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TRIANGULAR CONVEXITY [1)]

by

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In this paper a condition modifying the standard convexity conditions in mathematical literature up to now and so called "(m,n)-triangular convexity" has been introduced and obtained some results about it.

CHAPTER ONE

GENERALITIES

I. The prolific nature of the definition of ordinary convexity has given rise to a considerable amount of research and this because the condition imposing that the line segments determined by any two points of a set belong to the set has the possibility of being weakened in various ways to reveal several interesting new properties.

One way of doing this is to consider the property mentioned above to be true only for the intersection of the set with the neighborhood of a point belonging to the set which, in fact, gives rise to local convexity at the point. Some other ways are the consequences of certain orders and restrictions imposed on the points belonging to the set.

The cases which have proved to be the most interesting up to now are the ones that correspond to one of the following conditions :

i) For each m distinct points of a set S $(m \geq 2)$, at least one of the $\binom{m}{2}$ possible segments determined by those points is contained in S.

ii) For each m distinct points of a set S $(m \geq 2)$, at least n of the $\binom{m}{2}$ possible segments determined by those points for $1 \leq n \leq \binom{m}{2}$ are contained in S.

For $m = 2$ the above condition reduces to the one used to define convexity in the ordinary sense.

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iii) For each pair of points in S there exists a sequence of no more than m-1 points in S such that the given two points may be joined by a polygonal line having m or fewer sides, all contained in S.

iv) For each m distinct points of S, with m odd, taken in a given order and such that the line joining any two consecutive points is not in S, the initial and final points of this m-ple are joined by a line contained in S.

Of the above, case *i* has been examined by several authors, mainly by Valentine [3] for $m = 3$ and it is named the *three point convexity property* whereas for arbitrary m points the set is said *m-convex*: alternatively it is often said that the set has in this case the *property* P_m [2]; case *ii* has been examined in detail mainly by David C. Kay and Merle D. Guay [1], and called by these authors *(m, n)-convexity*. If a set has the property defined by the condition *iii* this set is said to have the *property* L_m, whereas if a set verifies the condition mentioned in the case *iv* it is said to have the *property* P_0. Sets having the property L_m have been examined in the papers [1], [2], [5] and sets having the property P_0 in [7] and [8].

All the conditions mentioned above use different combinations of line segments defined by points belonging to the set and require that these segments lie entirely in the set or at least guarantee this for a certain number of them. Hence new combinations appear to be inevitable foundation of new properties.

It is the purpose of this work to introduce a new condition, reached in a similar manner, and to deduce properties from this condition as well as interrelations between this property and the ones listed above.

Although a Hausdorff linear space L over the reals can be assumed as a setting for what follows, it may be sometimes necessary to restrict the discussion to sets S contained in 2-dimensional Euclidean space E^2.

Throughout notations used by F.A. Valentine in [9] will be used. Furthermore a closed triangle with vertices x, y, z will be denoted by xyz and similarly an open triangle will be denoted by (xyz).

CHAPTER TWO

TRIANGULAR CONVEXITY

1. Our main concern will be the study of the consequences of the condition expressed in the following definition:

1.1. Definition. In any linear space $\mathscr{X}$, a set S will be said to have the T_m^n *property* (or to be *(m, n)-triangular convex*) if it contains at least m points

[2]) By extension, sets that are (m, n)-convex will be said to have the property P_m^n.

such that $m \geq 3$ and if for each m distinct points of S at least n of the $\binom{m}{3}$ possible closed triangles determined by those points are contained in S, m being the lowest and n being the highest integer giving rise to such a property.

Obviously this condition has a meaning for values of the integers m and n related by the inequality

$$1 \leq n \leq \frac{m(m-1)(m-2)}{6} . \tag{1.1.1}$$

Furthermore it is necessary to assume that

$$\dim_{IR} \mathscr{X} \geq 2.$$

1.2. Remarks :

i) If a set S is $(m, 1)$-triangular convex (or equivalently, has the T_m^1 property) it will be said to be *m-triangular convex* or to have the T_m *property.*

ii) *Every convex set K in $\mathscr{X}$ has the T_3 property.*

iii) Conversely, *if a set S in $\mathscr{X}$ has the property T_3, S must be convex.*

2. It is possible to exhibit examples of sets that are (m,n)-convex without having the T_r^s property for some pairs of integers (r, s) satisfying a condition of the form $3 \leq r \leq m$, $s \leq n$.

2.1. Consider a set S which is the union of a convex set K and p isolated points $k_1, \ldots, k_p$

$$S = K \cup \{k_1, \ldots, k_p\} .$$

This set is obviously $(p+2,1)$-convex, where $m = p + 2$, $n = 1$. However S lacks any T_r^s property for $3 \leq r \leq p + 2$ since S is $(p + 3)$-triangular convex.

2.2. A similar situation can arise with connected sets as can be seen in the following example: Let $A = (abc)$ denote an open triangle and call S the set obtained as the union of A with its three vertices

$$S = A \cup \{a, b, c\} .$$

Then S is neither T_4 nor T_3 though it is (4,3)-convex and (5,3)-triangular convex.

2.3. An open polygon to which are added some of its vertices chosen in a convenient order can supply a suitable example of a set having the T_m^n property for any pair of integers m, n satisfying (1.1.1). In particular the utilization of regular polygons in such examples causes no loss of generality.

For instance the set S obtained as the union of an open hexagon and the set formed by the odd (or even) numbered vertices has the T_3 property and in order to obtain an example of a set having the property T_s $(s > 3)$ it is sufficient to add to the set considered above s-3 isolated points. Similarly, an open heptagon together with four of its vertices (chosen so that three non consecutive vertices are omitted) yields an example of a set that is T_4^2. In order to obtain a set having the property T_s^2 $(s > 4)$, again it is sufficient to add s-4 isolated points to the above set.

3. In order to make a comparison between (m,n)-triangular convexity and (p, q)-convexity it is convenient to return to example 2.2. S is an example of a set that has the P_4^3 property without having any one of the properties P_2, P_3, P_4 [3]. Although for every choice of 4 points in S at least 3 segments, determined by pairs chosen amongst these 4 points are contained in S, it is obvious that any quatuor of points, taken in S does not define even one closed triangle contained in S. This example is sufficient to show that the property T_m^n is a stronger condition (m, s)-convexity.

3.1. Proposition. The connected union of m closed convex sets which is $(m + 1,1)$-convex is not $(s,1)$-triangular convex for any s satisfying the inequality $3 \leq s < 2m + 1$.

Before giving a proof of the above statement it is helpful to go over an example to show that the condition of $(m+1,1)$ convexity, imposed on the union of the m closed convex sets, is necessary.

The closed star-like pentagon which can be expressed as the union of no less than 3 closed convex sets has the P_3 property but it has neither T_5, T_4 nor T_3 property since it is $(6,1)$-triangular convex whereas the set which consists of the union of two closed convex sets is $(5,1)$-triangular convex and accordingly which has not T_4, T_3 property still has the P_3 property. In the first example the set is connected union of 3 closed convex sets and it does not fulfil the requirement of $(4,1)$-convexity hence the assertion of the proposition does not hold. However in the second example since the required condition is satisfied, that is, since the union S has the P_3 property for $m = 2$, the set has also, as claimed, the T_n^s property for $s = 2m + 1$.

