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ON SOME WEAK FORMS OF CONTINUITY FOR MULTIFUNCTIONS

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Abstract

The authors define a multifunction $F : (X, m_X) \to (Y, \sigma)$ to be weakly m-continuous if for each point $x \in X$ and any open sets G_1, G_2 of Y such that $F(x) \subset G_1$ and $F(x) \cap G_2 \neq \emptyset$, there exists an m_X-open set U of X containing x such that $F(u) \subset \mathrm{Cl}(G_1)$ and $F(u) \cap \mathrm{Cl}(G_2) \neq \emptyset$ for every $u \in U$. They obtain several characterizations and properties concerning weakly m-continuous multifunctions.

1 Introduction

Semi-open sets, preopen sets, α-open sets, β-open sets and δ-open sets play an important role in the study of generalizations of continuity in topological spaces. By using these sets several authors introduced and studied various types of weak forms of continuity for functions and multifunctions. In 1961, Marcus [15] introduced the notion of quasicontinuity in topological spaces. Bânzaru [6] and Bânzaru and Crivăţ [7] extended it to the notion of quasicontinuity for multifunctions. The present authors introduced and studied weakly quasi-continuous multifunctions [20], weakly precontinuous multifunctions [35] and weakly β-continuous multifunctions [36]. These multifunctions have similar properties. The analogy in their definitions and results suggests the need of formulating a unified theory.

In this paper, in order to unify several characterizations and properties of weakly quasi-continuous multifunctions, weakly precontinuous multifunctions and weakly β-continuous multifunctions, we introduce a new notion of weakly m-continuous multifunctions defined on the domain satisfying minimal conditions.

2 Preliminaries

Let (X, τ) be a topological space and A a subset of X. The closure of A and the interior of A are denoted by $\mathrm{Cl}(A)$ and $\mathrm{Int}(A)$, respectively. A subset A is said to be *regular closed* (resp. *regular open*) if $\mathrm{Cl}(\mathrm{Int}(A)) = A$ (resp. $\mathrm{Int}(\mathrm{Cl}(A)) = A$).

Definition 2.1 A subset A of a topological space (X, τ) is said to be *α-open* [19] (resp. *semi-open* [13], *preopen* [16], *β-open* [1] or *semi-preopen* [3]) if $A \subset \mathrm{Int}(\mathrm{Cl}(\mathrm{Int}(A)))$ (resp. $A \subset \mathrm{Cl}(\mathrm{Int}(A))$, $A \subset \mathrm{Int}(\mathrm{Cl}(A))$, $A \subset \mathrm{Cl}(\mathrm{Int}(\mathrm{Cl}(A)))$).

A subset A is said to be *δ-open* [42] if for each $x \in A$ there exists a regular open set G such that $x \in G \subset A$. A point $x \in X$ is called a *δ-cluster* point of A if $\mathrm{Int}(\mathrm{Cl}(V)) \cap A \neq \emptyset$ for every open set V containing x. The set of all δ-cluster points of A is called the *δ-closure* of A and is denoted by $\mathrm{Cl}_\delta(A)$. The set $\{x \in X : x \in U \subset A$ for some regular open set U of $X\}$ is called the *δ-interior* of A and is denoted by $\mathrm{Int}_\delta(A)$.

Definition 2.2 A subset A of X is said to be *δ-preopen* [41] (resp. *δ-semi-open* [22]) if $A \subset \mathrm{Int}(\mathrm{Cl}_\delta(A))$ (resp. $A \subset \mathrm{Cl}(\mathrm{Int}_\delta(A))$).

The family of all semi-open (resp. preopen, α-open, β-open, δ-preopen, δ-semi-open) sets in X is denoted by $\mathrm{SO}(X)$ (resp. $\mathrm{PO}(X)$, $\alpha(X)$, $\beta(X)$, $\delta\mathrm{PO}(X)$, $\delta\mathrm{SO}(X)$).

Definition 2.3 The complement of a semi-open (resp. preopen, α-open, β-open, δ-preopen, δ-semi-open) set is said to be *semi-closed* [8] (resp. *preclosed* [10], *α-closed* [17], *β-closed* [1], *δ-preclosed* [41], *δ-semi-closed* [22]).

If A is both semi-open and semi-closed, then it is said to be *semi-regular* [9]. The set of all semi-regular sets of X is denoted by $\mathrm{SR}(X)$.

Definition 2.4 The intersection of all semi-closed (resp. preclosed, α-closed, β-closed, δ-preclosed, δ-semi-closed, semi-regular) sets of X containing A is called the *semi-closure* [8] (resp. *preclosure* [10], *α-closure* [17], *β-closure* [2], *δ-preclosure* [41], *δ-semi-closure* [22], *semi-θ-closure* [18]) of A and is denoted by $\mathrm{sCl}(A)$ (resp. $\mathrm{pCl}(A)$, $\alpha\mathrm{Cl}(A)$, ${}_\beta\mathrm{Cl}(A)$, $\mathrm{pCl}_\delta(A)$, $\mathrm{sCl}_\delta(A)$, $\mathrm{sCl}_\theta(A)$).

Definition 2.5 The union of all semi-open (resp. preopen, α-open, β-open, δ-preopen, δ-semi-open, semi-regular) sets of X contained in A is called the *semi-interior* (resp. *preinterior*, *α-interior*, *β-interior*, *δ-preinterior*, *δ-semi-interior*, *semi-θ-interior*) of A and is denoted by $\mathrm{sInt}(A)$ (resp. $\mathrm{pInt}(A)$, $\alpha\mathrm{Int}(A)$, ${}_\beta\mathrm{Int}(A)$, $\mathrm{pInt}_\delta(A)$, $\mathrm{sInt}_\delta(A)$, $\mathrm{sInt}_\theta(A)$).

Let (X,τ) be a topological space. A point $x \in X$ is called a *θ-cluster* of a subset A point of A if $\mathrm{Cl}(V) \cap A \neq \emptyset$ for every open set V containing x. The set of all θ-cluster points of A is called the *θ-closure* [42] of A and is denoted by $\mathrm{Cl}_\theta(A)$. If $A = \mathrm{Cl}_\theta(A)$, then A is said to be θ-closed. It is shown in [42] that $\mathrm{Cl}(V) = \mathrm{Cl}_\theta(V)$ for every open set V and $\mathrm{Cl}_\theta(S)$ is closed for every subset$S$ of X.

