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Bornological spaces of entire functions represented by Dirichlet series having slow growth

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ABSTRACT

The study of spaces of entire functions was initiated by V.G. Iyer [6] and the space of entire functions represented by Dirichlet series has been studied by Hussein and Kamthan [4] and others. Patwardhan [9] has successfully studied bornological properties of the spaces of entire function in terms of the coefficients of Taylor series expansions. In this paper we have used another norm and study the bornological aspects of the space Γ of all entire Dirichlet series $\alpha(s) = \sum_{n=1}^{\infty} a_n \exp(s\lambda_n)$ of order zero.

1. INTRODUCTION

Let C denote the field of complex numbers equipped with usual topology. Let Γ denote the family of all transformations:

$\alpha : C \rightarrow C$ such that

$$\alpha(s) = \sum_{n=1}^{\infty} a_n \exp(s\lambda_n), \tag{1.1}$$

where $\lambda_{n+1} > \lambda_n$, $\lambda_1 \geq 0$, $\lim\limits_{n\to\infty} \lambda_n = \infty$, $s = \sigma + it$ (σ, t real variables), and $\{a_n\}_1^\infty$ is any sequence of complex numbers. Set

(1.2) $$\limsup_{n\to\infty} \frac{\log n}{\lambda_n} = D^*.$$

Let γ and τ be the abscissa of convergence and abscissa of absolute convergence of $\alpha(s)$.

Then Bernstein [1,p.4], proved that

(1.3) $$0 < \gamma - \tau < D^*,$$

and

(1.4) $$\gamma = \limsup_{n\to\infty} \frac{\log |a_n|^{-1}}{\lambda_n}.$$

Thus if $D^* < \infty$ and $\gamma = \infty, \alpha(s)$ represents an entire function and by (1.3), $\tau = \infty$ so that the series (1.1) converges absolutely at every point of the finite complex plane. Further, for $D^* = 0$, we get

(1.5) $$\tau = \gamma = \limsup_{n\to\infty} \frac{\log |a_n|^{-1}}{\lambda_n}.$$

It is well known that the function $\alpha(s)$ is an analytic function in the half plane $\sigma < \tau$ $(-\infty < \tau < \infty)$. We now assume that $D^* = 0$ and

(1.6) $$\tau = \limsup_{n\to\infty} \frac{\log |a_n|^{-1}}{\lambda_n}.$$

It is well known that the transformation given by (1.1) represents an entire function for $\tau = \infty$. Kamthan and Gautam ([7] and [8]) gave the set Γ the topology of uniform convergence. They studied the properties of bases of the space Γ using the
growth properties of the entire Dirichlet series. For $\alpha \in \Gamma$, we set

$$M(\sigma,\alpha) \equiv M(\sigma) \equiv \underset{-\infty<t<\infty}{l.u.b.} |\alpha(\sigma+it)|.$$

Let ρ be a non-zero positive real number and Γ now denote the family of all entire Dirichlet functions α having Ritt order ρ, [11]. Then every $\alpha \in \Gamma$ can be characterized by the condition

$$\text{(1.7)} \qquad \limsup_{n\to\infty} \frac{\lambda_n \log \lambda_n}{\log |a_n|^{-1}} = \rho.$$

It is obvious that the above class of entire functions leaves a big subclass i.e. those entire functions for which $\rho = 0$. To further study the growth of such entire functions, the notion of logarithmic order is used [5]. Thus an entire function $\alpha(s)$ is said to be of logarithmic order ρ if

$$\limsup_{\sigma\to\infty} \frac{\log\log M(\sigma)}{\log \sigma} = \rho,\ 1 \le \rho \le \infty.$$

2. DEFINITIONS

The bornological aspect for entire function have been studied by Patwardhan [9] and others. The authors studied these properties for spaces of entire functions represented by Dirichlet series. So far in the study of these growth properties of entire functions have not been taken into consideration. The present paper is an effort in this direction.

In this section we give some definitions. We have

2.1. A bornology on a set X is a family **B** of subsets of X satisfying the following axioms:

(i) **B** is a covering of X, i.e. $X = \bigcup_{B \in \mathbf{B}} B$;

(ii) **B** is hereditary under inclusion, i.e. if $A \in \mathbf{B}$ and B is a subset of X contained in A, then $B \in \mathbf{B}$;

(iii) **B** is stable under finite union.