To prove the statement, let S be the connected union of m closed, connected, convex sets K_i $(i=1,\ldots,n)$ which is $(m+1,1)$-convex. In this case since $S = \bigcup_{i=1}^{m} K_i$ is connected, none of the components of S can be isolated points. Therefore it is always possible, because of the convexity of the sets K_i, to find two distinct

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[3]) It is sometimes argued that any set that has the P_m^n property has the P_m property; however in this paper the definition will be accepted to mean that n is the integer that denotes the greatest number of closed segments that any m-ple of points chosen in S can define, where m is the least integer giving rise to such a property.

sets of m points belonging to S, such that none of the segments determined by any two points of one such set is contained in S. Let any two such subsets of S containing exactly m points be

$$\Sigma_1 = \{p_1, p_2, \ldots, p_m\}$$
$$\Sigma_2 = \{q_1, q_2, \ldots, q_m\},$$

where $p_i, q_i \in K_i$ and $p_i p_j \not\subset S$, $q_i q_j \not\subset S$ for $1 \leq i, j \leq m$. Thus it is possible to get m segments contained in S without forming a triangle contained in S by means of the points belonging to the sets Σ_1 and Σ_2. To verify this assertion call S_0 the subset of S defined as follows:

$$S_0 = \Sigma_1 \cup \Sigma_2 = \{p_1, q_1, \ldots, p_m, q_m\}.$$

Since K_i is convex for each $i = 1, \ldots, m$, every pair (p_i, q_i) of S_0 determines a segment $p_i q_i \subset K_i$. Hence $p_i q_i$ is contained in S. However for pairs such as (p_i, q_j) no such thing can be asserted. For any triple $\{p_i, p_j, q_k\}$ or $\{p_i, q_j, q_k\}$ (where $i \neq j \neq k$) the segments $p_i p_j$ or $q_j q_k$ will not be contained in S and trivially no segments will be contained in S for $\{p_i, p_j, p_k\}$ or $\{q_i, q_j, q_k\}$ $(i, j, k = 1, \ldots, m)$, because of the hypothesis. Therefore no triangle contained in S can be obtained when using as vertices 3 of the $2m$ points of S_0. So S_0 which is formed by $2m$ points of S gives rise to at least m segments all contained in S, but to no triangle. If we add any $s \in S$ to S_0 and denote this new subset of S as

$$S_1 = S_0 \cup \{s\}$$

we will obtain two or more segments in addition to the m segments all contained in S, depending whether $s \in K_i \setminus \text{Ker}.S$ or $s \in \text{Ker}.S$. In the first case, that is if $s \in K_i \setminus \text{Ker}.S$, s, $p_i, q_i \in K_i$ will form $sp_i q_i \subset K_i \subset S$ due to the convexity of K_i. In the second case a greater number of triangles will be obtained but this is really irrelevant since we are interested in computing the least number of triangles contained in S by using $2m+1$ points of S. Thus S has the T_{2m+1} property.

4. The previous remarks enable us to conclude that though there exist sets that are (m, n)-convex without being (r, s)-triangular convex, the converse is never true: *if a set is (m, n)-triangular convex then it is (r, s)-convex.* However a determination of some limitation for the integers r and s appears to be quite difficult. Only for sets which are $(m, 1)$-triangular convex the following inequalities hold:

$$3 \leq r < m \quad , \quad 1 \leq s < 3 .$$

For example a closed set formed by two closed triangles having a vertex in common is $(5,1)$-triangular convex and $(3,1)$-convex. Nevertheless a closed triangle and two isolated points also form a set which is $(5,1)$-triangular convex but $(4,1)$-convex.

When r is taken equal to m, s will have as lower bound 3, but as its upper bound it will have some value M, such that

$$M \geq 3n.$$

It is possible to express M in terms of m. The following proposition introduces M in terms of m.

4.1. Proposition. A set $S \subset \mathscr{X}$ having the property T_m $(m \geq 3)$ is also (m,s)-convex, where

$$3 \leq s \leq m + \frac{1+(-1)^m}{2}\left[\frac{m-3}{2}\right] + \left[\frac{m-3}{2}\right]^2$$

($[r]$ denotes the greatest integer $\leq r$).

Proof. Consider a set S having the property T_m ; then any subset formed by m points chosen at random in S defines one closed triangle contained in S. The problem is to establish the highest possible number of closed segments contained in S without forming any other triangles.

For this purpose assume that the vertices of the single existing triangle are the points

(a) $$x_1 , x_2 , x_3$$

and the remaining points of the m-ple are

(b) $$x_4 , x_5 , \ldots, x_m .$$

Let σ_1 denote the set formed by the points in (a) , σ_2 that formed by the points shown in (b) and let

$$\Sigma = \sigma_1 \cup \sigma_2 .$$

Furthermore let σ_{2i} denote the set having cardinality n_i of points of σ_2 that are joined to the vertex x_i . Thus

$$\sigma_2 = \sigma_{21} \cup \sigma_{22} \cup \sigma_{23}$$

and

$$\text{card. } \sigma_2 = n_1 + n_2 + n_3 = m - 3.$$

At this stage the question is to investigate all possible combinations amongst m points and, eliminating certain cases after making comparisons between them, to reach the optimal combination that can occur.

i) Suppose n is the maximum number of segments that can be drawn amongst $m - 3$ points of σ_2 without these segments forming a triangle: then the set Σ may define $n + 3$ segments contained in S but this number can be increased by joining a point of σ_2 to a point of σ_1 because any such segment will not give rise to a configuration containing a second triangle. Hence it would be false to assume that the maximum number of segments contained in S is reached by a configuration in which points of o_2 are not joined to points of σ_1. This implies that the configuration having the greatest number of closed segments is reached when all the points of σ_2 are joined to some point of σ_1.

Observe that from any point of σ_2 one can draw at most one segment to a point of σ_2 : indeed if $x_i \in \sigma_2 = \bigcup_{i=1}^{3} \sigma_{2i}$, only one of the segments

$$x_i x_1, \qquad x_i x_2, \qquad x_i x_3$$

belongs to S, otherwise, as $x_2 x_3$, $x_3 x_1$ and $x_2 x_1$ are contained in S by our initial assumption, if more than one of the segments $x_i x_j$ $(j = 1, 2, 3)$ from x_i were drawn, a second closed triangle contained in S would be formed. Therefore it is clear that the segments such as xx_j for $j \neq i$ where $x \in \sigma_{2i}$ and $x_j \in o_1$, $i, j \in \{1, 2, 3\}$ would not be counted when the maximum number of segments contained in S are considered. The only segments that satisfy the restrictions mentioned above and hence suitable to our purpose are the ones that can be drawn between the points of any two of o_{2i}, $i \in \{1, 2, 3\}$; otherwise (that is if we take three points each element of one of the o_{2i}, $i \in \{1, 2, 3\}$) a second triangle, different from $x_1 x_2 x_3$ would be formed against the initial assumption. Under these circumstances in order to obtain the optimal combination it is sufficient to choose the set σ_{2i} with the greatest cardinality amongst $\sigma_{21}, \sigma_{22}, \sigma_{23}$ and multiply its cardinality with the cardinality of the remaining two sets of σ_2 respectively and add to their sum the cardinality of Σ.