Throughout the present paper (X,τ) and (Y,σ) always denote topological spaces and $F : (X,\tau) \to (Y,\sigma)$ presents a multivalued function. For a multifunction $F : (X,\tau) \to (Y,\sigma)$, we shall denote the upper and lower inverses of a subset B of a space Y by $F^+(B)$ and $F^-(B)$, respectively, that is,

$$F^+(B) = \{x \in X : F(x) \subset B\} \text{ and } F^-(B) = \{x \in X : F(x) \cap B \neq \emptyset\}.$$

Let $\mathcal{P}(Y)$ be the collection of all nonempty subsets of Y. For any set V of Y, we denote $V^+ = \{B \in \mathcal{P}(Y) : B \subset V\}$ and $V^- = \{B \in \mathcal{P}(Y) : B \cap V \neq \emptyset\}$ [40].

Definition 2.6 A multifunction $F : (X,\tau) \to (Y,\sigma)$ is said to be *quasi-continuous* [6], [7], [23] (resp. *precontinuous* [33], *α-continuous* [29], *β-continuous* [31]) if for each point $x \in X$ and each open set V_1, V_2 of Y such that $F(x) \in V_1^+ \cap V_2^-$, there exists a semi-open (resp. preopen, α-open, β-open) set U of X containing x such that $F(u) \in V_1^+ \cap V_2^-$ for every $u \in U$.

Definition 2.7 A multifunction $F : (X,\tau) \to (Y,\sigma)$ is said to be *weakly quasi-continuous* [20] (resp. *weakly precontinuous* [35], *weakly β-continuous*

[31]) if for each point $x \in X$ and each open set V_1, V_2 of Y such that $F(x) \in V_1^+ \cap V_2^-$, there exists a semi-open (resp. preopen, β-open) set U of X containing x such that $F(u) \in (\mathrm{Cl}(V_1))^+ \cap (\mathrm{Cl}(V_2))^-$ for every $u \in U$.

3 Characterizations

Definition 3.1 A subfamily m_X of the power set $\mathcal{P}(X)$ of a nonempty set X is called a *minimal structure* (briefly *m-structure*) on X if $\emptyset \in m_X$ and $X \in m_X$. Each member of m_X is said to be *m_X-open* and the complement of an m_X-open set is said to be *m_X-closed.*

Remark 3.1 Let (X, τ) be a topological space. Then the families τ, $\mathrm{SO}(X)$, $\mathrm{PO}(X)$, $\alpha(X)$, $\beta(X)$, $\delta\mathrm{PO}(X)$, $\delta\mathrm{SO}(X)$ and $\mathrm{SR}(X)$ are all m-structures on X.

Definition 3.2 Let X be a nonempty set and m_X an m-structure on X. For a subset A of X, the *m_X-closure* of A and the *m_X-interior* of A are defined in [14] as follows:

(1) $m_X\text{-}\mathrm{Cl}(A) = \cap\{F : A \subset F, X - F \in m_X\}$,

(2) $m_X\text{-}\mathrm{Int}(A) = \cup\{U : U \subset A, U \in m_X\}$.

Remark 3.2 Let (X, τ) be a topological space and A a subset of X. If $m_X = \tau$ (resp. $\mathrm{SO}(X)$, $\mathrm{PO}(X)$, $\alpha(X)$, $\beta(X)$, $\delta\mathrm{PO}(X)$, $\delta\mathrm{SO}(X)$, $\mathrm{SR}(X)$), then we have

(1) $m_X\text{-}\mathrm{Cl}(A) = \mathrm{Cl}(A)$ (resp. $\mathrm{sCl}(A)$, $\mathrm{pCl}(A)$, $\alpha\mathrm{Cl}(A)$, ${}_\beta\mathrm{Cl}(A)$, $\mathrm{pCl}_\delta(A)$, $\mathrm{sCl}_\delta(A)$, $\mathrm{s}_\theta\mathrm{Cl}(A)$),

(2) $m_X\text{-}\mathrm{Int}(A) = \mathrm{Int}(A)$ (resp. $\mathrm{sInt}(A)$, $\mathrm{pInt}(A)$, $\alpha\mathrm{Int}(A)$, ${}_\beta\mathrm{Int}(A)$, $\mathrm{pInt}_\delta(A)$, $\mathrm{sInt}_\delta(A)$, $\mathrm{s}_\theta\mathrm{Int}(A)$) .

Lemma 3.1 (Maki [14]). *Let X be a nonempty set and m_X a minimal structure on X. For subsets A and B of X, the following properties hold:*

(1) $m_X\text{-}\mathrm{Cl}(X - A) = X - m_X\text{-}\mathrm{Int}(A)$ *and* $m_X\text{-}\mathrm{Int}(X - A) = X - m_X\text{-}\mathrm{Cl}(A)$,

(2) If $(X - A) \in m_X$, *then* $m_X\text{-}\mathrm{Cl}(A) = A$ *and if* $A \in m_X$, *then* $m_X\text{-}\mathrm{Int}(A) = A$,

(3) $m_X\text{-}\mathrm{Cl}(\emptyset) = \emptyset$, $m_X\text{-}\mathrm{Cl}(X) = X$, $m_X\text{-}\mathrm{Int}(\emptyset) = \emptyset$ *and* $m_X\text{-}\mathrm{Int}(X) = X$,

(4) If $A \subset B$, *then* $m_X\text{-}\mathrm{Cl}(A) \subset m_X\text{-}\mathrm{Cl}(B)$ *and* $m_X\text{-}\mathrm{Int}(A) \subset m_X\text{-}\mathrm{Int}(B)$,

(5) $A \subset m_X\text{-Cl}(A)$ *and* $m_X\text{-Int}(A) \subset A$,

(6) $m_X\text{-Cl}(m_X\text{-Cl}(A)) = m_X\text{-Cl}(A)$ *and* $m_X\text{-Int}(m_X\text{-Int}(A)) = m_X\text{-Int}(A)$.

Definition 3.3 A multifunction $F : (X, m_X) \to (Y, \sigma)$, where X is a nonempty set with an m-structure m_X, is said to be *weakly m-continuous* (resp. *m-continuous* [39]) at a point $x \in X$ if for each point $x \in X$ and each open sets V_1, V_2 of Y such that $F(x) \in V_1^+ \cap V_2^-$, there exists $U \in m_X$ containing x such that $F(u) \in (\text{Cl}(V_1))^+ \cap (\text{Cl}(V_2))^-$ (resp. $F(u) \in V_1^+ \cap V_2^-$) for every $u \in U$.

Remark 3.3 Let $F : (X, \tau) \to (Y, \sigma)$ be a multifunction.

