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MIXED PROBLEMS FOR LINEAR AND QUASILINEAR PSEUDOPARABOLIC EQUATIONS*

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Abstract

In this article, the integral operator method has been used for the solution of mixed problems for an inhomogeneous pseudoparabolic equations.The obtained solutions are constructed on the basis of the solutions of corresponding problems for the parabolic equations and are represented by the integral of the multiplication of these solutions by a generalized functions, which are the generalized solutions of spesific differential equation of hyperbolic type.This functions are determining the hereditary properties of the medium which fills the region where the physical event is examined shows the effect of this property to the solutions we are trying to find. The formulas which were obtained are considered as the consistency principles of the solutions of the problems which were put forward for parabolic and pseudoparabolic equations. Corresponding problems for pseudoparabolic quasilinear equation with linear principal part are investigated on the basis of having constructed solutions for linear problems.

Keywords: Pseudoparabolic equation, mixed problem, deneralized function, fundamental solution, quasilinear equation.

AMS subject classification: 35K15; 35K20; 35Q35.

I.Introduction

The theory of heat conduction in medium with memory based on the postulates of balance of energy and the entropy production inequality has already been proposed and studied in many works [1,2]. Various models and the generalized cases of this theory to R^n are investigated in [3-7] (see also the referance of this work).

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The purpose of this work is to investigate the solutions of mixed problems for the pseudoparabolic equation

$$Lu + \nu Lu_t + f(x,t) = u_t,\ x = (x_1, \cdots, x_n) \qquad (1.1)$$

where ν is a constant, L is an elliptic operator while the coefficients are independent of t. This equation describes the heat conduction in medium with memory corresponding to the Kelvin's model [1]. In [8] Kelvin's model used for the problem of sound propagation in a viscous gas (see also [9]).

In [10] the authors solved a generalized mixed boundary value problem for the pseudoparabolic equation of the form $Mu_t + Lu = f$, where M and L are second order differential operators in the space variable and M is elliptic. They proved existance, uniqueness and regulariti of the solution and discussed its asymptotic behavior. Some theoretical problems of these and other partial differential equations are investigated in the monograph [11], which we refer for discussion and bibliography of work concerning problems of this type. We represent the solution of considered problem by the integral

$$u(x,t) = \int_0^{\infty} \upsilon(x,\tau) w(t,\tau) d\tau \qquad (1.2)$$

where $\upsilon(x,\tau)$ is the solution of corresponding problem for the parabolic equation arised from (1.1) when $\nu = 0$ and $w(t,\tau)$ is the generalized solution of an auxiliary problem for a linear differential equation of hyperbolic type.

We call the relation (1.2) the correspondence principle of the solutions of mixed problems for the equations of pseudoparabolic and parabolic type. The solutions of the parabolic equations which were obtained by exact, approximate of numerical methods don't cause any difficulty while one transfers them to the proper solutions of pseudoparabolic equations. The function $w(t,\tau)$ depends neither on the form of medium (domain) nor on the initial and boundary conditions. It does not also depend on the medium being homogeneous or inhomogeneous (on the coefficients of equation), but depends only on the hereditary property of medium, and characterizes the influence of this property on solutions. It will be called the kernel of influence.

The homogeneous boundary value problem, inhomogeneous pseudoparabolic equation and inhomogeneous boundary conditions will be investigated separately. Behaviour of solution when $\nu \to 0$, and for large and small values of time will be investigated, too. Solutions of corresponding problems for a quasilinear pseudoparabolic equation

$$Lu + \nu Lu_t + F(x,t,u,u_x) = u_t, \; u_x = (u_{x_1}, \cdots, u_{x_n}) \tag{1.3}$$

with linear boundary conditions will be found using constructed solutions for linear problems.

As an example the mixed problem for equation (1.3) has been considered when L is a self-adjoint operator of the second order, on the bounded domain. In the special case of $L = \Delta$ (Laplace operator) Cauchy problem is considered as well as the mixed problem with Dirichlet and Neumann boundary conditions for a half space and infinite strip. In [12] considering method has been used for solving some nonstationary dynamic problems for viscoelastic materials.

II. Statement of the Problem

Let $\Omega \subseteq R^n$ be a nonempty open bounded or unbounded (half-space, infinite strip, etc.) domain with the boundary S, which we will assume to be a piecewise smooth surface. In a cylinder $Q_T = \Omega \times (0,T)$, let us consider a mixed problem for the linear pseudoparabolic equation

$$Lu + \nu Lu_t + f(x,t) = u_t \quad (x,t) \in Q_T, \tag{2.1}$$

$$u|_{t=+0} = 0 \quad x \in \Omega, \tag{2.2}$$

$$Bu = 0 \quad x \in S, \; t \in (0,T) \tag{2.3}$$

where $L = L(x, \partial | \partial x)$ is an elliptic operator, ν is a positive constant, $B = B(x, \partial | \partial x)$ is a linear boundary operator and T is a positive finite number or $+\infty$.

