables,Bubnov-Galerkin method, etc.) the problems of dynamic stability of viscoelastic plates,shells and columns are reduced to investigation of stability of trivial solution of following equation

$$y''(t) + \left[\lambda^2 + \mu\varphi(t)\right]y(t) = \varepsilon\lambda^2\int_0^t \Gamma(t-r)y(r)dr \qquad (11)$$

and the initial conditions $y(0) = 0,\ y'(0) = 0$ [4,6], where λ is the frequence of elastic vibrations, $\mu\varphi(t)$ is the time varying load and $\varepsilon\Gamma(t)$ is the relaxation kernel.Investigation of this problem for any $\Gamma(t)$ is practically very important. In [6] this problem has been solved by using the average method. For the kernel $\varepsilon\Gamma(t) = -\frac{\eta}{E}\dot{\delta}(t)$, where $\dot{\delta}(t)$ is Dirac delta, Eq.(10) is obtained from (11).

The Laplace transform of the solution of Eq.(11) is calculated as

$$\bar{y} = \frac{-\mu\left(\overline{\varphi y}\right)}{s^2 + \lambda^2 - \varepsilon\lambda^2\bar{\Gamma}} \tag{12}$$

where the line over the function denotes its Laplace transformation with the complex parameter s. In order to calculate the invers transformation of (12) it is necessary to know the poles of this function which are the roots of the equation

$$s^2 + \lambda^2 - \varepsilon\lambda^2\bar{\Gamma} = 0 \tag{13}$$

This equation has two complex-conjugated roots $-\alpha \pm i\beta$, which satisfy the system of equations

$$\alpha^2 + \lambda^2 - \beta^2 = \varepsilon\lambda^2\int_0^\infty e^{\alpha r}\Gamma(r)\cos\beta r dr, \quad 2\alpha\beta = \varepsilon\lambda^2\int_0^\infty e^{\alpha r}\Gamma(r)\sin\beta r dr \tag{14}$$

For sufficiently small ε we found the following solutions of this system by iterations

$$\alpha = \frac{\varepsilon\lambda\Gamma_s}{2} + \varepsilon^2\lambda\varpi_1 + \varepsilon^3\theta_1 + \cdots, \quad \beta = \lambda - \gamma, \quad \gamma = \frac{\varepsilon\lambda\Gamma_c}{2} + \varepsilon^2\lambda\varpi_2 + \varepsilon^3\theta_2 + \cdots, \tag{15}$$

where

$$\varpi_1 = \frac{1}{4}\left(\Gamma_s\Gamma_c - \lambda\Gamma_c\Gamma_{1c} + \lambda\Gamma_s\Gamma_{1s}\right), \quad \varpi_2 = \frac{1}{8}\left(\Gamma_c^{\,2} - \Gamma_s^{\,2} + 2\lambda\Gamma_s\Gamma_{1c} + 2\lambda\Gamma_c\Gamma_{1s}\right),$$

$$\theta_1 = \frac{\lambda}{2}\left[\Gamma_s\varpi_2 + \Gamma_c\varpi_1 - \lambda\varpi_2\Gamma_{1c} + \lambda\varpi_1\Gamma_{1s} - \frac{\lambda^2}{4}\Gamma_s\Gamma_c\Gamma_{2c} + \frac{\lambda^2}{8}\Gamma_{2s}\left(\Gamma_s^{\,2} - \Gamma_c^{\,2}\right)\right],$$

$$\theta_2 = \frac{\lambda}{2}\left[\Gamma_c\varpi_2 - \Gamma_s\varpi_1 + \lambda\varpi_1\Gamma_{1c} + \lambda\varpi_2\Gamma_{1s} + \frac{\lambda^2}{4}\Gamma_s\Gamma_c\Gamma_{2s} + \frac{\lambda^2}{8}\Gamma_{2s}\left(\Gamma_s^{\,2} - \Gamma_c^{\,2}\right)\right],$$

$$\Gamma_{kc} = \int_0^\infty t^k\Gamma(t)\cos\lambda t dt, \quad \Gamma_{ks} = \int_0^\infty t^k\Gamma(t)\sin\lambda t dt, \qquad k = 0, 1, 2$$

and $\Gamma_{0c}=\Gamma_c and \quad \Gamma_{0s}=\Gamma_s$ denotes the cos- and sin-Fourier transformations of $\Gamma(t)$, respectively.