(1) If $m_X = \text{SO}(X)$ (resp. $\text{PO}(X)$, $\alpha(X)$, $\beta(X)$) and $F : (X, m_X) \to (Y, \sigma)$ is m-continuous, then F is quasicontinuous (resp. precontinuous, α-continuous, β-continuous),

(2) If $m_X = \text{SO}(X)$ (resp. $\text{PO}(X)$, $\beta(X)$) and $F : (X, m_X) \to (Y, \sigma)$ is weakly m-continuous, then F is weakly quasicontinuous (resp. weakly precontinuous, weakly β-continuous).

Theorem 3.1 *For a multifunction* $F : (X, m_X) \to (Y, \sigma)$*, the following properties are equivalent:*

(1) F *is weakly m-continuous;*

(2) $F^+(G_1) \cap F^-(G_2) \subset m_X\text{-Int}(F^+(\text{Cl}(G_1)) \cap F^-(\text{Cl}(G_2)))$ *for every open sets* G_1, G_2 *of* Y*;*

(3) $m_X\text{-Cl}(F^-(\text{Int}(K_1)) \cup F^+(\text{Int}(K_2))) \subset F^-(K_1) \cup F^+(K_2)$ *for every closed sets* K_1, K_2 *of* Y*;*

(4) $m_X\text{-Cl}(F^-(\text{Int}(\text{Cl}(B_1))) \cup F^+(\text{Int}(\text{Cl}(B_2)))) \subset F^-(\text{Cl}(B_1)) \cup F^+(\text{Cl}(B_2))$ *for every subsets* B_1, B_2 *of* Y*;*

(5) $F^+(\text{Int}(B_1)) \cap F^-(\text{Int}(B_2)) \subset m_X\text{-Int}(F^+(\text{Cl}(B_1)) \cap F^-(\text{Cl}(B_2))))$ *for every subsets* B_1, B_2 *of* Y*;*

(6) $m_X\text{-Cl}(F^-(G_1) \cup F^+(G_2)) \subset F^-(\text{Cl}(G_1)) \cup F^+(\text{Cl}(G_2))$ *for every open sets* G_1, G_2 *of* Y.

Proof. (1) $\Rightarrow$ (2): Let G_1, G_2 be any open sets in Y such that $x \in F^+(G_1) \cap F^-(G_2)$. Then $F(x) \in G_1^+ \cap G_2^-$ and hence there exists $U \in m_X$ such that $x \in U \subset F^+(\text{Cl}(G_1)) \cap F^-(\text{Cl}(G_2))$. Since $U \in m_X$, we have$x \in m_X\text{-Int}(F^+(\text{Cl}(G_1)) \cap F^-(\text{Cl}(G_2)))$.

(2) $\Rightarrow$ (3): Let K_1, K_2 be any closed sets in Y. Then, $Y - K_1$ and $Y - K_2$ are open sets in Y and by (2) and Lemma 3.1, we have

$$X - (F^-(K_1) \cup F^-(K_2)) = (X - F^-(K_1)) \cap (X - F^+(K_2)) = F^+(Y - K_1) \cap F^-(Y - V_2) \subset m_X\text{-}\mathrm{Int}(F^+(\mathrm{Cl}(Y - K_1)) \cap F^-(\mathrm{Cl}(Y - K_2))) = m_X\text{-}\mathrm{Int}[(X - F^-(\mathrm{Int}(K_1)) \cap (X - F^+(\mathrm{Int}(K_2)))] = m_X\text{-}\mathrm{Int}(X - [F^-(\mathrm{Int}(K_1)) \cup F^+(\mathrm{Int}(K_2))]).$$ Therefore, we obtain $$m_X\text{-}\mathrm{Cl}(F^-(\mathrm{Int}(K_1)) \cup F^+(\mathrm{Int}(K_2))) \subset F^-(K_1) \cup F^+(K_2).$$

(3) $\Rightarrow$ (4):Let B_1, B_2 be any subsets of Y. Then $\mathrm{Cl}(B_1)$, $\mathrm{Cl}(B_2)$ are closed sets of Y and by (3) we obtain m_X-$\mathrm{Cl}(F^-(\mathrm{Int}(\mathrm{Cl}(B_1))) \cup F^+(\mathrm{Int}(\mathrm{Cl}(B_2)))) \subset F^-(\mathrm{Cl}(B_1)) \cup F^+(\mathrm{Cl}(B_2))$.

(4) $\Rightarrow$ (5): Let B_1, B_2 be any subsets in Y. Then by (4) and Lemma 3.1 we have
$$F^-(\mathrm{Int}(B_1)) \cap F^+(\mathrm{Int}(B_2)) = X - [F^+(\mathrm{Cl}(Y - B_1)) \cup F^-(\mathrm{Cl}(Y - B_2))] \subset X - m_X\text{-}\mathrm{Cl}(F^+(\mathrm{Int}(\mathrm{Cl}(Y - B_1))) \cup F^-(\mathrm{Int}(\mathrm{Cl}(Y - B_2)))) = X - m_X\text{-}\mathrm{Cl}(F^+(Y - \mathrm{Cl}(\mathrm{Int}(B_1))) \cup F^-(Y - \mathrm{Cl}(\mathrm{Int}(B_2)))) = X - m_X\text{-}\mathrm{Cl}[(X - F^-(\mathrm{Cl}(\mathrm{Int}(B_1)))) \cup (X - F^+(\mathrm{Cl}(\mathrm{Int}(B_2))))] = X - m_X\text{-}\mathrm{Cl}(X - [F^-(\mathrm{Cl}(\mathrm{Int}(B_1)))) \cap F^+(\mathrm{Cl}(\mathrm{Int}(B_2)))]) = m_X\text{-}\mathrm{Int}(F^-(\mathrm{Cl}(\mathrm{Int}(B_1))) \cap F^+(\mathrm{Cl}(\mathrm{Int}(B_2)))).$$
Thus, we obtain $F^+(\mathrm{Int}(B_1)) \cap F^-(\mathrm{Int}(B_2)) \subset m_X$-$\mathrm{Int}(F^+(\mathrm{Cl}(B_1)) \cap F^-(\mathrm{Cl}(B_2))))$.

(5) $\Rightarrow$ (2): This is obvious.