We say that $u = u(x,t)$ is a solution of the problem (2.1)-(2.3) if all the derivatives of u which occur in (2.1) are continuous functions in Q_T, all the derivatives of u which occur in (2.3) are continuous on $\overline{\Omega} \times (0,T)$, $\overline{\Omega} = \Omega \cup S$, and satisfies (2.1) at each point $(x,t) \in Q_T$, satisfies (2.2) at each point $x \in \Omega$, and satisfies (2.3) at each point $(x,t) \in S \times (0,T)$.

We will suppose that data (the function fand the coefficients of the operators L and B) of the problem satisfy the necessary smoothness conditions for existence of a unique classical (or generalized) solution of corresponding problem for parabolic equation, which arises from (2.1)-(2.3) when [13].Under the same conditions we will construct a unique classical (or generalized) solution of corresponding problem for parabolic equation (2.1)-(2.3)

III.Solution of the Problem

Let us construct the solution of the problem (2.1)-(2.3). We call the solution $G^{\nu}(x,\xi,t)$ of the problem

$$G^{\nu}_{t} - LG^{\nu} - \nu LG^{\nu}_{t} = \delta(t)\delta(x-\xi), \quad (x,t) \in Q_{T}, \qquad (3.1)$$

$$G^{\nu}(x,\xi,+0) = 0, \quad x \in \Omega, \qquad (3.2)$$

$$BG^{\nu}|_{S} = 0, \qquad (3.3)$$

the Green function of the problem (2.1)-(2.3).

Let us suppose that the solution $G(x,\xi,\tau)$ of the following problem for parabolic quation

$$LG = G_{\tau}, \quad x \in \Omega, \tau > 0, \qquad (3.4)$$

$$G(x,\xi,+0) = \delta(x-\xi), \quad x \in \Omega, \qquad (3.5)$$

$$BG|_{S} = 0, \qquad (3.6)$$

is known.

Now let us consider a solution $\omega\ (t,\tau)$ of the following problem

$$\omega_{\tau} + \omega_{t} + \nu\omega_{t\tau} = \delta(t)\delta(\tau), \qquad (3.7)$$

$$\omega(0,\tau) = 0; \omega\ (t,0) = 0, \omega_{t}(t,0) = 0, \omega \to 0, \tau \to \infty. \qquad (3.8)$$

Theorem 3.1. *The generalized solution of the problem (3.1)-(3.3) is*

$$G^{\nu}(x,\xi,t) = \int_{0}^{\infty} G(x,\xi,\tau)\omega\ (t,\tau)d\tau \qquad (3.9)$$

where $G(x,\xi,\tau)$ and $\omega\ (t,\tau)$ *are the generalized solutions of the problems (3.4)-(3.6) and (3.7),(3.8) respectively.*

Proof: Putting (3.9) into (4.1), and assuming the possibility of differentiation with respect to coordinates and t under the integral sign, we obtain

$$\int_0^\infty G\omega_t d\tau - \int_0^\infty \omega\, LGd\tau - \nu \int_0^\infty \omega_t LGd\tau = \delta(x-\xi)\delta(t).$$

Using the equation (3.4) and the conditions (3.8), after integrating by parts we have

$$\int_0^\infty G(\omega_t + \omega_\tau + \nu\omega_{t\tau})d\tau = \delta(x-\xi)\delta(t).$$

Taking into account (3.5) and (3.7), this equation is satisfied identically. The initial condition (3.2) is satisfied by virtue of the condition $\omega\ (0,\tau) = 0$, and the boundary condition (3.3) follows from (3.6).

The theorem is proved.

Let us construct the solution of problem (3.7),(3.8). The Laplace transformations with respect to t and τ gives

$$\hat{\omega}\ (p,\lambda) = \frac{1}{p+\lambda+\nu p\lambda}.$$

After inverse transformations we get

$$\omega\ (t,\tau) = \frac{1}{\nu} H(t)H(\tau)e^{-(t+\tau)/\nu} I_0 (\frac{2}{\nu}\sqrt{t\tau}) \tag{3.10}$$