Let us rewrite (12) in the form

$$\bar{y}\left[(s+\alpha)^2+\beta^2\right]+\frac{\mu\left(\overline{\varphi y}\right)}{1-\bar{B}(s)}=0 \tag{16}$$

where

$$\bar{B}(s)=\frac{\varepsilon\lambda^2\bar{\Gamma}+2\alpha s+\alpha^2+\beta^2-\lambda^2}{(s+\alpha)^2+\beta^2}.$$

Using (14) we found the following inverse transformation of this function

$$B(t)=\frac{\varepsilon\lambda^2}{\beta}e^{-\alpha t}\int_t^{\infty}\Gamma(r)e^{\alpha r}\sin\beta(r-t)dr$$

From (14) it follows that $B(0)=2\alpha$.As we see the function $B(t)$ may be written as $B(t)=e^{-\alpha t}L^{-1}\tilde{B}(s)$,where

$$\tilde{B}(s)=\frac{\varepsilon\lambda^2\bar{\Gamma}(s-\alpha)+2\alpha s+\beta^2+\alpha^2-\lambda^2}{s^2+\beta^2}$$

is a slowly changing function of s.Using Shapery method

$$\tilde{B}(t)=\left(s\tilde{B}(s)\right)\Big|_{s=\frac{1}{2t}}$$

we found $B(t)\approx\frac{c}{t}e^{-\alpha t}, \quad c=const$. As we see the function $B(t)$ as well as $t^{-1}e^{-\alpha t}$ tends to zero for $t\to\infty$.Since the dynamic stability take place for large values of t,we may neglect the function B(t).Then from (16) we have found the following equation

$$y''+2\alpha y'+\left[\alpha^2+\beta^2+\mu\varphi(t)\right]y=0 \tag{17}$$

The problem of dynamic stability analysis of viscoelastic material is now reduced to the determination of the stability criteria for the nul solutions of Eq.(17).The advantage of this equation is that it contain the real properties of viscoelastic materials and similarly to the Eq.(5) which we may solved. Depending on the material parameters α,β and μ, and nature of loading the criterion for stability varies.If we put $\omega^2=\alpha^2+\beta^2$ in (17) the Eq.(5) is obtained.Therefore the condition (9) is written as

$$\alpha^2 + \left(\beta - \frac{k\pi}{T} + \mu c_k\right)^2 < r_k{}^2 \mu^2 \qquad (18)$$

Here the parameter β has defined in (15).

For the function $\varphi(t) = \cos\theta t$ and $k = 1$ we found $T = \frac{2\pi}{\theta}$ *ve* $\lambda = \frac{\theta}{2}$. Using the values obtained in the second section,from (17) we found

$$\mu^2 > 4\theta^2(\alpha^2 + \gamma^2) \qquad (19)$$

If we take into account only the linear terms with respect to ε in (15) we will get the result

$$\mu^2 > (\varepsilon\lambda\theta)^2(\Gamma_s{}^2 + \Gamma_c{}^2),$$

which is obtained in[6] by Bogolyubov's averaging method.

In addition to the results obtained in the third section we may noted that if$\theta = 2\lambda$ and

a) if$\mu^2 > 4\theta^2(\alpha^2 + \gamma^2)$ the vibrations are unstable;there is parametric resonance;
b) if $\mu^2 = 4\theta^2(\alpha^2 + \gamma^2)$ the motion is stable;the solutions are tends to periodic;
c) if $\mu^2 < 4\theta^2(\alpha^2 + \gamma^2)$ the motion of vibration is asymptotic stable.

