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ON ENTIRE FUNCTIONS OF IRREGULAR GROWTH DEFINED BY DIRICHLET SERIES

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Lef $f(s) = \Sigma_{n \in N} a_n e^{s\lambda_n}$ be an entire function defined by an everywhere convergent Dirichlet series whose exponents are subjected to the condition that $\limsup_{n \to +\infty} \frac{\log n}{\lambda_n} = D \in R_+ \cup \{0\}$ (R_+ is the set of positive reals), and E be the set of all such entire functions. An entire function $f \in E$ is said to be of irregular growth if its lower order is not equal to its Ritt order. In this paper we have studied certain properties of such functions

1. Let E be the set of mappings $f: C \to C$ (C is the complex field) such that the image of an element $s \in C$ under f is $f(s) = \sum_{n \in N} a_n e^{s\lambda_n}$ with $\limsup_{n \to +\infty} \frac{\log n}{\lambda_n} = D \in R_+ \cup \{0\}$ (R_+ is the set of positive reals), and $\sigma_c^f = +\infty$ (σ_c^f is the abscissa of convergence of the Dirichlet series defining f); N is the set of natural numbers $0, 1, 2, \ldots$, $< a_n \mid n \in N >$ is a sequence in C, $s = \sigma + it$, $\sigma, t \in R$ (R is the field of reals), and $< \lambda_n \mid n \in N >$ is a strictly increasing unbounded sequence of nonnegative reals. Since the Dirichlet series defining f converges for each $s \in C$, f is an entire function. Also, since $D \in R_+ \cup \{0\}$, we have ([1], p.168) $\sigma_a^f = +\infty$ (σ_a^f is the abscissa of absolute convergence of the Dirichlet series defining f) and that f is bounded on each vertical line $Re(s) = \sigma_0$.

Let

$$M(\sigma, f) = \sup_{t \in R} \{ | f(\sigma + it) | \}, \ \forall \sigma < \sigma_c^f \tag{1.1}$$

be the maximum modulus of an entire function $f \in E$ on any vertical line $Re(s) = \sigma$,

$$\mu(\sigma, f) = \max_{n \in N} \{ | a_n | e^{\sigma\lambda_n} \}, \quad \forall \sigma < \sigma_c^f \tag{1.2}$$

be the maximum term, for $Re(s) = \sigma$, in the Dirichlet series defining f and

$$N(\sigma, f) = \max_{n \in N} \{ n : \mu(\sigma, f) = | a_n | e^{\sigma\lambda_n} \}, \ \forall \sigma < \sigma_c^f \tag{1.3}$$

be the rank of the maximum term.

An entire function $f \in E$ is said to be of irregular growth if its lower order is not equal to its Ritt order. In this paper we study a few results pertaining to such functions.

2. Kamthan has shown ([2], Thm. 4) that :

Theorem A. If $f \in E$ is an entire function of Ritt order $\rho \in R_+$ and type $T \in R_+$, then

$$\limsup_{\sigma \to +\infty} \frac{\mu'(\sigma, f)}{\mu(\sigma, f)\, T \rho\, e^{\rho\sigma}} \leq e,$$

where μ' is the derivative of μ with respect to σ.

Remark. Theorem A has been proved under condition that $D = 0$, but it is true for any $D \in R_+ \cup \{0\}$; that is why we have mentioned it in this improved form.

We first show that :

Theorem 1. For every entire function $f \in E$ of Ritt order $\rho \in R_+$ and type $T \in R_+$,

$$\liminf_{\sigma \to +\infty} \frac{\mu'(\sigma, f)}{\mu(\sigma, f)\, \rho\, T e^{\rho\sigma}} \geq 1. \tag{2.1}$$

Proof. We know ([3], Lemma 1) that, for almost all values of σ,

$$\frac{\mu'(\sigma, f)}{\mu(\sigma, f)} = \lambda_{N(\sigma, f)}.$$

Let

$$\limsup_{\sigma \to +\infty} \frac{\lambda_{N(\sigma, f)}}{e^{\rho\sigma}} = \gamma.$$

Then, for an infinite sequence of values of σ, and any given $\varepsilon \in R_+$, $\lambda_{N(\sigma,f)} > (\gamma - \varepsilon)\, e^{\rho\sigma} \geq (\rho T - \varepsilon)\, e^{\rho\sigma}$, since $\gamma \geq \rho\, T$ ([4], p. 141). Hence

$$\liminf_{\sigma \to +\infty} \frac{\mu'(\sigma, f)}{\mu(\sigma, f)\, \rho T\, e^{\rho\sigma}} \geq 1.$$

Next we show that :

Theorem 2. For every entire function $f \in E$ of irregular growth of Ritt order $\rho \in R_+$ and type $T \in R_+$,

$$\lim_{\sigma \to +\infty} \frac{\mu'(\sigma, f)}{\mu(\sigma, f)\, \rho\, T e^{\rho\sigma}} = 1. \tag{2.2}$$

Proof. It is known ([5], p.250) that, for entire functions $f \in E$ of irregular growth,

$$\lim_{\sigma \to +\infty} \begin{matrix} \sup \\ \inf \end{matrix} \frac{\log \mu(\sigma, f)}{e^{\rho\sigma}} = \begin{matrix} T \\ 0 \end{matrix}.$$