Theorem 3.2. *The function $\omega\ (t,\tau)$, represented by the formula (3.10), is a generalized solution of the problem (3.7),(3.8). This function is nonnegative, vanishes when $t \le 0$ and $\tau \le 0$, and when $t \to \infty, \tau \to \infty$, is infinitely differentiable at $(t,\tau) \ne (0,0)$, and is locally integrable in R^2. Besides,*

$$\int_0^\infty \omega\ (t,\tau)d\tau = H(t). \tag{3.11}$$

Proof: The function (3.10) satisfies the equation (3.7) because of the following expressions

$$\omega_t = \frac{1}{\nu}\delta(t)H(\tau)e^{-\tau/\nu} - \frac{1}{\nu^2}H(t)H(\tau)e^{-(t+\tau)/\nu}[I_0 - \sqrt{\frac{\tau}{t}}I_1],$$

$$\omega_\tau = \frac{1}{\nu}\delta(\tau)H(t)e^{-t/\nu} - \frac{1}{\nu^2}H(t)H(\tau)e^{-(t+\tau)/\nu}[I_0 - \sqrt{\frac{t}{\tau}}I_1],$$

$$\nu\omega_{t\tau} = \delta(\tau)\delta(t) - \frac{1}{\nu}\delta(\tau)H(t)e^{-t/\nu} - \frac{1}{\nu}\delta(t)H(\tau)e^{-\tau/\nu}$$

$$+\frac{1}{\nu^2}H(t)H(\tau)e^{-(t+\tau)/\nu}[2I_0 - \sqrt{\frac{t}{t}}I_1 - \sqrt{\frac{t}{t}}I_1].$$

The validity of the conditions (3.8) is clear. The infinite differentiability of (3.10) in the region $(t,\tau) \neq (0,0)$ comes from the infinite differentiability of the functions, included in (3.10). So, we have

$$\int_0^\infty \omega(t,\tau)d\tau = \frac{H(t)}{\nu}e^{-t/\nu}\int_{+0}^\infty e^{-\tau/\nu}I_0(\frac{2}{\nu}\sqrt{t}\sqrt{\tau})d\tau = H(t).$$

The proof of the theorem is complete.

The solution of the problem (2.1)-(2.3) is represented by the convolution of functions $G^\nu(x,\xi,t)$ and $f(x,t)$

$$u(x,t) = \int_\Omega\int_0^t G^\nu(x,\xi,t-\eta)f(\xi,\eta)d\xi d\eta$$

$$= \frac{1}{\nu}\int_\Omega\int_0^t\int_{+0}^\infty G(x,\xi,\tau)e^{-(t+\tau-\eta)/\nu}I_0(\frac{2}{\nu}\sqrt{\tau(t-\eta)})f(\xi,\eta)d\tau d\eta d\xi. \qquad (3.12)$$

Here $d\xi = d\xi_1 \cdots d\xi_n$.

Theorem 3.3. *If $f(x,t) \in C(\overline{Q}_T)$ and $f = 0$ for $t < 0$, then the function(3.12) is a solution of the problem (2.1)-(2.3).*

Proof: From (3.12) we get

$$u_t = \frac{1}{\nu}\int\limits_{\Omega}\int\limits_{+0}^{\infty} G(x,\xi,\tau)e^{-\tau/\nu}f(\xi,t)d\tau d\xi$$

$$-\frac{1}{\nu^2}\int\limits_{\Omega}\int\limits_{+0}^{\infty}\int\limits_{0}^{t} G(x,\xi,\tau)e^{-(t+\tau-\eta)/\nu}f(\xi,\eta)[I_0 - \sqrt{\frac{\tau}{t-\tau}}I_1]d\eta d\tau d\xi, \quad (3.13)$$

$$Lu = \frac{1}{\nu}\int\limits_{\Omega}\int\limits_{+0}^{\infty}\int\limits_{0}^{t} LGe^{-(t+\tau-\eta)/\nu}f(\xi,\eta)I_0 d\eta d\tau d\xi.$$

Here the argument $\frac{2}{\nu}\sqrt{\tau(t-\eta)}$ of Bessel functions I_0 *and* I_1 has not been written for the sake of the conciseness. Using (3.4) in the last equality we find

$$Lu = -\frac{1}{\nu}\int\limits_{0}^{t} e^{-(t-\eta)/\nu}f(x,\eta)d\eta$$

$$+\frac{1}{\nu^2}\int\limits_{\Omega}\int\limits_{0}^{t}\int\limits_{+0}^{\infty} Gfe^{-(t+\tau-\eta)/\nu}[I_0 - \sqrt{\frac{t-\eta}{\tau}}I_1]d\tau d\eta d\xi, \quad (3.14)$$

which is obtained by integrating by parts.
By the same way from (3.13) we find

$$\nu Lu_t = -f(x,t) + \frac{1}{\nu}\int\limits_{0}^{t} e^{-(t-\eta)/\nu}f(x,\eta)d\eta + \frac{1}{\nu}\int\limits_{\Omega}\int\limits_{+0}^{\infty} Gf(\xi,t)e^{-\tau/\nu}d\tau d\xi$$

$$-\frac{1}{\nu^2}\int\limits_{\Omega}\int\limits_{0}^{t}\int\limits_{+0}^{\infty} Gfe^{-(t+\tau-\eta)/\nu}[2I_0 - \sqrt{\frac{\tau}{t-\eta}}I_1 - \sqrt{\frac{t-\eta}{\tau}}I_1]d\tau d\eta d\xi. \quad (3.15)$$

Substituting (3.13), (3.14) and (3.15) in (2.1) we show that the function (3.12) satisfies the equation (2.1).