Suppose

$$\text{card. } \sigma_{21} = n_1 \leq \text{card. } \sigma_{22} = n_2 \leq \text{card. } \sigma_{23} = n_3$$

then

$$(*) \qquad s(n_1, n_2, n_3) = n_3 n_2 + n_3 n_1 + m$$

will denote the possible maximum number of line segments.

ii) The second stage of our proof is to observe the various kinds of possible distributions of the $m-3$ points according to the segments joining them to one of the three vertices. This distribution can be as follows :

a) Every point belonging to σ_2 can be joined to a single vertex.

b) The $m - 3$ points belonging to σ_2 can be distributed between any two of the three vertices.

c) The distribution of these points may occur between all three vertices.

iii) Assume all the points of σ_2 are joined to the same vertex of σ_1. Then no other segment can be added to the configuration without forming a further closed triangle contained in S, because this segment would be joining two points of σ_2, already joined to one of the points of σ_1. Thus the assumption that all the points of σ_2 are joined to the same vertex of σ_1 yields as maximum number of segments the value m since by (*)

$$s(o, o, m-3) = \text{card.}\, \Sigma\,.$$

This value is not the maximum one attainable (unless $m = 4$, because if $m = 4$ there is no other possibility), as can be shown by considering the case in which one point of σ_2 is joined, say to x_2 and all the remaining $m-4$ points to x_1. Obviously the maximum number of segments that can be drawn in this case is obtained when x_4 (the point joined to x_2) is also joined to the $m-4$ points of σ_2 so that the configuration has $2m-4$ segments. Clearly for $m \geq 5$

$$2m - 4 > m$$

so that the maximum is not reached when all the points of σ_2 are joined to the same vertex, chosen in σ_1.

iv) Under these circumstances it is necessary to investigate the case where the points of o_2 are joined to the two vertices of the triangle $x_1\, x_2\, x_3$.

An "equal distribution" or "almost equal distribution" of the $m-3$ points between any two vertices gives a higher value than an arbitrary distribution.

α) Assume $m - 3 = 2\text{k}$. Then by "equal distribution" we shall understand that k points are to be joined to each of the two vertices. In this case, again from (*)

$$\text{s}(o, k, k) = k^2 + m.$$

If, in fact, $k+r$ points were joined to one vertex and $k-r$ points were joined to any one of the other two vertices, then the maximum number of lines joined that could be drawn for this arbitrary distribution would be

$$s(o, k-r, k+r) = m + k^2 - r^2$$

and

$$s(o, k-r, k+r) < s(o, k, k)\,.$$

β) Assume $m - 3 = 2k + 1$. Then for an "almost equal distribution" (whereby we understand that $k+1$ points are joined to one of the three vertices and k points are joined to another one of the remaining two), we have

$$s(o, k+1, k) = k^2 + k + m\,.$$

Suppose again an arbitrary distribution in which $k+r+1$ points are joined to one vertex and $k-r$ points to the other. The maximum number of lines that could be drawn would be

$$s\,(o, k+r+1, k-r) = k^2 + k - r^2 + m$$

and again

$$s(o, k + r + 1, k - r) \leq s(o, k, k + 1).$$

On the other hand if we are to consider an arbitrary distribution in which $k + r$ points are joined to one vertex and $k + 1 - r$ points are joined to any one of the remaining two

$$\begin{aligned} s(o, k + r, k + 1 - r) &= k^2 + k + m + r - r^2 \\ &\leq s(o, k, k + 1)\,. \end{aligned}$$

It follows that no matter whether $m - 3$ is odd or even, the maximum number of segments is obtained by means of an "equal" or "almost equal distribution".

The results obtained in (α) and (β) can be expressed by means of a single formula by writing

$$s^{(2)}_{max} = m + \frac{1 + (-1)^m}{2}\left[\frac{m-3}{2}\right] + \left[\frac{m-3}{2}\right]^2 .$$

v) An equal distribution of the $m - 3$ points of Σ between the three vertices leads to a number of line segments that is smaller than the one obtained in the case of an equal or almost equal distribution between only two vertices.

Assume $m - 3 = 3p$ and let $s^{(3)}_{max}$ denote the number of segments reached in this case. Then

$$s^{(3)}_{max} = m + p^2 + p^2 = m + 2p^2 .$$

Since

$$p = \frac{m-3}{3}$$

$$s^{(2)}_{max} = m + \frac{1 + (-1)^m}{2}\left[\frac{m-3}{2}\right] + \left[\frac{m-3}{2}\right]^2$$

then

$$\begin{aligned} s^{(2)}_{max} &\geq m + \left[\frac{m-3}{2}\right]^2 = m + \frac{1}{4}(m-3)^2 \\ &> m + \frac{2}{9}(m-3)^2 = s^{(3)}_{max} \end{aligned}$$

$$s^{(2)}_{max} \geq s^{(3)}_{max} .$$

If the distribution between the three vertices is not equal then there are two alternatives. That is either

$$n_1 = n_2 \neq n_3$$

or

$$n_1 \neq n_2 \neq n_3 ,$$

where n_i $(i = 1, 2, 3)$ indicates the cardinality of each subset of Σ such that the elements of these subsets are joined to x_i. Assume that $n_1 = n_2 \neq n_3$ then either

$$n_1 = n_2 > n_3$$

or

$$n_1 = n_2 < n_3 ,$$

where

$$\text{card.}\, \sigma_2 = n_1 + n_2 + n_3 .$$

Assume $n_1 = n_2 > n_3$.

(*i*) First claim is $s(n^*, n^*, (m-3) - 2n^*) \geq s(n_1, n_2, n_3)$, where

$$n^* = \max \{n_i \mid 2n_i + n_j = m - 3, \quad i, j \in (1, 2, 3), i \neq j\} .$$

In order to prove the first claim it is sufficient to show

$$s(n_1, n_1, n_3) < s(n_1 + 1, n_1 + 1, n_3 - 2)$$

for $n_1 = n_2 > n_3$.

$$s(n_1, n_1, n_3) = m + n_1^2 + n_1 n_3$$

$$s(n_1 + 1, n_1 + 1, n_3 - 2) = m + (n_1 + 1)^2 + (n_1 + 1)(n_3 - 2)$$

and

$$m + n_1^2 + n_1 n_3 \leq m + n_1^2 + n_1 n_3 - 1 + n_3 .$$

Equality holds for $n_3 = 1$ and $n_3 = 0$ is excluded since this particular case corresponds to a distribution between two vertices.

(*ii*) Second claim is $s(n^*, n^*, (m-3) - 2n^*) > s(n_3, n_1, n_1)$ for $n_1 = n_2 < n_3$.