A pair $(X, \mathbf{B})$ consisting of a set X and a bornology **B** on X is called a bornological space, and the elements of **B** are called the bounded subsets of X

2.2. A base of a bornology **B** on X is any subfamily $\mathbf{B}_0$ of **B** such that every element of **B** is contained in an element of $\mathbf{B}_0$. A family $\mathbf{B}_0$ of subsets of X is a base for a bornology on X if and only if $\mathbf{B}_0$ covers X and every finite union of elements of $\mathbf{B}_0$ is contained in a member of $\mathbf{B}_0$. Then the collection of those subsets of X, which are contained in an element of $\mathbf{B}_0$ defines a bornology **B** on X having $\mathbf{B}_0$ as a base. A bornology is said to be a bornology with a countable base if it possesses a base consisting of a sequence of bounded sets. Such a sequence can always be assumed to be increasing.

2.3. Let E be a vector space over the complex field C. A bornology **B** on E is said to be a vector hornology on E, if **B** is stable under vector addition, homothetic transformations and the formation of circled hulls or, in other words, if the sets $A+B$, λA, $\bigcup_{|\eta|\leq 1} \eta A$ belong to **B**, whenever A and B belong to **B** and $\lambda \in C$. Any pair (E,**B**) consisting of vector space E and a vector bornology on E is called a bornological vector space.

2.4. A vector bornology on a vector space E is called a convex vector hornology if it is stable under the formation of convex hulls. Such a bornology is also stable under the formation of disked hulls, since the convex hull of a circled set is circled. A bornological vector space (E, **B**) whose bornology **B** is convex is called a convex bornological vector space.

2.5. A separated bornological vector space (E, **B**) or (a separated bornology **B**) is one where $\{O\}$ is the only bounded vector subspace of E .

2.6. A set P is said to be bornivorous if for every bounded set B there exists a $t \in C$ $t \neq O$ such that $\mu B \subset P$ for all $\mu \in C$ for which $|\mu| \leq |t|$.

2.7. Let E be a vector space and let A be a disk in E not necessarily absorbent in E. We denote by E_A the vector space spanned by A, i.e. the space $\bigcup_{\lambda > o} \lambda A = \bigcup_{\lambda \in k} \lambda A$.

2.8. Let E be a bornological vector space. A sequence $\{x_n\}$ in E is said to be M-convergent to a point $x \in E$ if there exists a decreasing sequence $\{t_n\}$ of positive real number tending to zero such that the sequence $\left\{\frac{x_n - x}{t_n}\right\}$ is bounded.

2.9. Let E be a separated convex bornological space. A sequence $\{x_n\}$ in E is said to be a bornological Cauchy sequence (or a Mackey-Cauchy sequence) in E if there exists a bounded disk $B \subset E$ such that $\{x_n\}$ is a Cauchy sequence in E_B. For more details we refer to [3].

3. THE SPACE Γ

Let $\rho > 1$ be any positive real number. Further we assume that Γ denotes the space of all entire Dirichlet series satisfying (1.1) to (1.7) and

$$(3.1) \qquad \limsup_{\sigma\to\infty} \frac{\log\log M(\sigma)}{\log\sigma} \le \rho < +\infty.$$

It is known [10] that (3.1) is satisfied if and only if

$$(3.2) \qquad \limsup_{\sigma\to\infty} \frac{\log\lambda_n}{\log\log|a_n|^{-1/\lambda_n}} \le \rho - 1.$$

For an entire function $\alpha(s)$, define the number $\|\alpha\|$ by

$$(3.3) \qquad \|\alpha\| = l.u.b.\|a_n\|^{1/\lambda_n}, n > 1.$$

For each $\alpha \in \Gamma$, we define

$$(3.4) \qquad \|\alpha : \rho+\delta\| = \sum_1^{\infty} |a_n| e^{-\lambda_n(\rho+\delta/\rho+\delta-1)},$$

where $\delta > 0$ is arbitrary. On account of (3.2), (3.4) is clearly well defined. Let $\Gamma(\rho,\delta)$ denote the space Γ equipped with the norm $\|. : \rho+\delta\|$. We define a bornology on Γ with the help of $\|\cdot\|$ defined by (3.3). We denote by B_k the set $\{\alpha \in \Gamma : \| \alpha \| \leq k\}$. Then the family $\mathbf{B}_0 = \{B_k : k = 1,2,\ldots\}$ forms a base for a bornology $\mathbf{B}$ on Γ.