The validity of the initial condition (2.2) is clear, and the boundary condition (2.3) is satisfied by virtue of (3.6).

It is easily shown that the functions under the integral sign in (3.12)-(3.15) are continuous on the corresponding domains. Moreover, using the boundedness of the integral (see [15])

$$\int\limits_{\Omega} G(x,\xi,\tau)d\xi,$$

and the value of integrals

$$\frac{1}{\nu}\int_0^\infty I_0\left(\frac{2}{\nu}\sqrt{t\tau}\right)e^{-(t+\tau)/\nu}d\tau=1,$$

$$\frac{1}{\nu}\int_0^\infty\sqrt{\frac{t}{\tau}}I_1\left(\frac{2}{\nu}\sqrt{t\tau}\right)e^{-(t+\tau)/\nu}d\tau=1-e^{-t/\tau},\quad t>0,$$

$$\int_0^\infty e^{-(t+\tau-\eta)/\nu}\sqrt{\frac{\tau}{t-\eta}}I_1\left(\frac{2}{\nu}\sqrt{\frac{\tau}{(t-\eta)}}\right)d\tau=\nu, \tag{3.16}$$

it is easily proved that the integrals included in the formulas (3.12)-(3.15) are uniformly convergent on Q_T. The theorem is proved.

Carrying out the same procedure for the function $\omega(t,\tau)$, we get the following asimptotic expressions

$$\omega\,(t,\tau)\sim\ 2(t\tau)^{-\frac{1}{4}}H(t)H(\tau)\delta(\sqrt{\tau}-\sqrt{t}) \tag{3.17}$$

for $\nu\to 0$, and

$$\omega\ (t,\tau)\sim\ \frac{H(\tau)}{2\tau}\left(\frac{\tau}{t}\right)^{3/4}\delta\left(1-\sqrt{\frac{\tau}{t}}\right) \tag{3.18}$$

for t>>1

Using (3.18), we find

$$\int_0^t\omega\ (t-\eta,\tau)d\eta=1\quad for\quad \tau<t\text{, and }=0\text{ for }\tau\geq t\ , \tag{3.19}$$

For t<<1 we get

$$\int_0^t\omega\ (t,\tau)d\tau=e^{-\tau/\nu}(1-e^{-t/\nu}). \tag{3.20}$$

Using (3.17) and (3.18) in (3.9), in both cases we get $G^\nu\sim G$, i.e. in the first case the solution of the mixed problem for pseudoparabolic equation changes into the solution of corresponding problem for parabolic equation. The second result shows that for sufficiently large values of time the effect of hereditary property of medium vanishes.

IV. Quasilinear Equation

Let us consider the problem

$$Lu + \nu L u_t + F(x,t,u,u_x) = u_t \quad (x,t) \in Q_T, \qquad (4.1)$$
$$u|_{t=+0} = 0 \qquad x \in \Omega, \qquad (4.2)$$
$$Bu = 0 \quad x \in S, t \in (0,T), \qquad (4.3)$$

where $u_x = (u_{x_1}, \cdots, u_{x_n})$.

We shall assume the following condition on the function F:

1^o. The function F is defined and continuous on the set

$$\{(x,t,u,p) \mid (x,t) \in \overline{Q}_T, -\infty < u < \infty, -\infty < p_i < \infty, i = \overline{1,n}, p = (p_1, \cdots, p_n)\};$$

2^o. For each $c > 0$ and for $|u|, |p_i| < c$ $i = \overline{1,n}$, the function F is uniformly H"lder continuous with respect to x and t on each compact subset of Q_T;

3^o. There exists a constant C_F such that

$$|F(x,t,u_1,p_1^1,\cdots,p_n^1) - F(x,t,u_2,p_1^2,\cdots,p_n^2)|$$
$$\leq C_F[|u_1 - u_2| + \sum_{i=1}^{n} |p_i^1 - p_i^2|] \qquad (4.4)$$

holds for all $(u_i, p_1^i, \cdots, p_n^i), i = 1,2;$

4^o. For unbounded Q_T, we add the assumption that F is bounded for bounded u and p.