(2) $\Rightarrow$ (1): Let G_1, G_2 be any open sets of Y such that $F(x) \in G_1^+ \cap G_2^-$. Then $x \in F^+(G_1) \cap F^-(G_2) \subset m_X$-$\mathrm{Int}(F^+(\mathrm{Cl}(G_1)) \cap F^-(\mathrm{Cl}(G_2)))$. Then there exists $U \in m_X$ such that $x \in U \subset F^+(\mathrm{Cl}(G_1)) \cap F^-(\mathrm{Cl}(G_2))$. Therefore, $F(u) \subset \mathrm{Cl}(G_1)$ and $F(u) \cap \mathrm{Cl}(G_2) \neq \emptyset$ for every $u \in U$. Hence F is weakly m-continuous.

(4) $\Rightarrow$ (6): Let G_1, G_2 be any open sets of Y. Then we obtain m_X-$\mathrm{Cl}(F^-(G_1) \cup F^+(G_2)) \subset m_X$-$\mathrm{Cl}(F^-(\mathrm{Int}(\mathrm{Cl}(G_1))) \cup F^+(\mathrm{Int}(\mathrm{Cl}(G_2)))) \subset F^-(\mathrm{Cl}(G_1)) \cup F^+(\mathrm{Cl}(G_2))$.

(6) $\Rightarrow$ (2): Let G_1, G_2 be any open sets of Y. Then we have
$$F^+(G_1) \cap F^-(G_2) \subset F^+(\mathrm{Int}(\mathrm{Cl}(G_1))) \cap F^-(\mathrm{Int}(\mathrm{Cl}(G_2))) = X - [F^-(\mathrm{Cl}(Y - \mathrm{Cl}(G_1))) \cup F^+(\mathrm{Cl}(Y - \mathrm{Cl}(G_2)))] \subset X - m_X\text{-}\mathrm{Cl}[F^-(Y - \mathrm{Cl}(G_1)) \cup F^+(Y - \mathrm{Cl}(G_2))] = m_X\text{-}\mathrm{Int}(F^+(\mathrm{Cl}(G_1)) \cap F^-(\mathrm{Cl}(G_2))).$$
Therefore, we obtain $F^+(G_1) \cap F^-(G_2) \subset m_X$-$\mathrm{Int}(F^+(\mathrm{Cl}(G_1)) \cap F^-(\mathrm{Cl}(G_2)))$.

Theorem 3.2 *For a multifunction $F : (X, m_X) \to (Y, \sigma)$, the following are equivalent:*

(1) F is weakly m-continuous;

(2) m_X-$\mathrm{Cl}(F^-(\mathrm{Int}(\mathrm{Cl}_\theta(B_1))) \cup F^+(\mathrm{Int}(\mathrm{Cl}_\theta(B_2)))) \subset F^-(\mathrm{Cl}_\theta(B_1)) \cup F^+(\mathrm{Cl}_\theta(B_2))$ *for every subsets B_1, B_2 of Y;*

(3) m_X-$\mathrm{Cl}(F^-(\mathrm{Int}(\mathrm{Cl}(B_1))) \cup F^+(\mathrm{Int}(\mathrm{Cl}(B_2)))) \subset F^-(\mathrm{Cl}_\theta(B_1)) \cup F^+(\mathrm{Cl}_\theta(B_2))$ *for every subsets B_1, B_2 of Y;*

(4) m_X-$\mathrm{Cl}(F^-(\mathrm{Int}(\mathrm{Cl}(G_1))) \cup F^+(\mathrm{Int}(\mathrm{Cl}(G_2)))) \subset F^-(\mathrm{Cl}(G_1)) \cup F^+(\mathrm{Cl}(G_2))$ *for every open sets G_1, G_2 of Y;*

(5) m_X-$\mathrm{Cl}(F^-(\mathrm{Int}(\mathrm{Cl}(V_1))) \cup F^+(\mathrm{Int}(\mathrm{Cl}(V_2)))) \subset F^-(\mathrm{Cl}(V_1)) \cup F^+(\mathrm{Cl}(V_2))$ *for every preopen sets V_1, V_2 of Y;*

(6) m_X-$\mathrm{Cl}(F^-(\mathrm{Int}(K_1)) \cup F^+(\mathrm{Int}(K_2))) \subset F^-(K_1) \cup F^+(K_2)$ *for every regular closed sets K_1, K_2 of Y.*

Proof. (1) $\Rightarrow$ (2): Let B_1, B_2 be any subsets of Y. Then $\mathrm{Cl}_\theta(B_1)$ and $\mathrm{Cl}_\theta(B_2)$ are closed in Y. Therefore, by Theorem 3.1 we obtain

$$m_X\text{-}\mathrm{Cl}[F^-(\mathrm{Int}(\mathrm{Cl}_\theta(B_1))) \cup F^+(\mathrm{Int}(\mathrm{Cl}_\theta(B_2)))] \subset F^-(\mathrm{Cl}_\theta(B_1)) \cup F^+(\mathrm{Cl}_\theta(B_2)).$$

(2) $\Rightarrow$ (3): This is obvious since $\mathrm{Cl}(B) \subset \mathrm{Cl}_\theta(B)$ for every subset B of Y.

(3) $\Rightarrow$ (4): This is obvious since $\mathrm{Cl}(G) = \mathrm{Cl}_\theta(G)$ for every open set G of Y.

(4) $\Rightarrow$ (5): Let V_1, V_2 be any preopen sets of Y. Then since $V_i \subset \mathrm{Int}(\mathrm{Cl}(V_i))$, we have $\mathrm{Cl}(V_i) = \mathrm{Cl}(\mathrm{Int}(\mathrm{Cl}(V_i)))$ for $i = 1, 2$. Now, set $G_i = \mathrm{Int}(\mathrm{Cl}(V_i))$, then G_i is open in Y and $\mathrm{Cl}(G_i) = \mathrm{Cl}(V_i)$. Therefore, by (4) we obtain m_X-$\mathrm{Cl}(F^-(\mathrm{Int}(\mathrm{Cl}(V_1))) \cup F^+(\mathrm{Int}(\mathrm{Cl}(V_2)))) \subset F^-(\mathrm{Cl}(V_1)) \cup F^+(\mathrm{Cl}(V_2))$.

(5) $\Rightarrow$ (6): Let K_1, K_2 be any regular closed sets of Y. Then we have $\mathrm{Int}(K_1) \in \mathrm{PO}(Y)$ and $\mathrm{Int}(K_2) \in \mathrm{PO}(Y)$ and hence by (5) m_X-$\mathrm{Cl}(F^-(\mathrm{Int}(K_1)) \cup F^+(\mathrm{Int}(K_2))) = m_X$-$\mathrm{Cl}(F^-(\mathrm{Int}(\mathrm{Cl}(\mathrm{Int}(K_1)))) \cup F^+(\mathrm{Int}(\mathrm{Cl}(\mathrm{Int}(K_2))))) \subset F^-(K_1) \cup F^+(K_2)$.