We now prove

Theorem 3.1. $(\Gamma, \mathbf{B})$ is a separated convex bornological vector space with a countable base.

Proof. Since the vector bornology $\mathbf{B}$ on the vector space Γ is stable under the formation of the convex hulls, it is a convex vector bornology. Since the convex hull of a circled set is circled. $\mathbf{B}$ is stable under the formation of disked hulls, and hence the bornological vector space $(\Gamma, \mathbf{B})$ is a convex bornological vector space. Now to show that $\{0\}$ is the only bounded vector subspaces of Γ, we must show that Γ contains no bounded open set. Let $\bigcup(\varepsilon)$ denote the set of all $\alpha \in \Gamma$ such that $\| \alpha \| < \varepsilon$.

To prove the result stated, it is enough to show that no $\bigcup(\varepsilon)$ is bounded, that is, given $\bigcup(\varepsilon)$, we have to prove that there exists $\bigcup(\eta)$ for which there is no $c > 0$ sucb that $\bigcup(\varepsilon) \subset c\bigcup(\eta)$. For this purpose, take $\eta = \frac{\varepsilon}{4}$. Given a positive number c, we can find a sufficiently large positive integer λ_m so that $|c|^{1/\lambda_m} < 2$. Let $\alpha = \left(\frac{\varepsilon}{2}\right)^{\lambda_m} \exp(s\,\lambda_m)$. Then $\| \alpha \| = \frac{\varepsilon}{2}$; so $\alpha \in \bigcup(\varepsilon)$ but $\left\| \frac{\alpha}{c} \right\| = \underset{m \geq 1}{l.u.b.}\left(\frac{\varepsilon}{2}|c|^{-1/\lambda_m}\right) > \frac{\varepsilon}{4} = \eta$, so that $c^{-1}\alpha$

does not belong to $\bigcup(\eta)$ that is, $\alpha \notin c\bigcup(\eta)$. This shows that $\bigcup(\varepsilon)$ is not bounded. Thus $\{0\}$ is the only bounded vector subspace of Γ, and hence $(\Gamma,\mathbf{B})$ is separated. Since **B** possesses a base consisting of increasing sequence of bounded sets, **B** is a bornology with a countable base. Thus $(\Gamma,\mathbf{B})$ is a separated convex bornological vector space with countable base. This proves Theorem 3.1.

Theorem 3.2. **B** contains no bornivorous set.

Proof. Suppose **B** contains a bornivorous set A. Then there exists a set $B_i \in$ **B** such that $A \subset B_i$ and consequently B_i is also bornivorous. We now assert that if $i_1 > i$, then $t\ B_{i_1} \not\subset B_i$ for any $t \in C$ which leads to a contradiction. If $i_1 > i$, it is easy to see that $t\ B_{i_1} \not\subset B_i$ for any $i \in C$ such that $|t| \geq 1$. Now we prove that $t\ B_{i_1} \not\subset B_i$ for any $t \in C$ such that $|t| < 1$ also. Let thus $|t| < 1$. Since $i_1 / i > 1$, we can choose n such that $1 < 1/|t| < (i_1/i)^{\lambda_n}$. Now let $a_n \in C$ be such that $i^{\lambda_n}/|t| < |a_n| \leq i_1^{\lambda_n}$ and let $\alpha = a_n e^{s\lambda_n}$. Then $\|\alpha\| = |a_n|^{1/\lambda_n} \leq i_1$ and hence $\alpha \in B_{i_1}$. Now $\|t\alpha\| = \|t\,a_n e^{s\lambda_n}\| = |t\,a_n|^{1/\lambda_n} > i$ and hence $t\alpha \notin B_i$. Thus $t\,B_{i_1} \not\subset B_i$ for any $t \in C$. This proves Theorem 3.2.

The following result is due to H. Hogbe and Nlend [2].