Theorem 4.1. *Let there exists unique solution $G(x,\xi,\tau)$ of the linear problem (3.4)-(3.6), for the parabolic equation, and the conditions $1^o - 4^0$ holds for the function F. Then, there exists a unique bounded solution $u = u(x,t)$ of the problem (4.1)-(4.3) that is the unique solution with bounded continuous derivative u_x of the integrodifferential equation*

$$u(x,t)=\frac{1}{\nu}\int\limits_{+0}^{\infty}\int\limits_{\Omega}\int\limits_{0}^{t}G(x,\xi,\tau)e^{-(t+\tau-\eta)/\nu}I_{0}\left(\frac{2}{\nu}\sqrt{\tau(t-\eta)}\right)$$
$$\times F(\xi,\eta,u(\xi,\eta),u_{\xi}(\xi,\eta))d\eta d\xi d\tau. \qquad (4.5)$$

Proof. The equation (4.5) is obtained from the solution (3.12) of the linear problem (2.1)-(2.3) by formally substituting the function $f(x,t)$ to the $F(x,t,u,u_{x})$. From (4.5) we find

$$u_{t}=\int\limits_{+0}^{\infty}\int\limits_{\Omega}Ge^{-\tau/\nu}F(\xi,t,u(\xi,t),u_{\xi}(\xi,t))d\xi d\tau$$
$$-\frac{1}{\nu^{2}}\int\limits_{+0}^{\infty}\int\limits_{\Omega}\int\limits_{0}^{t}Ge^{-(t+\tau-\eta)/\nu}[I_{0}-\sqrt{\frac{\tau}{t-\eta}}I_{1}]F(\xi,\eta,u(\xi,\eta),u_{\xi}(\xi,\eta))d\eta d\xi d\tau,$$
$$Lu=-\frac{1}{\nu}\int\limits_{0}^{t}e^{-(t+\tau-\eta)/\nu}F(x,\eta,u(x,\eta),u_{x}(x,\eta))d\eta$$
$$+\frac{1}{\nu^{2}}\int\limits_{+0}^{\infty}\int\limits_{\Omega}\int\limits_{0}^{t}Ge^{-(t+\tau-\eta)/\nu}[I_{0}-\sqrt{\frac{t-\tau}{\tau}}I_{1}]F(\xi,\eta,u(\xi,\eta),u_{\xi}(\xi,\eta))d\eta d\xi d\tau, \quad (4.6)$$

$$\nu Lu_{t}=-F(x,t,u,u_{x})+\frac{1}{\nu}\int\limits_{0}^{t}e^{-(t-\eta)/\nu}F(x,\eta,u(x,\eta),u_{x}(x,\eta))d\eta$$
$$+\frac{1}{\nu}\int\limits_{+0}^{\infty}\int\limits_{\Omega}Ge^{-\tau/\nu}F(\xi,t,u(\xi,t),u_{\xi}(\xi,t))d\xi d\tau$$
$$-\frac{1}{\nu^{2}}\int\limits_{+0}^{\infty}\int\limits_{\Omega}\int\limits_{0}^{t}Ge^{-(t+\tau-\eta)/\nu}[2I_{0}-\sqrt{\frac{\tau}{t-\tau}}I_{1}-\sqrt{\frac{t-\eta}{\tau}}I_{1}]F(\xi,\eta,u(\xi,\eta),u_{\xi}(\xi,\eta))d\eta d\xi d\tau.$$

Continuities of the under the integral signs follows from the continuities of the functions $G(x,\xi,\tau), exp(-y/\nu), I_{0}(y)$ and $I_{1}(y)/y$ on the corresponding domains and from the conditions 1^{o} and 2^{o} for the function F. Uniform convergence of these integrals follows from the boundedness of the functions above on the corresponding domains, and the conditions 3^{o} and 4^{o} for the function F and

$$\int_{\Omega} G(x,\xi,\tau)d\xi \leq M_0, \qquad (4.7)$$

where M_0 is a positive constant (see [13]-[15]).

By virtue of (4.6), the function $u(x,t)$ satisfies the equation (4.1). From (4.5) we get

$$u(x,+0)=0,$$

$$Bu|_S = \frac{1}{\nu}\int_{+0}^{\infty}\int_{\Omega}\int_{0}^{t} BG|_S \, e^{-(t+\tau-\eta)/\nu} I_0 F d\eta d\xi d\tau = 0$$