(6) $\Rightarrow$ (1): Let G_1, G_2 be any open sets of Y. Then $\mathrm{Cl}(G_1)$ and $\mathrm{Cl}(G_2)$ are regular closed sets of Y. Therefore, we obtain m_X-$\mathrm{Cl}(F^-(G_1) \cup F^+(G_2)) \subset m_X$-$\mathrm{Cl}[F^-(\mathrm{Int}(\mathrm{Cl}(G_1))) \cup F^+(\mathrm{Int}(\mathrm{Cl}(G_2)))] \subset F^-(\mathrm{Cl}(G_1)) \cup F^+(\mathrm{Cl}(G_2))$. It follows from Theorem 3.1 that F is weakly m-continuous.

Theorem 3.3 *For a multifunction $F : (X, m_X) \to (Y, \sigma)$, the following are equivalent:*

(1) F is weakly m-continuous;

(2) m_X-$\mathrm{Cl}(F^-(\mathrm{Int}(\mathrm{Cl}(G_1))) \cup F^+(\mathrm{Int}(\mathrm{Cl}(G_2)))) \subset F^-(\mathrm{Cl}(G_1)) \cup F^+(\mathrm{Cl}(G_2))$ *for every $G_1, G_2 \in \beta(Y)$;*

(3) m_X-$\mathrm{Cl}(F^-(\mathrm{Int}(\mathrm{Cl}(G_1))) \cup F^+(\mathrm{Int}(\mathrm{Cl}(G_2)))) \subset F^-(\mathrm{Cl}(G_1)) \cup F^+(\mathrm{Cl}(G_2))$ *for every* $G_1, G_2 \in \mathrm{SO}(Y)$.

Proof. (1) $\Rightarrow$ (2): Let $G_1, G_2 \in \beta(Y)$. Then $G_i \subset \mathrm{Cl}(\mathrm{Int}(\mathrm{Cl}(G_i)))$ and $\mathrm{Cl}(G_i) = \mathrm{Cl}(\mathrm{Int}(\mathrm{Cl}(G_i)))$ for $i = 1, 2$. Since $\mathrm{Cl}(G_1)$ and $\mathrm{Cl}(G_2)$ are regular closed sets, by Theorem 3.2 we have m_X-$\mathrm{Cl}(F^-(\mathrm{Int}(\mathrm{Cl}(G_1))) \cup F^+(\mathrm{Int}(\mathrm{Cl}(G_2)))) \subset F^-(\mathrm{Cl}(G_1)) \cup F^+(\mathrm{Cl}(G_2))$.

(2) $\Rightarrow$ (3): This is obvious since $\mathrm{SO}(Y) \subset \beta(Y)$.

(3) $\Rightarrow$ (1): For any $G \in \mathrm{PO}(Y)$, $\mathrm{Cl}(G)$ is regular closed and $\mathrm{Cl}(G) \in \mathrm{SO}(Y)$. Then m_X-$\mathrm{Cl}(F^-(\mathrm{Int}(\mathrm{Cl}(G_1))) \cup F^+(\mathrm{Int}(\mathrm{Cl}(G_2)))) \subset F^-(\mathrm{Cl}(G_1)) \cup F^+(\mathrm{Cl}(G_2))$. By Theorem 3.2, F is weakly m-continuous.

Remark 3.4 If $F : (X, \tau) \to (Y, \sigma)$ a multifunction and $m_X = \mathrm{SO}(X)$ (resp. $\mathrm{PO}(X)$, $\beta(X)$), then by Theorems 3.1-3.3 we can obtain characterizations established in [20], [37] (resp. [35], [36]).

4 Weakly m-continuity and m-continuity

Definition 4.1 A multifunction $F : (X, m_X) \to (Y, \sigma)$ is said to be *almost m-continuous* if for each point $x \in X$ and each open sets G_1, G_2 of Y such that $F(x) \in G_1^+ \cap G_2^-$, there exists $U \in m_X$ containing x such that $F(u) \subset \mathrm{Int}(\mathrm{Cl}(G_1))$ and $F(u) \cap \mathrm{Int}(\mathrm{Cl}(G_2)) \neq \emptyset$ for every $u \in U$.

Remark 4.1 For a multifunction the following implications hold:

m-continuity $\Rightarrow$ almost m-continuity $\Rightarrow$ weak m-continuity.

Theorem 4.1 *If $F : X \to Y$ is weakly m-continuous and $F(x)$ is open in Y for each point $x \in X$, then F is almost m-continuous.*

Proof. Let $x \in X$ and G_1, G_2 be open sets in Y such that $F(x) \in G_1^+ \cap G_2^-$. Since F is weakly m-continuous, there exists $U \in m_X$ containing x such that $F(u) \subset \mathrm{Cl}(G_1)$ and $F(u) \cap \mathrm{Cl}(G_2) \neq \emptyset$ for every $u \in U$. Since $F(x)$ is open for each $x \in X$, $F(u)$ is open and $F(u) \subset \mathrm{Int}(\mathrm{Cl}(G_1))$. Moreover, $F(u) \cap \mathrm{Cl}(G_2) \neq \emptyset$ implies $F(u) \cap \mathrm{Int}(\mathrm{Cl}(G_2)) \neq \emptyset$ for each $u \in U$. Therefore, F is almost m-continuous.

Remark 4.2 If $F : (X, \tau) \to (Y, \sigma)$ a multifunction and $m_X = \mathrm{SO}(X)$ (resp. $\mathrm{PO}(X)$, $\beta(X)$), then by Theorem 4.1 we can obtain the results established in [34] (resp. [32], [33]).

Definition 4.2 A subset A of a topological space (X, τ) is said to be

(1) *α-regular* [11] if for each $a \in A$ and each open set U containing a, there exists an open set G of X such that $a \in G \subset \mathrm{Cl}(G) \subset U$,

(2) *α-almost regular* [12] if for each $a \in A$ and each regular open set U containing a, there exists an open set G of X such that $a \in G \subset \mathrm{Cl}(G) \subset U$,

(3) *α-paracompact* [40] if every X-open cover of A has an X-open refinement which covers A and is locally finite for each point of X.