Theorem 3.3. The Mackey-convergence in a bornological vector space E is topologisable if and only if E has a bounded bornivorous set.

Combining Theorems 3.2 and 3.3, we get the following:

Corollary 3.1. The Mackey-convergence of Γ is not topologisable.

Proof. Suppose the Mackey-convergence of Γ is topologisable. Then by Theorem 3.3 Γ has a bounded bornivorous set and this contradicts Theorem 3.2.

4. δ-NORMS ON Γ

We define, for each $\delta > 0$, the expression

(4.1) $$\|\alpha : \rho + \delta\| = \sum_{n=1}^{\infty} |a_n| e^{-\lambda_n(\rho+\delta)/(\rho+\delta-1)}, \alpha \in \Gamma.$$

It is easily seen that, for each (ρ,δ), (4.1) defines a norm on the class of entire functions represented by Dirichlet series. We shall denote by $\Gamma(\rho,\delta)$ the space Γ endowed with this norm. We denote by $\mathbf{B}_\delta$ the hornology on Γ consisting of the sets bounded in the sense of the norm $\|\alpha : \rho + \delta\|$. We now prove

Theorem 4.1. $\mathbf{B} = \bigcup_{\delta > 0} \mathbf{B}_\delta$.

Proof. Let $B \in \mathbf{B}$. Then there exists a constant J such that $\|\alpha\| \leq J$ for all

$$\alpha = \sum_{n=1}^{\infty} a_n e^{s\lambda_n} \in B. \text{Then } |a_n|^{1/\lambda_n} \leq J \text{ for all } n.$$

Choose $\delta > 0$, such that $e^{-(\rho+\delta)/(\rho+\delta-1)} < \frac{1}{J}$.

Then $J^{\lambda_n} < e^{\lambda_n(\rho+\delta)/(\rho+\delta-1)}$,

or

$|a_n| \, e^{-\lambda_n(\rho+\delta_1)/(\rho+\delta_1-1)} < J^{\lambda_n} e^{-\lambda_n(\rho+\delta_1)/(\rho+\delta_1-1)}$, $\delta_1 > \delta$ where $J > 0$, we have

$$\sum_{n=1}^{\infty} |a_n| e^{-\lambda_n(\rho+\delta_1)/(\rho+\delta_1-1)} < \sum_{n=1}^{\infty} e^{\lambda_n\left(\frac{\rho+\delta}{\rho+\delta-1} - \frac{\rho+\delta_1}{\rho+\delta_1-1}\right)}, \; \delta_1 > \delta,$$

$$< \infty .$$

Hence $B \in \mathbf{B}_\delta$ and so $\mathbf{B} \subset \bigcup_{\delta>0} \mathbf{B}_\delta$.

For the reverse inclusion let $B \in \mathbf{B}_\delta$, then there exists a constant J such that for all $\alpha \in \mathbf{B}$, $\|\alpha : \rho+\delta\| \le J$,

i.e. $\sum_{n=1}^{\infty} |a_n| e^{-\lambda_n(\rho+\delta)/(\rho+\delta-1)} \le J$

or $|a_n| \le J e^{\lambda_n(\rho+\delta)/(\rho+\delta-1)}$

i.e. $|a_n|^{1/\lambda_n} \le J^{1/\lambda_n} \left(e^{\lambda_n(\rho+\delta)/(\rho+\delta-1)}\right)^{1/\lambda_n}$, $n \ge 1$

i.e. $\|\alpha\| \le l.u.b.\{J^{1/\lambda_n} e^{(\rho+\delta)/(\rho+\delta-1}\} < \infty$.

Thus $B \in \mathbf{B}$ and hence $\bigcup_{\delta>0} \mathbf{B}_\delta \subset \mathbf{B}$. This completes proof of Theorem 4.1.

Lemma 4.1. In the topological dual Γ' of Γ, every functional is of the form

$f(\alpha) = \sum_{n=1}^{\infty} c_n\, a_n \,,\ \alpha = \sum_{n=1}^{\infty} a_n\, e^{s\lambda_n}$, if and only if the sequence $\left\{ \left| c_n \right| e^{-\lambda_n (\rho+\delta)/(\rho+\delta-1)} \right\}$ is bounded.