Thus, if $u(x,t)$ is the solution of integrodifferential equation (4.5), it is also a solution of the problem (4.1)-(4.3). Let us show that the solution of the integrodifferential equation (4.5) exists. The set

$$E_\gamma = \{z(x,t) \mid z, z_x \in C(Q_\gamma), \| z \|_\gamma < \infty\},$$

where $Q_\gamma = \Omega \times (0,\gamma]$, and

$$\| z \|_\gamma = \sup_{(x,t)\in \bar{Q}_\gamma} | z(x,t)| + \sum_{i=1}^{n} \sup_{(x,t)\in \bar{Q}_\gamma} | z_{x_i}(x,t)|,$$

is a Banach space. The mapping

$$Az(x,t) = \frac{1}{\nu}\int_{+0}^{\infty}\int_{\Omega}\int_{0}^{t} G(x,\xi,\tau) e^{-(t+\tau-\eta)/\nu} I_0\left(\frac{2}{\nu}\sqrt{\tau(t-\eta)}\right) F(\xi,\eta,z(\xi,\eta),z_\xi(\xi,\eta)) d\eta d\xi d\tau$$

for $(x,t)\in Q_\gamma$, maps E_γ into E_γ. By virtue of (4.4) and (4.7), we see that

$$|Az_1(x,t) - Az_2(x,t)| \leq \begin{cases} C_F \| z_2 - z_1 \|_\gamma M_0 \gamma & \text{for } 1 << t \leq \gamma, \\ C_F \| z_2 - z_1 \|_\gamma M_0 \nu (1 - e^{-\gamma/\nu}) & \text{for } 0 \leq t \leq \gamma << 1, \end{cases}$$

and

$$\left|\frac{\partial Az_1}{\partial x_i}(x,t)-\frac{\partial Az_2}{\partial x_i}(x,t)\right| \le \begin{cases} C_F \| z_1 - z_2 \|_\gamma \, M_i \gamma & for \ 1 << t \le \gamma, \\ C_F \| z_1 - z_2 \|_\gamma \, M_i \nu(1-e^{-\gamma/\nu}) & for \ 0 \le t \le \gamma << \ 1, \end{cases}$$

where (see [13]-[15])

$$\int_\Omega \left| \frac{\partial G}{\partial x_i}(x,\xi,\tau) \right| d\xi \le M_i$$

Combining these formulas, it follows that

$$\| Az_1 - Az_2 \|_\gamma \le \begin{cases} C_F \| z_1 - z_2 \|_\gamma \, M\gamma & for \ 1 << \ t \le \gamma, \\ C_F \| z_1 - z_2 \|_\gamma \, M\nu(1-e^{-\gamma/\nu}) & for \ 0 \le t \le \gamma << 1, \end{cases}$$

where $M = \sum_{i=0}^{n} M_i$. We select γ such that

$$C_F M\gamma < 1 \, and \quad or \, C_F M\nu(1-e^{-\gamma/\nu}) < 1.$$

Hence, A is a contraction of E_γ *into* E_γ. Thus, A has a unique fixed point. In other words, there exists a unique function u such that $u = Au$.

Existence of the solution of the integrodifferential equation (4.5) for any finite T may be shown as was done in [15]. Solution of the problem (4.1)-(4.3) may be constructed by the Picard's iteration method from (4.5).

V. Examples

1. Let us consider the Cauchy problem

$$\Delta u + \nu \Delta u_t + F(x,t,u,u_x) = u_t \quad x \in R^n, \ t \in (0,T]; \ u|_{t=+0} = 0 \ x \in R^n, \tag{5.1}$$

where

$$\Delta = \frac{\partial^2}{\partial x_1^{\,2}} + \cdots + \frac{\partial^2}{\partial x_n^{\,2}}$$

is the Laplacian. If the function F satisfies conditions $1^o - 4^o$ of sec.4, u_0 is continuously differentiable and u_0 *and* u_{0x} are bounded, the problem (5.1) posseses a unique solution $u = u(x,t)$ that is the unique solution with bounded continuous derivative u_x of the integrodifferential equation

$$u(x,t)=\frac{1}{\nu}\int\limits_{+0}^{\infty}\int\limits_{R^n}u_0(\xi)[\Gamma_n(x-\xi,\tau)-\nu\Gamma_{n\tau}(x-\xi,\tau)]e^{-(t+\tau)/\nu}I_0(\frac{2}{\nu}\sqrt{t\tau})d\xi d\tau$$

$$+\frac{1}{\nu}\int\limits_{+0}^{\infty}\int\limits_{0}^{t}\int\limits_{R^n}\Gamma_n(x-\xi,\tau)e^{-(t+\tau-\eta)/\nu}I_0(\frac{2}{\nu}\sqrt{\tau(t-\eta)})F(\xi,\eta,u(\xi,\eta),u_\xi(\xi,\eta))d\xi d\eta d\tau.$$

Where

$$\Gamma_n(x,t)=\frac{1}{(4\pi t)^{n/2}e^{-|x|^2/4t}},\qquad t>0, \qquad (5.2)$$

is a fundamental solution of the heat conduction equation, $|x|^2=x_1^2+\cdots+x_n^2$.