(a) If $m - 3 = 2p$ then

$$s(n^*, n^*, (m-3) - 2n^*) = s(p, p, o) = m + p^2$$

and

$$s(n_3, n_1, n_1) = m + 2n_3 n_1 .$$

Since

$$2n_1 + n_3 = 2p \quad \text{and} \quad 2n^* = 2p$$

$$n^* = \frac{2n_1 + n_3}{2} = p .$$

To prove the claim it is sufficient to show

$$m + p^2 \geq m + \frac{(2n_1 + n_3)^2}{4} \geq m + 2n_3 n_1$$

or

$$4n_1^2 + 4 n_1 n_3 + n_3^2 \geq 8n_3 n_1 .$$

Since $(2n_1 - n_3)^2 \geq 0$ the above inequality always holds.

(b) If $m - 3 = 2p + 1$ then

$$s(n^*, n^*, (m-3) - 2n^*) = s(p, p, 1) = m + p^2 + p$$

and for $n_1 = n_2 < n_3$

$$s(n_3, n_1, n_1) = m + 2n_3 n_1 .$$

We can prove the claim using the same argument as before :

$$2n_1 + n_3 = 2p + 1$$

implies

$$s(n_3, n_1, n_1) = m + 2pn_3 + n_3 - n_3^2$$

and this reduces the question to show the validity of the following inequality

$$m + p^2 + p \geq m + 2pn_3 + n_3 - n_3^2 ;$$

since $(p - n_3)^2 \geq n_3 - p$ is always true, the claim is proved.

We can now conclude by the previous claims (*i*) and (*ii*) that when $n_1 = n_2 \neq n_3$ the maximum number of segments can be obtained for the distribution of $m - 3$ points between the three vertices, when $n_1 = n_2$ represents the greatest possible equal distribution out of $m - 3$ points, for the two vertices.

The last step of the proof is the comparison of $s(n^*, n^*, (m-3) - 2n^*)$ with $s(n_1, n_2, n_3)$ for $n_1 \neq n_2 \neq n_3$. Before that a remark has to be made concerning $s(n^*, n^*, (m-3) - 2n^*)$:

$$s(n^*, n^*, (m-3) - 2n^*) = s^{(2)}_{max} .$$

To verify this assertion, it suffices to consider that $(m-3) - 2n^*$ is equal to 0 or 1 because of the definition of n^*, which in fact amounts to the two cases investigated in (α) and (β).

To complete the proof let $n_1 \neq n_2 \neq n_3$ and $n_1 < n_2 < n_3$, then

$$s(n_1, n_2, n_3) = m + n_3 n_2 + n_3 n_1 = s^{(3)}_{max} .$$

For $m - 3$ odd

$$s(n^*, n^*, 1) = m + n^{*2} + n^*$$

and for $m - 3$ even

$$s(n^*, n^*, 0) = m + n^{*2} .$$

Now to compare $s^{(3)}_{max}$ with $s(n^*, n^*, 0)$ consider that

$$n_1 + n_2 + n_3 = 2n^* .$$

Then

$$n_3 (n_2 + n_1) = n_3 (2n^* - n_3) = 2n^* n_3 - n_3^2 \leq n^{*2} ,$$

since

$$(n^* - n_3)^2 \geq 0 .$$

This gives

$$\underset{max}{s^{(3)}} = m + n_3 (n_2 + n_1) \leq m + n^2 = \underset{max}{s^{(2)}} .$$

The same argument can be used for $m - 3$ odd. Therefore no matter whether $m - 3$ is even or odd

$$\underset{max}{s^{(3)}} \leq \underset{max}{s^{(2)}} .$$

This completes the proof.

4.1.1. Remark. The limit values in the statement of the proposition can be effectively reached, as is shown by the following examples :

(α) Let (abc) be an open triangle and p_0 be any point of (abc). Consider the set

$$S = \{(abc) \cup ab \cup \{c\}\} \setminus \{p_0\} .$$

This set is T_5 as can readily be seen by drawing the lines ap_0 and bp_0 and taking on them two points d and e respectively, both lying in (abc) on the segment opposed to p_0 with respect to a and b. Then the set $\{a, b, c, d, e\}$, consisting of five points of S, defines only one closed triangle edc, contained in S, and six segments namely ab, ae, ec, bd, dc and ed all contained in S. Therefore the set is (5,6)-convex, 6 being exactly the maximum number of line segments that can be attained according to the first proposition.

(β) The minimum number 3 is obtained by considering a set S' such that

$$S' = (abc) \cup \{k_1 , k_2\}$$

where k_1 , k_2 are isolated points.

An intermediate value, for example $s = 4$, is obtained in a set which is the union of two many-pointed convex sets.

4.1.2. Result. Remark 4.1.1 shows that the inequality of proposition 4.1 can not be sharpened.

4.1.3. Remark. For $m = 4$

$$3 \leq s \leq 4 .$$

If we furthermore assume that the set S considered in the above proposition is closed and connected, by Theorem I of Kay and Guay [1] , S is convex.

5.1. Theorem. If $S \subset \mathscr{X}$ and S is closed, (m,n)-triangular convex set, with $n > \frac{1}{3} \cdot \frac{(m-2)^3}{2^3} - \frac{m-2}{24}$, then either S is convex or else S is the union of a closed convex set S_1 and of k isolated points not in S_1 , where k satisfies the condition

$$\binom{m-k}{3} \geq n .$$

Proof. If S is convex, there is nothing to prove. If S is not convex, then there exists at least a pair of points x and y, such that $xy \not\subset S$. If furthermore S is assumed to be connected, it is possible to find two sequences of points,

$$X = \{x_i\}_{i\in D} \quad , \quad Y = \{y_j\}_{j\in D}$$

and two directed sets D, E such that for all $i \in D$, $j \in E$, x_i and y_j belong to the set S and

$$\lim_{i\in D} x_i = x \quad , \quad \lim_{j\in E} y_j = y .$$

Since S is closed, there would also have to exist $i_0 \in D$ and $j_0 \in E$ such that for all $i \geq i_0$ and for all $j \geq j_0$

$$x_i\, y_j \not\subset S .$$

Then it would be possible to obtain m points, adequately chosen from these points x_i and y_j, all belonging to S such that the number of triangles defined by this particular m-ple, contained in S, is at most

$$2\binom{r}{3}$$

if m is even and equal to $2r$,

$$\binom{r}{3} + \binom{r+1}{3} = 2\binom{r}{3} + \binom{r}{3}$$

if m is odd and equal to $2r+1$.

In the first case,

$$2\binom{r}{3} = \frac{2r(r-1)(r-2)}{6} = \frac{2r(2r-2)(2r-4)}{24}$$

$$= \frac{m(m-2)(m-4)}{24} = \frac{(m-2)(m^2-4m)}{24}$$

$$= \frac{1}{3} \cdot \frac{(m-2)^3}{2^3} - \frac{m-2}{6}$$

and in the second case

$$2\binom{r}{3} + \binom{r}{2} = \frac{2r(2r-2)(2r-4)}{24} + \frac{r(r-1)}{2}$$

$$= \frac{(m-1)(m-3)(m-5)}{24} + \frac{2r(2r-2)}{8}$$

$$= \frac{(m-2)(m-1)(m-3)}{24}$$

$$= \frac{(m-2)(m^2-4m+3)}{24} = \frac{1}{3} \cdot \frac{(m-2)^3}{2^3} - \frac{m-2}{24} .$$

Since

$$n > \frac{1}{3} \cdot \frac{(m-2)^3}{2^3} - \frac{m-2}{24} > \frac{1}{3} \cdot \frac{(m-2)^3}{2^3} - \frac{m-2}{6}$$

the above results contradict in each case the hypothesis that S is (m,n)-triangular convex; hence S must be convex.