Lemma 4.1 (Kovačević [11]) *If A is an α regular α-paracompact set of a topological space (X, τ) and U is an open neighborhood of A, then there exists an open set G of X such that $A \subset G \subset \mathrm{Cl}(G) \subset U$.*

Lemma 4.2 (Popa and Noiri [31]) *If A is an α-almost regular α-paracompact set of X and U is a regular open neighborhood of A, then there exists an open set G of X such that $A \subset G \subset \mathrm{Cl}(G) \subset U$.*

Lemma 4.3 (Popa [26]) *If A is an α-almost regular set of a topological space (X, τ) and U is a regular open set such that $U \cap A \neq \emptyset$, then there exists an open set G of X such that $A \cap G \neq \emptyset$ and $\mathrm{Cl}(G) \subset U$.*

Lemma 4.4 (Popa [27]) *If A is an α-regular set of a topological space (X, τ) and U is a regular open set such that $U \cap A \neq \emptyset$, for every open set D which intersect A, there exists and open set D_A such that $A \cap D_A \neq \emptyset$ and $Cl(D_A) \subset D$.*

Theorem 4.2 *A multifunction $F : (X, m_X) \to (Y, \sigma)$ is almost m-continuous if and only if $F^+(G_1) \cap F^-(G_2) = m_X\text{-}\mathrm{Int}(F^+(G_1) \cap F^-(G_2))$ for every regular open sets G_1, G_2 of Y.*

Proof. *Necessity.* Let G_1, G_2 be regular open sets of Y and $x \in F^+(G_1) \cap F^-(G_2)$. Then there exists $U \in m_X$ containing x such that $F(u) \subset G_1$ and $F(u) \cap G_2 \neq \emptyset$ for every $u \in U$. Therefore, we have $x \in U \subset F^+(G_1) \cap F^-(G_2)$ and hence $F^+(G_1) \cap F^-(G_2) \subset m_X\text{-}\mathrm{Int}(F^+(G_1) \cap F^-(G_2))$. By Lemma 3.1, it follows that $F^+(G_1) \cap F^-(G_2) = m_X\text{-}\mathrm{Int}(F^+(G_1) \cap F^-(G_2))$.

Sufficiency. Let G_1, G_2 be any open sets of Y such that $F(x) \in G_1^+ \cap G_2^-$. Then $\mathrm{Int}(\mathrm{Cl}(G_1))$, $\mathrm{Int}(\mathrm{Cl}(G_2))$ are regular open sets of Y and

$$F(x) \in (\mathrm{Int}(\mathrm{Cl}(G_1)))^+ \cap (\mathrm{Int}(\mathrm{Cl}(G_2)))^-.$$

By hypothesis, we have

$$x \in F^+(\mathrm{Int}(\mathrm{Cl}(G_1))) \cap F^-(\mathrm{Int}(\mathrm{Cl}(G_2))) = m_X\text{-}\mathrm{Int}(F^+(\mathrm{Int}(\mathrm{Cl}(G_1))) \cap F^-(\mathrm{Int}(\mathrm{Cl}(G_2)))).$$

There exists $U \in m_X$ containing x such that $F(u) \subset \mathrm{Int}(\mathrm{Cl}(G_1))$ and $F(u) \cap \mathrm{Int}(\mathrm{Cl}(G_2)) \neq \emptyset$ for every $u \in U$. Thus F is almost m-continuous.

Theorem 4.3 *If $F : (X, m_X) \to (Y, \sigma)$ is weakly m-continuous and $F(x)$ is an α-almost regular α-paracompact set of Y for each point $x \in X$, then F is almost m-continuous.*

Proof. Let V_1, V_2 be regular open sets of Y and $x \in F^+(V_1) \cap F^-(V_2)$. Then $F(x) \subset V_1$ and $F(x) \cap V_2 \neq \emptyset$. Since $F(x)$ is α-almost regular α-paracompact, by Lemma 4.2 there exists an open set W_1 such that $F(x) \subset W_1 \subset \mathrm{Cl}(W_1) \subset V_1$. By Lemma 4.3, there exists an open set W_2 of Y such that $F(x) \cap W_2 \neq \emptyset$ and $\mathrm{Cl}(W_2) \subset V_2$. Since F is weakly m-continuous, there exists $U \in m_X$ containing x such that $F(u) \subset \mathrm{Cl}(W_1) \subset V_1$ and $F(u) \cap \mathrm{Cl}(W_2) \neq \emptyset$ for every $u \in U$. Therefore, we have $x \in U \subset F^+(V_1) \cap F^-(V_2)$. Hence $F^+(V_1) \cap F^-(V_2) \subset m_X\text{-}\mathrm{Int}(F^+(V_1) \cap F^-(V_2))$. By Lemma 3.1, $F^+(V_1) \cap F^-(V_2) = m_X\text{-}\mathrm{Int}(F^+(V_1) \cap F^-(V_2))$. It follows from Theorem 4.2 that F is almost m-continuous.

Remark 4.3 Let $F : (X, \tau) \to (Y, \sigma)$ be a multifunction. If $m_X = \mathrm{SO}(X)$ (resp. $\mathrm{PO}(X)$, $\beta(X)$), then by Theorem 4.3 we obtain the results established in [34] (resp. [35], [36]).

Theorem 4.4 *For a multifunction $F : (X, m_X) \to (Y, \sigma)$ such that $F(x)$ is an α-regular α-paracompact set of Y for each point $x \in X$, the following are equivalent:*

(1) F is m-continuous;

(2) F is almost m-continuous;

(3) F is weakly m-continuous.

Proof. We show only the implication (3) $\Rightarrow$ (1) since the others are obvious. Suppose that F is weakly m-continuous. Let $x \in X$ and V_1, V_2 be any open sets of Y such that $x \in F^+(V_1) \cap F^-(V_2)$. Then $F(x) \subset V_1$ and $F(x) \cap V_2 \neq \emptyset$. Since $F(x)$ is α-regular α-paracompact, by Lemmas 4.1 and 4.4 there exist open sets W_1, W_2 of Y such that $F(x) \subset W_1 \subset \mathrm{Cl}(W_1) \subset V_1$, $F(x) \cap W_2 \neq \emptyset$ and $\mathrm{Cl}(W_2) \subset V_2$. Since F is weakly m-continuous, there exists $U \in m_X$ containing x such that $F(u) \subset \mathrm{Cl}(W_1)$ and $F(u) \cap \mathrm{Cl}(W_2) \neq \emptyset$ for every $u \in U$. Then $F(u) \subset V_1$ and $F(u) \cap V_2 \neq \emptyset$ for every $u \in U$. This shows that F is m-continuous.

Remark 4.4 Let $F : (X,\tau) \to (Y,\sigma)$ be a multifunction. If $m_X = \mathrm{PO}(X)$ (resp. $\beta(X)$), then by Theorem 4.4 we obtain the results established in [21] (resp. [36]).