2. Suppose that Ω is a half-space $\Omega=\{x\,|\,x'=(x_1,\cdots,x_{n-1})\in R^{n-1},x_n\in(0,\infty)\}$. Let us consider the problem

$$\begin{aligned}&\Delta u+\nu\Delta u_t+F(x,t,u,u_x)=u_t \qquad (x,t)\in Q_T=\Omega\times(0,T],\\ &u|_{t=+0}=0 \qquad x\in\Omega, \\ &u|_{x_n=0}=0 \quad x'\in R^{n-1},\ t\in(0,T].\end{aligned} \qquad (5.3)$$

If the function F satisfies conditions 1^o-4^o of sec.4, than the problem (5.1) posseses a unique solution $u=u(x,t)$ that is the unique solution with bounded continuous derivative u_x of the integrodifferential equation

$$u(x,t)=\frac{1}{\nu}\int\limits_{+0}^{\infty}\int\limits_{0}^{t}\int\limits_{\Omega}K(x,\xi,\tau)e^{-(t+\tau-\eta)/\nu}\times I_0(\frac{2}{\nu}\sqrt{\tau(t-\eta)})F(\xi,\eta,u(\xi,\eta),u_\xi(\xi,\eta))d\xi d\eta d\tau,$$

where

$$K(x,\xi,\tau)=\Gamma_{n-1}(x'-\xi',\tau)[\Gamma_1(x_n-\xi_n,\tau)-\Gamma_1(x_n+\xi_n,\tau)],\ \tau>0.$$

3. In the half-space Ω let us consider the problem (see [13] p.265)

$$\Delta u + \nu \Delta u_t + F(x,t,u,u_x) = u_t \qquad (x,t) \in Q_T,$$

$$u|_{t=+0} = 0 \quad x \in \Omega, \qquad (5.4)$$

$$\sum_{i=1}^{n} b_i \frac{\partial u}{\partial x_i}|_{x_n=0} = 0 \qquad x' \in R^{n-1}, \qquad t \in (0,T],$$

where b_i are certain real constants with $b_n \neq 0$. If the function F satisfies conditions $1^o - 4^o$ of sec.4 then there exists a unique bounded solution $u = u(x,t)$ of the problem (5.4) that is the unique solution with bounded continuous derivative u_x of the integrodifferential equation

$$u(x,t) = \frac{1}{\nu} \int_{+0}^{\infty}\int_{0}^{t}\int_{\Omega} K_1(x,\xi,\tau) e^{-(t+\tau-\eta)/\nu} I_0\left(\frac{2}{\nu}\sqrt{\tau(t-\eta)}\right) F(\xi,\eta,u(\xi,\eta),u_\xi(\xi,\eta))\,d\xi\,d\eta\,d\tau,$$

where

$$K_1(x,\xi,\tau) = G(x'-\xi', x_n - \xi_n, \tau) + G(x'-\xi', x_n + \xi_n, \tau), \; t > 0.$$

4. Suppose that Ω is an infinite strip $\Omega = \{x \mid x' \in R^{n-1}, x_n \in (0,1)\}$. Let us consider the problem

$$\Delta u + \nu \Delta u_t + F(x,t,u,u_x) = u_t \quad (x,t) \in Q_T,$$

$$u|_{t=+0} = 0 \quad x \in \Omega,$$

$$u|_{x_n=0} = 0 \quad x' \in R^{n-1}, \quad t \in (0,T], \qquad (5.5)$$

$$u|_{x_n=1} = 0 \quad x' \in R^{n-1}, \quad t \in (0,T].$$

If the function F satisfies conditions $1^o - 4^o$ of sec.4 then there exists a unique bounded solution $u = u(x,t)$ of the problem (5.5) that is the unique solution with bounded continuous derivative u_x of the integrodifferential equation

$$u(x,t) = \frac{1}{\nu} \int_{+0}^{\infty}\int_{0}^{t}\int_{\Omega} [\Phi(x'-\xi', x_n - \xi_n, \tau) - \Phi(x'-\xi', x_n + \xi_n, \tau)] e^{-(t+\tau-\eta)/\nu}$$

$$\times I_0\left(\frac{2}{\nu}\sqrt{\tau(t-\eta)}\right) F(\xi,\eta,u(\xi,\eta),u_\xi(\xi,\eta))\,d\xi\,d\eta\,d\tau,$$

where

$$\Phi(x,\tau)=\sum_{m=-\infty}^{\infty}\Gamma_n(x',x_n+2m,\tau)=\Gamma_{n-1}(x',\tau)\sum_{m=-\infty}^{\infty}\Gamma_1(x_n+2m,\tau),\ \tau>0,$$

and $\Gamma_n(x,t)$ is defined by (5.2).