If S is not connected, by the previous argument, each of its components has to be convex. But then the condition given for n implies that only one of the convex components of S can have infinity of elements and all the remaining have to reduce to singletons. Then let S_1 denote the convex component of S and p_1, $p_2, \ldots, p_k$ be the isolated points of S; as S is (m,n)-triangular convex, the integer k has to satisfy the condition

$$n \leq \binom{m-k}{3} .$$

5.1.1. Remark. If $m = 4$

$$\frac{1}{3} \cdot \frac{(m-2)^3}{2^3} - \frac{m-2}{24} = \frac{1}{3} - \frac{1}{12} = \frac{1}{4} .$$

So for $n = 1$ the assertion of the theorem is satisfied, hence any closed connected set S having the property T_4 is convex or else, if not connected S is the union of a convex set S_1 and a single isolated point $\{k_1\}$, that is

$$S = S_1 \cup \{k_1\}$$

which confirms Remark 4.1.3. Furthermore, since for $m = 5$

$$\frac{1}{3} \cdot \frac{(m-2)^3}{8} - \frac{m-2}{24} = 1 ,$$

Theorem 5.1 can not be applied to a closed set having the property T_5. Indeed the union of two closed convex sets (e.g. two triangles with a common vertex) is (5,1)-triangular convex and is neither convex nor the union of a convex set and some isolated points.

5.2. Theorem. Any closed set $S \subset \mathscr{X}$ having the property P_0 has either the property T_4 or T_5.

Proof. Let $S \subset \mathscr{X}$ be a set having the property P_0. Then by the results of Mc Kinney [7], S is the union of two convex sets S_1 and S_2 such that the union is not convex. Therefore there exists at least a pair of points x and y of S such that $xy \not\subset S$. We know also by the definition of P_0 that it is possible to find a triple $\{x, y, z\} \not\subset S$, satisfying the following inclusion and non-inclusion relations:

$$xz \subset S \qquad \text{and} \qquad yz \not\subset S.$$

There are now two possibilities:

(*a*) either this relation may hold only for a single fixed point $y \in S$, or else

(*b*) for a variable point y in some subset of S.

If we assume y as a fixed point in S, then for every $z \in S$ yz's will be the only segments that are not contained in S. Therefore for every finite subset $\{x, y, z, v\} \subset S$, the segments xz, zv, xv would be contained in S, hence the triangle $xvz \subset S$. This implies under this assumption that the set S has the T_4 property.

If we assume y as a varying point in S, then for arbitrary point sets such as $\Sigma_1 = \{x, y, z\}$ and $\Sigma_2 = \{x, t, k\}$ the only segments contained in S would be xz and xk. Let Σ be the union of Σ_1 and Σ_2,

$$\Sigma = \Sigma_1 \cup \Sigma_2 = \{x, y, z, t, k\}$$

then by the P_0 property we will have also yt and zk contained in S. That means that for a certain choice of five points belonging to $S \setminus \mathrm{Ker}.S$ we can have only one triangle contained in S. The purpose of a such choice is evident since we can only get the least number of triangles contained in S by choosing points belonging to $S_1 \setminus \mathrm{Ker}.S$ and $S_2 \setminus \mathrm{Ker}.S$. If we choose four points, two of them belonging to $S_1 \setminus \mathrm{Ker}.S$ and the other two to $S_2 \setminus \mathrm{Ker}.S$, no such triangle can be found. The fifth point can either be in the $\mathrm{Ker}.S$ or else in one of the sets $S_1 \setminus \mathrm{Ker}.S$ or $S_2 \setminus \mathrm{Ker}.S$. If we take the fifth point in $\mathrm{Ker}.S$ then it may occur that more than one triangle contained in S is obtained. So it is only when out of five points two of them are chosen in $S_1 \setminus \mathrm{Ker}.S$ and the remaining three in $S_2 \setminus \mathrm{Ker}.S$ that we get a single triangle. Because of this brief remark, on returning to the last step of our proof, we can claim that the set S has the property T_5 since 5 is the minimum number of points for which only one triangle contained in S is always to be found.

5.2.1. Remark. The converse of the theorem is not true. That is, a set may have the T_5 property or the T_4 property without having the P_0 property. Take any set having the property T_5. Then any quintuple of points $\mathscr{F} = \{a, b, c, d, e\}$ in S is such that at least one of the closed triangles defined by three of these points is contained in S. Call a, b, c the vertices of any such triangle and let d, e denote the remaining points of the quintuple $\mathscr{F}$. The assumption

$$ad, ae, de \not\subset S$$

does not contradict the fact that S has the property T_5, yet

$$ad, ae \not\subset S$$

does not imply

$$de \subset S.$$

Hence the property P_0 does not necessarily hold.

An example evidencing such a situation is supplied by a set S consisting of the union of the singletons formed by the two isolated points i_1 and i_2 with any convex set C.

5.3. Theorem. Any closed, connected T_5-convex set $S \subset \mathscr{X}$ has the P_0 property.

Proof. Since S has the T_5 property, for any quintuple $\mathscr{F} = \{x_1, x_2, x_3, x_4, x_5\}$ in S, at least one closed triangle having as vertices three of these points has to be contained in S. Let the three vertices of such a triangle be x_1, x_3, x_5. If we can show that whenever

$$(\alpha) \qquad x_1 x_4, \quad x_3 x_4, \quad x_4 x_5, \quad x_2 x_1, \quad x_3 x_2, \quad x_2 x_5 \not\subset S$$

implies

$$x_2 x_4 \subset S$$

then the proof will be complete.

Since S is closed, there is a polygonal arc P in S, joining any two points of S having minimal length. Let k denote the number of sides of P and let

$$y_0, y_1, \ldots, y_k$$

be the consecutive vertices of P.

a) We claim that $k = 2$.

Assume $k > 2$ and take

$$x_3 = y_0, y_1, \ldots, y_k = x_4.$$

Consider the set of five points

$$\mathscr{F}_1 = \{x_5, y_0, y_2, y_k, x_2\}.$$

$y_0 y_2$, $y_2 y_k$ and $y_0 y_k$ can not belong to S, because had this been the case it would be possible to shorten the polygonal arc P, eliminating y_1 in the first case, $y_3, \ldots, y_{k-1}$ in the second case and $y_1, y_2, y_3, \ldots, y_{k-1}$ in the last case: in all three cases this would contradict the hypothesis of minimal length of P. Under these conditions the quintuple $\mathscr{F}_1$ fails to determine any triangle contained in S, against the hypothesis that S has the T_5 property. This last contradiction shows that the assumption $k > 2$ is not true. Hence $k = 2$, as claimed.

b) We now claim that if (α) holds $x_2 x_4 \subset S$.