6.Triangle plate

It is well-known that if the plate is loaded by periodic forces in its plane, instability occurs for some frequency and amplitude of the applied force.Using similar approaches to the ones for free vibration, the dynamic stability of equilataral triangle viscoelastic plate under the action $N_0 + N_1\cos\theta t$ in its plane is reduced to the equation

$$A'' + 2\alpha A' + S_1{}^2(1 - 2\mu_1\cos\theta t)A = 0$$

in [4],where

$$\alpha = \frac{\varepsilon\lambda\Gamma_s}{2S},\quad S_1{}^2 = \lambda^2 S^2\left(1 - \frac{\varepsilon\Gamma_c}{\lambda^2 S^2}\right),\quad \mu_1 = \frac{\mu}{1 - \frac{\varepsilon\Gamma_c}{\lambda^2 S^2}},\quad S^2 = 1 - \frac{N_0}{N_*},\quad \mu = \frac{N_1}{2(N_* - N_0)}$$

$$N_* = \frac{4\pi^2}{3H^2} D(m^2+n^2+mn), \quad \lambda^2 = \frac{4\pi^2}{3H^2}\frac{D}{\rho}(m^2+n^2+mn)^2, \quad m,n = 1,2,\cdots; D = \frac{Eh^3}{12(1-\gamma^2)},$$

and H is the height of the triangle and h is the thickness of plate. Comparison with the Eq.(5) gives

$$\omega = S_1, \quad \mu = 2\mu_1 S^2, \quad \varphi(t) = -\cos\theta t\,.$$

Using the result of (9) for $\varphi(t) = \cos\theta t$ we found

$$N_1 > 2\varepsilon\lambda\Gamma_s\left(1-\frac{\varepsilon\Gamma_s}{\lambda^2 S}\right)N_*\left(1-\frac{N_0}{N_*}\right)^{\frac{1}{2}}$$

So that the parametric resonance take place if the amplitude of the time varying part of external actions greater than the right hand side of obtained inequality.This condition is not obtained in [4].The inspection of parametric vibrations of plate can be accomplished by the solution of Eq.(17) under the same initial conditions used before.The condition for the parametric resonance may be obtained from $\mu^2 > 4\theta^2(\alpha^2+\beta^2)$ for appropriate notations.The results obtained in this section and in [4] are the special cases of mentioned inequality.

For the Kelvin material we have

$$\varepsilon\Gamma_c = 0, \quad \varepsilon\Gamma_s = \frac{\mu\lambda}{E}, \quad \alpha = \frac{\mu\lambda^2}{2E}\left(1-\frac{N_0}{N_*}\right)^{-\frac{1}{2}},$$

therefore the last inequality takes the form

$$N_1 > \frac{2\lambda^2\mu N_*}{E}\left(1-\frac{N_0}{N_*}\right)^{\frac{1}{2}}.$$

REFERENCES

[1] Yu.V.Bolotin, Dynamic Stability of Elastic Systems. Holden Day Inc.,San Francisco, 1964.

[2] N.W.Mclachlan, Theory and Aplication of Mathieu Functions, Oxford, 1947.

[3] G.Ahmadi, P.G.Glockner, Dynamic Stability of Kelvin-Viscoelastic Column, J.Engin. Mech. 109(4)(1983), 990-998.

[4] M.H.Ilyasov, A.Y.Aköz,The Vibration and Dynamic Stability of Viscoelastic Plates, Int. J. Engin.Science, 38(2000),695-714.

[5] A.P.Seyranyan, Resonance Domains for Damped Hill Equation, Dokl.Akad.Nauk,376(2001), No1, 44-47 (in Russian).

[6] V.I.Matyash,Dynamic Stability of Freely Fixed Rod, Mechanics of Polymers, No2, 1971 (in Russian).

[7] D.K.Arrowsmith, C.M.Place, Ordinary Differential Equations.A qualitative approach with appications. L.,N.-Y., 1982.

[8] C.Shuping, L.Kangsheng, L.Zhuangyi, Spectrum and Stability for Elastic Systems with global or local Kelvin-Voigt Damping. SIAM J. Appl. Math. Vol.59 (1998), No.2, 651-668.

[9] A.O.Ignatyev,Stability of a Linear Oscillator with Variable Parameters. EJDE, No. 17 (1997),!-6.

Baku State University,Faculty of Mechanics and Mathematics,
Baku, AZERBAİJAN.