Proof. Suppose that $f(\alpha)$ is a continuous linear functional on Γ. Then there exists $k > 0$ such that

$$\left| f(\alpha) \right| \le k \left\| \alpha : \rho + \delta \right\| \text{ for every } \alpha .$$

Let $\psi_n = e^{s\lambda_n}$ and $f(\psi_n) = c_n$ $(n \ge 1)$.

In Γ', $\alpha = \sum_{n=1}^{\infty} a_n\, e^{s\lambda_n} = \lim_{n\to\infty} \sum_{i}^{n} a_i \psi_i$. Since f is continuous, we have

$$f(\alpha) = f\left(\lim_{n\to\infty} \sum_{i=1}^{n} a_i\, e^{s\lambda_n} \right)$$

$$= \lim_{n\to\infty} \sum_{i}^{n} a_i\, f(\psi_i)$$

$$= \lim_{n\to\infty} \sum_{i}^{n} a_i\, c_i$$

$$= \sum_{i}^{\infty} a_n\, c_n \,.$$

Also $\left| c_n \right| \le k \left\| \psi_n : \rho + \delta \right\| = k\, e^{-\lambda_n (\rho+\delta)/(\rho+\delta-1)}$.

Then $\left\{ \left| c_n \right| e^{-\lambda_n (\rho+\delta)/(\rho+\delta-1)} \right\} = k\, e^{-2\lambda_n (\rho+\delta)/(\rho+\delta-1)}$.

Hence $\{|c_n|e^{-\lambda_n(\rho+\delta)/(\rho+\delta-1)}\}$ is bounded for all $n = 1,2,\ldots$.

Conversely let $\{|c_n|e^{-\lambda_n(\rho+\delta)/(\rho+\delta-1)}\}$ be bounded for all $n = 1,2,\ldots$.

Let f be defined by

$$f(\alpha) = \sum_{n=1}^{\infty} c_n a_n , \quad \alpha = \sum_{n=1}^{\infty} a_n e^{s\lambda_n}$$, then f is linear functional and

$$|f(\alpha)| \le \sum_{n=1}^{\infty} |c_n| |a_n|$$

$$< \sum_{n=1}^{\infty} k e^{-\lambda_n(\rho+\delta)/(\rho+\delta-1)} |a_n| \quad \text{for some } k > 0$$

$$= k\|\alpha : \rho + \delta\| \quad \text{for all } \alpha .$$

Hence $f(\alpha)$ is continuous on Γ. This proves Lemma 4.1.

Theorem 4.2. The bornological dual Γ^* of Γ is the same as its topological dual Γ' .

Proof. Let $\overline{\Gamma}$ be the vector space of all linear functionals on the vector space Γ. For every $f \in \overline{\Gamma}$ and $\alpha \in \Gamma$ we denote by $< \alpha, f >$ the scalar f (α), i.e. the value of linear functional f at the function α , the map $\Gamma \times \overline{\Gamma} \to K$ defined by $(\alpha, f) \to < \alpha, f >$ is a bilinear form on $\Gamma \times \overline{\Gamma}$ called the canonical bilinear form. Let $\overline{\Gamma}$ be the topological dual of Γ, i.e. the vector space of all continuous linear functionl on Γ. Since Γ' is a subspace of $\overline{\Gamma}$, the restriction of the canonical bilinear form induces a duality between

Γ and Γ'. Since Γ is locally convex space and separated, it follows from Corollary 2 to Theorem 1 of [3] that this duality is separated in Γ.

Let the vector space Γ^* be the set of all bounded linear functionals in the sense of bornology on Γ which is a separated by Theorem 3.1. We can induce a duality between Γ and Γ^* by using the bilinear form $(\alpha, f^*) \to <\alpha, f^*> = f^*(\alpha)$, for $\alpha \in \Gamma$ and $f^* \in \Gamma^*$. This duality is called the bornological duality between Γ and Γ^*.

Let t_Γ be the space Γ, endowed with the locally convex topology associated with the hornology of Γ. Since, algebraically, $\Gamma^* = (t_\Gamma)'$ [the topological dual of Γ], we see that the bornological duality between Γ and Γ^*, is identical to the topological duality between t_Γ and $(t_\Gamma)'$.
Thus the proof of Theorem 4.2 is complete.

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