Consider again the quintuple $\mathscr{F} = \{x_1, x_2, x_3, x_4, x_5\}$ under the hypothesis that $x_1 x_3 x_5 \subset S$ and that

$$(\alpha) \qquad x_1 x_4, \quad x_3 x_4, \quad x_4 x_5, \quad x_2 x_1, \quad x_3 x_2, \quad x_2 x_5 \not\subset S:$$

we claim that $x_2 x_4 \subset S$.

Suppose $x_2 x_4 \not\subset S$. Because of (α) neither x_2 nor x_4 is cantained in $x_1 x_3 x_5$. Then, as x_2, $x_4 \in S$ and S is connected, there exists a polygonal arc P joining x_2 and x_4 having minimal length and by (a) exactly two sides. Let y_1 denote the point of intersection of the two sides of this polygonal arc. Then $x_2 y_1$, $y_1 x_4 \subset S$.

Let $y_2 \in x_2 y_1$, $y_4 \in y_1 x_4$; then y_2, $y_4 \in S$. Consider the subset

$$\mathscr{F} = \{x_1, x_2, y_2, x_4, y_4\} \subset S.$$

Because of the T_5 property this subset must determine at least one triangle completely contained in S. Since $x_1 x_2$, $x_4 x_1 \not\subset S$ the only possible triangles are

$$x_1 y_2 y_4, \qquad x_2 y_2 y_4, \qquad x_4 y_2 y_4.$$

However if $x_2 y_2 y_4 \subset S$ then $y_4 x_2 \subset S$ and the polygonal arc with vertices x_2, y_4, x_4 (which is shorter then P) would join x_2 to x_4 against the minimality condition; therefore $x_2 y_2 y_4 \not\subset S$.

A similar process shows that $x_4 y_2 x_4 \not\subset S$, therefore $\mathscr{F}_2$ determines exactly one triangle belonging to S,

$$(*) \qquad x_1 y_2 y_4 \subset S.$$

Since y_2 and y_4 are chosen arbitrarily on the segments $x_2 y_1$ and $y_1 x_4$ it is possible to define two nets of points

$$Y_{(2)} = \{y_{2,i}\}_{i \in D}, \qquad Y_{(4)} = \{y_{4,j}\}_{j \in E}$$

and two directed sets D, E such that for all $i \in D$ and all $j \in E$, $y_{2,i}$, $y_{4,j} \in S$ and

$$\lim_{i \in D} y_{2,i} = x_2, \quad \lim_{j \in E} y_{4,j} = x_4.$$

Furthermore, by (*),

$$\beta y_{2,i} + (1-\beta) y_{4,j} \in y_{2,i} y_{4,j} \subset S, \quad 0 < \beta < 1.$$

Since S is closed

$$\lim_{\substack{i \in D \\ j \in E}} \beta y_{2,i} + (1-\beta) y_{4,j} = \beta x_2 + (1-\beta) x_4 \in S.$$

Hence

$$x_2 x_4 \subset S$$

as claimed.

5.3.1. Corollary. On a closed, connected set $S \subset \mathscr{X}$ properties P_0 and T_5 imply each other.

Proof. Since S has the P_0 property then it is the union of two convex sets S_1 and S_2. Let $x_1, x_2 \in S$ such that $x_1 x_2 \not\subset S$, where $S = S_1 \cup S_2$ and

$S_1 \cap S_2 \neq \phi$. If $x_1 \in S_1$ then $x_2 \notin S_1$ since S_1 is convex and trivially $x_1, x_2 \notin$ Ker.S, therefore $x_1 \in S_1 \setminus$ Ker.S and $x_2 \in S_2 \setminus$ Ker.S. Furthermore x_1 and x_2 are not the only pairs which satisfy the above conditions since S_1 and S_2 are the connected convex sets. Let $x_3, x_4 \in S$ be another pair such that $x_3 x_4 \not\subset S$, where $x_3 \in S_1 \setminus$ Ker.S, $x_4 \in S_2 \setminus$ Ker.S. Then for any quintuple of points in S such as

$$P' = \{x_1, x_2, x_3, x_4, x_5\}$$

P_0 implies $x_1 x_3 x_5 \subset S$.

By Theorem 5.3 it can be directly concluded that the T_5 property implies the P_0 property. Hence

$$T_5 \Leftrightarrow P_0 .$$

5.3.2. Corollary. Any closed, connected (5,1)-triangular convex set S is an L_2 set.

Proof. See a) in the proof of Theorem 5.3.

5.4. Theorem. Any closed, connected T_5 convex set $S \subset \mathscr{X}$ is (3,1)-convex.

Proof. By using Theorem 5.3 we can conclude that S has the P_0 property. However P_0 property is a generalization of P_3 property. Hence S is (3,1)-convex.

5.4.1. Remark. If a set is not P_3 it can not be P_0. To prove this statement it is sufficient to assume that some point triple, say $\beta' = \{x, y, z\} \subset S$ exists, such that

$$yz, zx, xy \not\subset S ,$$

so that S is not P_3.

Let $\beta'' = \{u, v\} \subset S$. Then the quintuple

$$\beta = \beta' \cup \beta'' = \{x, y, z, u, v\} \subset S$$

is such that no ordering of these five points satisfies the requirements of the P_0 property.

5.4.2. Remark. According to Theorem 4 of Kay and Guay [1], any closed, connected m-convex set $S \subset \mathscr{X}$ is an L_{m-1} set, so Theorem 5.4 confirms Corollary 5.3.1 of Theorem 5.3. However the converse of the Theorem 5.4 is not true. For a star-shaped pentagon is (3,1)-convex, yet it is not (5,1)-triangular convex.

5.5. Lemma. If S is $(m,1)$-triangular convex then S is $(m-1,1)$-convex.

Proof. For any m-ple in S at least one triangle, consequently at least three segments determined by these points are contained in S since S has the T_m property.

By eliminating any one point from the m-ple, depending as to whether this point is one of the vertices of the triangle or not at most two of the sides of the triangle contained in S are erased while the third side remains in S. Hence the set S is $(m-1,1)$-convex.

5.5.1. Remark. The following example shows that the preceding result is best possible.

Consider the set S as the union of two closed triangles xyz and ztp, each deprived of the sides (xy), (xz) and (tp), (zp) respectively.

5.6. Theorem. Any m-triangular convex set is the union of $m-2$ or fewer starshaped sets.

Proof. If S is m-triangular convex, S is $(m-1,1)$-convex by Lemma 5.5. Hence by Theorem 2 of Kay and Guay [1], S is the union of $m-2$ or fewer starshaped sets.

5.7. Theorem. If $S \subset \mathscr{X}$ is a closed, connected 4-triangular convex set then S has no lnc points[4].

Proof. Assume that S has at least one lnc point q in S. Since q is an lnc point of S there exist nets x_i and y_j in S over the directed sets, D and E such that

$$\lim_{i \in D} x_i = \lim_{j \in E} y_j = q\,,$$

where $x_i y_j \not\subset S$ for all $i \in D$, $j \in E$. Then for all the sets

$$S_{(ij)} = \{x_i\,, x_{i+1}\,, y_j\,, y_{j+1}\}\,,$$

where $i \in D$, $j \in E$,

$$x_i\, y_j\,, x_{i+1}\, y_{j+1} \not\subset S.$$

Therefore the quadruples $S_{(ij)}$ define no closed triangles contained in S, contradicting the hypothesis. Consequently the set S can not contain any lnc points.