Theorem 4.5 *If $F : (X, m_X) \to (Y,\sigma)$ is a closed valued multifunction and (Y,σ) is a normal T_1 space, then the following are equivalent:*
(1) F is m-continuous;
(2) F is almost m-continuous;
(3) F is weakly m-continuous.

Proof. As in Theorem 4.4, we prove only the implication (3) $\Rightarrow$ (1). Suppose that F is weakly m-continuous. Let $x \in X$ and G_1, G_2 be any open sets of Y such that $F(x) \in G_1^+ \cap G_2^-$. Since $F(x)$ is closed in Y, by the normality of Y there exists an open set D of X such that $F(x) \subset D \subset \mathrm{Cl}(D) \subset G_1$. Since every normal T_1 space is T_3 and $F(x) \cap G_2 \neq \emptyset$, there exists an open set E of Y such that $E \cap F(x) \neq \emptyset$ and $\mathrm{Cl}(E) \subset G_2$. Since F is weakly m-continuous and $F(x) \in D^+ \cap E^-$, there exists $U \in m_X$ containing x such that $F(u) \subset \mathrm{Cl}(D)$ and $F(u) \cap \mathrm{Cl}(E) \neq \emptyset$ for every $u \in U$. Therefore, we obtain $F(u) \subset G_1$ and $F(u) \cap G_2 \neq \emptyset$ for every $u \in U$. This shows that F is m-continuous.

Remark 4.5 Let $F : (X,\tau) \to (Y,\sigma)$ be a multifunction. If $m_X = \mathrm{PO}(X)$ (resp. $\beta(X)$), then by Theorem 4.5 we obtain the results established in [21] (resp. [36]).

5 Properties of weakly m-continuous multifunctions

For a multifunction $F : X \to (Y,\sigma)$, by $\mathrm{Cl}(F) : X \to (Y,\sigma)$ [5] we denote a multifunction defined as follows: $\mathrm{Cl}(F)(x) = \mathrm{Cl}(F(x))$ for each $x \in X$. Similarly, we denote $\mathrm{sCl}(F) : X \to (Y,\sigma)$ [23], $\mathrm{pCl}(F) : X \to (Y,\sigma)$ [25], $\alpha\mathrm{Cl}(F) : X \to (Y,\sigma)$ [28], $\beta\mathrm{Cl}(F) : X \to (Y,\sigma)$ [32].

Lemma 5.1 *If $F : (X, m_X) \to (Y,\sigma)$ is a multifunction such that $F(x)$ is α-regular and α-paracompact for each $x \in X$, then*
(1) $G^+(V) = F^+(V)$ for each open set V of Y,
(2) $G^-(K) = F^-(K)$ for each closed set K of Y,
where G denotes $\mathrm{Cl}(F)$, $\mathrm{pCl}(F)$, $\mathrm{sCl}(F)$, $\alpha\mathrm{Cl}(F)$ or $\beta\mathrm{Cl}(F)$.

Proof. (1) Let V be any open set of Y and $x \in G^+(V)$. Then $G(x) \subset V$ and $F(x) \subset G(x) \subset V$. We have $x \in F^+(V)$ and hence $G^+(V) \subset F^+(V)$. Conversely, let $x \in F^+(V)$. Then we have $F(x) \subset V$ and by Lemma 4.1 there exists an open set H of Y such that $F(x) \subset H \subset \mathrm{Cl}(H) \subset V$. Since $G(x) \subset \mathrm{Cl}(F(x))$, $G(x) \subset V$ and hence $x \in G^+(V)$. Thus we obtain $F^+(V) \subset G^+(V)$. Therefore, $G^+(V) = F^+(V)$.

(2) This follows from (1) immediately.

Lemma 5.2 *For a multifunction $F : (X, m_X) \to (Y, \sigma)$, the following properties hold:*

(1) $G^-(V) = F^-(V)$ for each open set V of Y,

(2) $G^+(K) = F^+(K)$ for each closed set K of Y,

where G denotes $\mathrm{Cl}(F)$, $\mathrm{pCl}(F)$, $\mathrm{sCl}(F)$, $\alpha\mathrm{Cl}(F)$ or $\beta\mathrm{Cl}(F)$.

Proof. (1) Let V be any open set of Y and $x \in G^-(V)$. Then $G(x) \cap V \neq \emptyset$ and hence $F(x) \cap V \neq \emptyset$ since V is open. We have $x \in F^-(V)$ and hence $G^-(V) \subset F^-(V)$. Conversely, let $x \in F^-(V)$. Then we have $\emptyset \neq F(x) \cap V \subset G(x) \cap V$ and hence $x \in G^-(V)$. Thus we obtain $F^-(V) \subset G^-(V)$. Therefore, $F^-(V) = G^-(V)$.

(2) This follows from (1) immediately.

Theorem 5.1 *Let $F : (X, m_X) \to (Y, \sigma)$ be a multifunction such that $F(x)$ is α-regular and α-paracompact for each $x \in X$. Then the following properties are equivalent:*

(1) F is weakly m-continuous;

(2) $\mathrm{Cl}(F)$ is weakly m-continuous;

(3) $\mathrm{sCl}(F)$ is weakly m-continuous;

(4) $\mathrm{pCl}(F)$ is weakly m-continuous;

(5) $\alpha\mathrm{Cl}(F)$ is weakly m-continuous;

(6) $\beta\mathrm{Cl}(F)$ is weakly m-continuous.

Proof. We put $G = \mathrm{Cl}(F)$, $\mathrm{pCl}(F)$, $\mathrm{sCl}(F)$, $\alpha\mathrm{Cl}(F)$ or $\beta\mathrm{Cl}(F)$ in the sequel.

Necessity. Suppose that F is weakly m-continuous. Then it follows from Theorem 3.1 and Lemmas 5.1 and 5.2 that for every open sets V_1 and V_2 of Y, $G^+(V_1) \cap G^-(V_2) = F^+(V_1) \cap F^-(V_2) \subset m_X\text{-Int}(F^+(\mathrm{Cl}(V_1)) \cap F^-(\mathrm{Cl}(V_2))) = m_X\text{-Int}(G^+(\mathrm{Cl}(V_1)) \cap G^-(\mathrm{Cl}(V_2)))$. By Theorem 3.1, G is weakly m-continuous.