5. In the strip Ω let us consider the problem

$$\Delta u+\nu\Delta u_t+F(x,t,u,u_x)=u_t \qquad (x,t)\in Q_T,$$

$$u|_{t=+0}=0 \quad x\in\Omega, \qquad (5.6)$$

$$\sum_{i=1}^{n}b_i\frac{\partial u}{\partial x_i}|_{x_n=0}=0 \qquad x'\in R^{n-1}, \qquad t\in(0,T],$$

$$\sum_{i=1}^{n}b_i\frac{\partial u}{\partial x_i}|_{x_n=1}=0 \qquad x'\in R^{n-1}, \qquad t\in(0,T],$$

If the function F satisfies assumptions 1^0-4^0 of sec.4 then there exists a unique bounded solution $u=u(x,t)$ of the problem (5.6) that is the unique solution with bounded continuous derivative u_x of the integrodifferential equation

$$u(x,t)=\frac{1}{\nu}\int_{+0}^{\infty}\int_{0}^{t}\int_{\Omega}[\Psi(x'-\xi',x_n-\xi_n,\tau)+\Psi(x'-\xi',x_n+\xi_n,\tau)]e^{-(t+\tau-\eta)/\nu}$$

$$\times I_0(\frac{2}{\nu}\sqrt{\tau(t-\eta)})F(\xi,\eta,u(\xi,\eta),u_\xi(\xi,\eta))d\xi d\eta d\tau,$$

where

$$\Psi(x,\tau)=\sum_{m=-\infty}^{\infty}G(x',x_n+2m,\tau),\ \tau>0,$$

and $G(x,t)$ is defined as

$$G(x,t)=\frac{1}{(4\pi t)^{n/2}b^2}\left[b_ne^{-\frac{x^2}{4t}}+\frac{b^2x_n-b_n(bx)}{|b|\sqrt{t}}e^{-\frac{x^2}{4t}+\frac{(bx)^2}{4tb^2}}\int_{\frac{bx}{2\sqrt{t}|b|}}^{\infty}e^{-z^2}dz\right],t>0.$$

References

1. R.M.CHRISTENSEN,"Theory of Viscoelasticity. An Introduction," Academic Press, New York, 1971.

2. M.E.GURTIN, A.C.PIPKIN, A General Theory of Heat Conduction With Finite Wave Speeds, *Rational Mech. and Analysis*, 31(1968),113-126.

3. F.R.NORWOOD, Transient Thermal Waves in the General Theory of Heat Conduction With Finite Wave Speeds, *ASME,J.Appl.Mech.*,39(1972),673.

4. L.W.WHITE, Point Control of Pseudoparabolic Problems, *J.Differential Equations* 42(1981),366-374.

5. A.I.KOZHANOV, Correctness of Boundary-Value Problems and Properties of Solutions to Parabolic Equations With Nonlinear Nonlocal Sources, *DAN* v.336 (1994) no.2,161-164.

6. H.ENGLER, On Some Parabolic Integro-Differential Equations: Existence and Asymptotics of Solutions, *Lect.Notes Math.*v.1017(1983),161-167.

7. D.BRANDON, W.J.HRUSA, Construction of a Class of Integral Models For Heat Flow in Materials With Memory, *J.Integral Equations Appl.*1(1988),no.2,175-201.

8. S.S.VOYT, Propagation of initial densities in viscous gas. Uch.zap. MGU (1954), 172, Mechanics, 125-142.

9. S.D.EIDELMAN, "Parabolic Systems", North-Holland Pub.Comp., 1969.

10. R.E.SHOWALTER, T.W.TING, Pseudoparabolic Partial Differential Equations, *SIAM.J.Math.Anal.* 1 (1970), 1-26.

11. R.W.CARROLL, R.E.SHOWALTER, "Singular and Degenerate Cauchy Problems", Academic Press, 1976.

12. M.H.ILYASOV,"Some Dynamical Problems of Linear Viscoelasticity", *Doctoral Dissertation*, Moscow.Gos.Univ., Moscow, 1985.

13. O.A.LADYZENSKAJA, V.A.SOLONNIKOV, N.N.URAL'CEVA,"Linear and Quasilinear Equations of Parabolic Type", *Amer.Math.Soc.Trans.* 23(1968).

14. A.FRIEDMAN,"Partial Differential Equations of Parabolic Type", Prentice-Hall, Englewood Cliffs, N.J., 1964.

15. J.R.CANNON,"The One-Dimensional Heat Equation", Addison-Wesley Publishing Company, 1984.

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