5.7.1. Corollary. If $S \subset \mathscr{X}$ is a closed, connected 4-triangular convex set then S is convex.

Proof. Theorem 5.7 implies that S has no lnc points. The fact that S is a convex set then follows immediately from Tietze's theorem [6].

5.8. Definition. Whenever the open segment joining two points x and y is contained in S, it will be said that "x sees y via S". A set T is "visually independent via S" if no two distinct members of T see each other via S.

[4]) A point of *local non convexity* (or lnc point) x of S is according to Kay and Guay [1], a point such that each neighbourhood V of x contains $y, z \in S$ such that $yz \not\subset S$. Such points are called points of strong lnc by Valentine [9], as pointed out by the above authors.

5.9. Theorem. A closed m-triangular convex set $S \subset \mathscr{X}$, which contains k Inc points which are visually independent via S, is the union of $m-2k-1$ or fewer starshaped sets.

Proof. Let $q_1, \ldots, q_k$ denote the k Inc points of S. Since S is closed, for each q_i $(i = 1, \ldots, k)$ there exist nets x_{i_l}, y_{i_j} over the directed sets E_i, D_i satisfying the conditions

$$\lim_{l \in E_i} x_{i_l} = \lim_{j \in D_i} y_{i_j} = q_i \tag{5.9.1}$$

and

$$x_{i_l} \, y_{i_j} \not\subset S \tag{5.9.2}$$

for every $j \in D_i$, $l \in E_i$.

Thus a set of $2k$ points

$$S_0 = \{x_{1l}, y_{1j}, \ldots, x_{kl}, y_{kj}\}$$

which is visually independent via S, is attained. Call "consecutive point" of x_{i_l} the point $x_{i_{l+1}}$ or of y_{i_j} the point $y_{i_{j+1}}$ and consider together with S_0 the set S_0^+ of all consecutive points of those in S_0. The set $S_0 \cup S_0^+ = S_1$ has $4k$ points which are no more visually independent via S. By (5.9.2) S_1 can at most define $2k$ segments contained in S, such that they span no triangle in S. Furthermore, since S is m-triangular convex there exists a set of $m-1$ points such that no point triple chosen amongst them defines a triangle contained in S. Hence

$$m - 1 \geq 4k.$$

Let

$$h = m - 1 - 4k.$$

Then $s \leq h$ will denote the number of points of S that will complete S_0 to the maximal visually independent set of S that is

$$S^* = S_{\max} \setminus S_0$$

for

$$S_2 = \{P_1, \ldots, P_h\} \qquad \text{where} \qquad S^* \subset S_2 .$$

In order to obtain the greatest number of starshaped sets the set $S_{\max}$ of visually independent points of S of maximal cardinality has to be determined. This set corresponds to the case $s = h$ because when $s = h$ $S^* = S_2$ and therefore

$$S_{\max} = S_0 \cup S_2$$

and card.$S_{\max} = 2k + h$. If, on the other hand some points of S_2 see only one other point of S_2 via S, $S_{\max} = S_0 \cup S_t$ and card.$S_{\max} = 2k + s$, s being the card. S_1. So, as previously stated the maximal cardinality is reached for $s = h$ which means that S_2 consists of all isolated points. Under this assumption the set is $(2k + h + 1)$-convex and by Theorem 2 of Kay and Guay it is the union of $2k + h$ or fewer starshaped sets.

5.9.1. Corollary. A closed m-triangular convex set $S \subset \mathscr{X}$, which contains k lnc points such that they are visually independent via S is a $(s,1)$-convex set for $\frac{m+1}{2} \leq s \leq m - 2k$.

Proof. By Theorem 5.9 $S_{\max}$ is the greatest possible maximal visually independent set that can be obtained, so in this case S is $(2k + h + 1)$-convex. If, on the other hand, $h = 0$ then $S_{\max} = S_0$ and consequently S is $(2k + 1,1)$-convex. In each of these cases it is sufficient to substitute $m - 1 - 4k$ in place of h to get the upper and the lower bounds for s as given in the statement of the corollary.

5.10. Lemma. If $S \subset \mathscr{X}$ is a closed, connected, m-triangular convex set, the integer m is odd.

Proof. The set S is connected and therefore S has no isolated points, so S does not have points that see only themselves via S.

Let S' denote a subset of S such that every point of S' sees exactly one single other point of S' via S. Call $\mathscr{S} = \{S'\}$ the class of such sets S'. $\mathscr{S}$ can be partially ordered by inclusion and therefore, by Zorn's Lemma, there exists a maximal set $S'_{\max} \in \mathscr{S}$. $S'_{\max}$ can be described as follows:

$$S'_{\max} = \{P_j, Q_j \in S \mid P_j \text{ sees only } Q_j \text{ and } Q_j \text{ sees only } P_j \text{ via } S;$$
$$j \in \{1, \ldots, h\}.$$

Consequently

$$\text{Card. } S'_{\max} = 2h$$

and $S'_{\max}$ is the union of two visually independent sets,

$$S_1' = \{P_1, \ldots, P_h\} \qquad \text{and} \qquad S_2' = \{Q_1, \ldots, Q_h\}.$$

We claim that

$$2h = m - 1 .$$

a) If $2h > m-1$ then $2h \geq m$ and the points of $S'_{\max}$, because of m-triangular convexity would have to define a triangle contained in S, in contradiction with the hypothesis that every point of $S'_{\max}$ sees exactly one other point of S' via S.

b) If $2h < m-1$ let $k = m-1 - 2h$ and suppose the P_i's are numbered so that

$$T = \{P_1, \dots, P_k\}$$

is the set consisting of all such points. Then no element of T sees any other element of the set S via S because otherwise maximality would be contradicted. This implies that all the points of T are isolated and this contradicts connectedness. Hence

$$2h < m - 1$$

is impossible.

5.11. Theorem. For $m > 3$ a closed, connected, m-triangular convex set S having a single lnc point q is the starshaped union of $\frac{m-1}{2}$ convex sets.