Sufficiency. Suppose that G is weakly m-continuous. Then it follows from Theorem 3.1 and Lemmas 5.1 and 5.2 that for every open sets V_1 and V_2 of Y, $F^+(V_1) \cap F^-(V_2) = G^+(V_1) \cap G^-(V_2) \subset m_X\text{-Int}(G^+(\text{Cl}(V_1)) \cap G^-(\text{Cl}(V_2))) = m_X\text{-Int}(F^+(\text{Cl}(V_1)) \cap F^-(\text{Cl}(V_2)))$. It follows from Theorem 3.1 that F is weakly m-continuous.

Remark 5.1 Let $F : (X, \tau) \to (Y, \sigma)$ be a multifunction. For a multifunction $F : (X, m_X) \to (Y, \sigma)$ and $m_X = \text{PO}(X)$ (resp. $\beta(X)$), by Theorem 5.1 we obtain the results established in [35] (resp. [36]).

Definition 5.1 Let X be a nonempty set and m_X an m-structure on X. For a subset A of X, the *m_X-frontier* of A, m_X-Fr(A), is defined as follows:

$$m_X\text{-Fr}(A) = m_X\text{-Cl}(A) \cap m_X\text{-Cl}(X - A).$$

Lemma 5.3 (Popa and Noiri [37]) *Let X be a nonempty set with an m-structure m_X and A a subset of X. Then, $x \in m_X\text{-Cl}(A)$ if and only if $U \cap A \neq \emptyset$ for every $U \in m_X$ containing x.*

Theorem 5.2 *The set of all points $x \in X$ at which a multifunction $F : (X, m_X) \to (Y, \sigma)$ is not weakly m-continuous is identical with the union of the m_X-frontiers of the intersection of upper/lower inverse images of the closures of open sets containing/meeting $F(x)$.*

Proof. Let x be a point of X at which F is not weakly m-continuous. Then there exist two open sets V_1, V_2 of Y such that $x \in F^+(V_1) \cap F^-(V_2)$ and $U \cap (X - (F^+(\text{Cl}(V_1)) \cap F^-(\text{Cl}(V_2)))) \neq \emptyset$ for every $U \in m_X$ containing x. By Lemma 5.3, $x \in m_X\text{-Cl}(X - (F^+(\text{Cl}(V_1)) \cap F^-(\text{Cl}(V_2))))$. On the other hand, we have $x \in F^+(V_1) \cap F^-(V_2) \subset (F^+(\text{Cl}(V_1)) \cap F^-(\text{Cl}(V_2))$. Thus we obtain $x \in m_X\text{-Fr}(F^+(\text{Cl}(V_1)) \cap F^-(\text{Cl}(V_2)))$.

Conversely, suppose that F is weakly m-continuous at $x \in X$ and let V_1, V_2 be any open sets of Y such that $x \in F^+(V_1) \cap F^-(V_2)$. Then by Theorem 3.1, we have $x \in m_X\text{-Int}(F^+(\text{Cl}(V_1)) \cap F^-(\text{Cl}(V_2)))$. Therefore, $x \notin m_X\text{-Fr}(F^+(\text{Cl}(V_1)) \cap F^-(\text{Cl}(V_2)))$. This completes the proof.

6 Applications

There are many modifications of open sets in topological spaces. Recently, many researchers are very interested in δ-preopen sets, δ-semi-open sets and b-open sets. The definition of b-open sets as follows: A subset A of a topological space (X, τ) is said to be *b-open* [4] if $A \subset \text{Int}(\text{Cl}(A)) \cup \text{Cl}(\text{Int}(A))$.

Among sets in Definition 2.1 and these sets, the following relationships are known:

$$\begin{array}{ccccccc}
 & & \text{open} \Rightarrow & \alpha\text{-open} & \Rightarrow & \text{preopen} & \Rightarrow \delta\text{-preopen} \\
 & & & \Downarrow & & \Downarrow & \\
\delta\text{-semi-open} & \Rightarrow & & \text{semi-open} & \Rightarrow & b\text{-open} & \Rightarrow \beta\text{-open}
\end{array}$$

For each of modifications of open sets containing sets stated above, we can define a new type of weakly m-continuous multifunctions. For instance, we can define as follows:

Definition 6.1. Let (X,τ) and (Y,σ) be topological spaces. A multifunction $F:(X,\tau)\to(Y,\sigma)$ is said to be *weakly super continuous* (resp. *weakly δ-precontinuous, weakly δ-semicontinuous, weakly b-continuous*) if for each point $x \in X$ and each open sets V_1, V_2 of Y such that $F(x) \in V_1^+ \cap V_2^-$, there exists a δ-open (resp. δ-preopen, δ-semi-open, b-open) set U of (X,τ) containing x such that $F(u) \in (\mathrm{Cl}(V_1))^+ \cap (\mathrm{Cl}(V_2))^-$ for every $u \in U$.

We can obtain their characterizations and properties from Sections 3 and 4. For example, in case $m_X = \delta\mathrm{SO}(X)$ by Theorem 3.1 we obtain characterizations of weakly δ-semicontinuous multifunctions.

Theorem 6.1 *For a multifunction $F:(X,\tau)\to(Y,\sigma)$, the following properties are equivalent:*

(1) F is weakly delta-semicontinuous-continuous;

(2) $F^+(G_1)\cap F^-(G_2) \subset \mathrm{sInt}_\delta(F^+(\mathrm{Cl}(G_1))\cap F^-(\mathrm{Cl}(G_2)))$ for every open sets G_1, G_2 of Y;

(3) $\mathrm{sCl}_\delta(F^-(\mathrm{Int}(K_1)) \cup F^+(\mathrm{Int}(K_2))) \subset F^-(K_1) \cup F^+(K_2)$ for every closed sets K_1, K_2 of Y;

(4) $\mathrm{sCl}_\delta(F^-(\mathrm{Int}(\mathrm{Cl}(B_1)))\cup F^+(\mathrm{Int}(\mathrm{Cl}(B_2)))) \subset F^-(\mathrm{Cl}(B_1))\cup F^+(\mathrm{Cl}(B_2))$ for every subsets B_1, B_2 of Y;

(5) $F^+(\mathrm{Int}(B_1))\cap F^-(\mathrm{Int}(B_2)) \subset \mathrm{sInt}_\delta(F^+(\mathrm{Cl}(B_1))\cap F^-(\mathrm{Cl}(B_2))))$ for every subsets B_1, B_2 of Y;

(6) $\mathrm{sCl}_\delta(F^-(G_1)\cup F^+(G_2)) \subset F^-(\mathrm{Cl}(G_1))\cup F^+(\mathrm{Cl}(G_2))$ for every open sets G_1, G_2 of Y.

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