Proof. 1) S is a starshaped set with respect to q. To prove this consider the following:

a) By Theorem 5.10 under the present hypothesis m is odd , so let $m = 2p + 1$.

b) S has a single lnc point q and has the T_m property. Therefore if S' is a subset of S such that every point of S' sees exactly one single other point of S' via S and if $\mathscr{S} = \{S'\}$, any maximal set $S'_{\max}$ of $\mathscr{S}$ has cardinality $m - 1 = 2p$.

c) As shown above it is always possible to choose a set $S'_{\max}$ so that it is the union of two visually independent sets

$$S_1 = \{x_1, \dots, x_p\} \quad , \quad S_2 = \{y_1, \dots, y_p\}$$

such that

$$x_i q \, , \; y_i q \, , \; x_i y_i \subset S \tag{5.11.1}$$

for every $i \in \{1,\dots,p\}$ and $x_i y_j \not\subset S$ for $i \neq j$. Then, because S is m-triangular convex for any $x \in S \setminus \{q\}$ the set $S = S'_{\max} \cup \{x\}$ has cardinality m and must define at least one triangle contained in S. Because of the choice of S_1 and S_2

$$x_i x_j \; , \; y_i y_j \; , \; x_i y_j \not\subset S$$

for any $(i, j) \in \{1,\dots,p\} \times \{1,\dots,p\}$ so this triangle has to be $x_i y_i x$. This implies that there is always at least one index i such that

$$x_i y_i x \subset S.$$

Furthermore this index is unique: if indeed another triangle, say $x_j y_j x \subset S$ then

$$xx_i \, , \, xx_j \subset S.$$

Since S has no lnc points except q, x is not a lnc point and by Corrollary 2 in Valentine [2] ,

$$x_i x_j \subset S$$

thus contradicting the assumption of visual independence in S_1. Thus

$$x_i y_i x \subset S$$

for a single value of i and

$$xx_i \subset S.$$

d) By (5.11.1)

$$x_i q \subset S$$

so, since $x_i \neq q$ is not a lnc point, again by Corollary 2 of Valentine

$$xq \subset S.$$

This proves that S is a starshaped set with respect to q.

2) S is the union of p convex sets. To prove this call T_i the triangle $x_i y_i q$ for every $i \in \{1,\ldots,p\}$. We claim that $S = \bigcup_{i=1}^{p} K_i$ where all the K_i are convex sets such that $T_i \subset K_i$ for all values of the index i.

Call l_i $(i = 1,\ldots,p)$ the rays issuing from q seperating two consecutive points of S_1 such that $l_i \cap S = \{q\}$. Call Δ_i the angular domain of vertex q, defined by l_i, l_{i+1} containing the point x_i $(l_{p+1} = l_1)$.

Call

$$K_i = \Delta_i \cap S\,;$$

then $x_i \in K_i$.

K_i is convex. In fact, for any $x, y \in K_i$ since S is starshaped

$$qx,\ qy \subset S,\ \Delta_i$$

hence

$$xy \subset S,\ \Delta_i$$

by Corollary 2 of Valentine applied to S and ordinary convexity to Δ_i. Thus

$$xy \subset K_i$$

as was to be proved.

5.11.1. Remark. The second part of the proof requires S to be a set in the Euclidean plane but clearly a generalization by means of polytops to E^n for any finite n can be obtained.

5.11.2. Corollary. The set S of Theorem 5.11 is an L_2 set.

Proof. Since S is starshaped with respect to q, every point $x \in S$ sees q via S. Then for any $x, y \in S$ the two sided polygonal arc xqy joins these two points, so S is L_2.

5.12. Theorem. If $S \subset \mathscr{L}$ is a closed, connected, m-triangular convex set then S is $\left(\frac{m+1}{2}, 1\right)$-convex.

Proof. Since S has the T_m property and S is connected, there exists a maximal set

$$S' = \{x_1, \ldots, x_{m-1}\}$$

in which each element sees only one other element of S' via S. By Theorem 5.10 $m-1$ is even, furthermore S has no isolated points and therefore S' can be expressed as the union of two visually independent sets with the same cardinality. In this case the cardinality of each set is equal to $\frac{m-1}{2}$. This proves the existence of a maximal visually independent set contained in S, with cardinality $\frac{m-1}{2}$. Hence S is $\frac{m+1}{2}$-convex.

5.13. Theorem. A closed, connected, m-triangular convex set $S \subset \mathscr{L}$ is an $L_{\frac{m-1}{2}}$ set.

Proof. If S is a m-triangular convex set, S is $\frac{m+1}{2}$-convex by Theorem 5.12. Hence by Theorem 4 of Kay and Guay [1], S is an $L_{\frac{m-1}{2}}$ set.

5.14. Theorem. In a finite dimensional linear space every connected m-triangular convex set is polygonally connected.

Proof. If S is a m-triangular convex set contained in a finite dimensional linear space, by Lemma 5.5 it is a $(m-1,1)$-convex set in this finite dimensional space, so that by Theorem 6 of Kay and Guay it is polygonally connected.

5.14.1. Corollary. In a finite dimensional linear space every connected m-triangular convex set is an L_{2m-5}.

Proof. Under the hypothesis of the Corollary, again by Lemma 5.5 S is a $(m-1,1)$-convex set in a finite dimensional linear space and therefore by Corollary 2 of Kay and Guay, S is an L_{2m-5}.

CHAPTER THREE
POSSIBILITIES OF FURTHER GENERALIZATIONS

The notion of triangular convexity introduced in the previous Chapters may be extended in a fairly obvious manner.

As a first step, consider *tetrahedral convexity*, assuming that the ambient linear space $\mathscr{X}$ has dimension ≥ 3.

Then in any such linear space $\mathscr{X}$, a set S will be said to have the $\mathscr{T}_m^n$ property (or to be *tetrahedrally* (m,n)*-convex*) if S contains at least m distinct points (with $m \geq 4$) and if for each subset of m distinct points of S at least n of the $\binom{m}{4}$ possible closed tetrahedra determined by these points are contained in S, m being the lowest and n being the highest integer giving rise to such a property. Obviously the integers m and n are related by the inequality

$$1 \leq n \leq \binom{m}{4}.$$

In particular, if a set S has the $\mathscr{T}_m^n$ property with $n = 1$, it will be said to have the $\mathscr{T}_m$ property.

It can be shown that

$$\mathscr{T}_4 \Leftrightarrow \text{convexity}$$

and it is clear that several of the theorems proved in Chapter Two can be extended (with, of course, some amendments) to tetrahedral convexity.

More in general, suppose that the linear space $\mathscr{X}$ has dimension $d > r$ and in any such linear space consider a set S, containing at least $r + 1$ distinct points. S will be said to be *r-simplicially* (m, n)*-convex* or to have the ${}_rS_m{}^n$ *property* if for each subset of m distinct points (with $m \geq r + 1$), at least n of the $\binom{m}{r+1}$ possible *r-simplici* determined by these points are contained in S, m being the lowest integer and n being the highest integer giving rise to such a property. Again, the integers m and n will be related by the inequality

$$1 \leq n \leq \binom{m}{r+1}.$$

In particular, if a set S has the ${}_rS_m{}^n$ property with $n = 1$, S will be said to have the ${}_rS_m{}^n$ property.

Again, it can be shown that

$${}_rS_{r+1} \Leftrightarrow \text{convexity}$$

and again, most of the results, adjusted as far as integers appearing in the bounds are concerned, obtained in Chapter Two can be carried on to r-simplicially (m, n)-convex sets.

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ÖZET

Bu çalışmada, bugüne kadar tanımlanan zayıflatılmış konvekslik çeşitleri ile bazı yakınlıkları olan, özel koşullar altında da konvekslik tanımı ile çakışan, (m,n) - üçgensel konvekslik adı verilen yeni bir kavram ithal edilmekte ve bununla ilgili bazı sonuçlar elde edilmektedir.