Hence, for any $\varepsilon \in R_+$ and sufficiently large σ,

$$-\varepsilon e^{\rho\sigma} < \log \mu(\sigma, f) < (T + \varepsilon) e^{\rho\sigma}. \tag{2.3}$$

Also, since ([6], p.67) log $\mu(\sigma, f)$ is an increasing convex function of σ, we may write, for arbitrary σ, σ_0 $(\sigma > \sigma_0)$,

$$\log \mu(\sigma, f) = \log \mu(\sigma_0, f) + \int_{\sigma_0}^{\sigma} \frac{\mu'(x, f)}{\mu(x, f)} dx. \tag{2.4}$$

Now, for any $k \in R_+ \cup \{0\}$, we have

$$\int_{\sigma}^{\sigma+k} \frac{\mu'(x,f)}{\mu(x,f)} dx = \int_{0}^{\sigma+k} \frac{\mu'(x,f)}{\mu(x,f)} dx - \int_{0}^{\sigma} \frac{\mu'(x,f)}{\mu(x,f)} dx$$

$$= \log \mu(\sigma + k, f) - \log \mu(\sigma, f), \text{ in view of (2.4)}$$

$$< (T + \varepsilon) e^{\rho(\sigma+k)} + \varepsilon e^{\rho\sigma}, \text{ in view of (2.3)}$$

$$= e^{\rho\sigma} (Te^{\rho k} + \varepsilon (e^{\rho k} + 1)). \tag{2.5}$$

But

$$\int_{\sigma}^{\sigma+k} \frac{\mu'(x,f)}{\mu(x,f)} dx \geq \frac{\mu'(\sigma,f)}{\mu(\sigma,f)} k. \tag{2.6}$$

Hence, from (2.5) and (2.6),

$$\frac{\mu'(\sigma,f)}{\mu'(\sigma,f) e^{\rho\sigma}} < \frac{Te^{\rho k} + \varepsilon (e^{\rho k} + 1)}{k}. \tag{2.7}$$

Since k is arbitrary but belongs to $R_+ \cup \{0\}$ and the left side of (2.7) is independent of k, it follows that

$$\limsup_{\sigma \to +\infty} \frac{\mu'(\sigma,f)}{\mu(\sigma,f) e^{\rho\sigma}} \leq \rho T. \tag{2.8}$$

Similarly, we can show that

$$\liminf_{\sigma \to +\infty} \frac{\mu'(\sigma,f)}{\mu(\sigma,f) e^{\rho\sigma}} \geq \rho T. \tag{2.9}$$

Combining (2.8) and (2.9), we get (2.2).

Remarks. (*i*) With the same argument, it can be shown that Theorem 2 is true for entire functions $f \in E$ of perfectly regular growth.

(*ii*) We conjecture, although we have not been able to prove, that Theorem 2 is true for entire functions $f \in E$ of regular growth but not of perfectly regular growth.

3. In the end we give a result regarding ordinary proximate linear order of entire functions $f \in E$ of irregular growth. We first recall its definition.

Definition ([7], p.64). A nonnegative extended real valued function ϕ of reals σ is called an ordinary proximate linear order of an entire function $f \in E$ of Ritt order $p \in R_+$, if

a) ϕ is eventually a continuous function,

b) ϕ is differentiable almost everywhere except at isolated points at which the left and right derivatives exist,

c) $\lim_{\sigma \to +\infty} \sigma \phi(\sigma) = 0$,

d) $\limsup_{\sigma \to +\infty} \phi(\sigma) = p$, and

e) $\limsup_{\sigma \to +\infty} \dfrac{\log M(\sigma, f)}{e^{\sigma \phi(\sigma)}} = 1$.

Theorem 3. For every entire function $f \in E$ of irregular growth of Ritt order $p \in R_+$, and ordinary proximate linear order ϕ, and any $m \in Z_+$ (Z_+ is the set of positive integers),

$$\liminf_{\sigma \to +\infty} \frac{\lambda_{N(\sigma, f^{(m)})}}{e^{\sigma \phi(\sigma)}} = 0 . \tag{3.1}$$

Proof. Let f be of lower order λ. Then $\lambda < p$. Since, by definition,

$$\lambda = \liminf_{\sigma \to +\infty} \frac{\log \log M(\sigma, f^{(m)})}{\sigma},$$

and ([8], Theorem 2.7)

$$\liminf_{\sigma \to +\infty} \frac{\log \log M(\sigma, f^{(m)})}{\sigma} = \liminf_{\sigma \to +\infty} \frac{\log \lambda_{N(\sigma, f^{(m)})}}{\sigma},$$

we have

$$\lambda = \liminf_{\sigma \to +\infty} \frac{\log \lambda_{N(\sigma, f^{(m)})}}{\sigma} .$$

Therefore, for any $\varepsilon \in R_+$ and sufficiently large σ, we get

$$\lambda_{N(\sigma, f^{(m)})} > e^{(\lambda - \varepsilon)\sigma} , \tag{3.2}$$

and, for an infinite sequence of values of σ,

$$\lambda_{N(\sigma, f^{(m)})} < e^{(\lambda + \varepsilon)\sigma} . \tag{3.3}$$

Dividing (3.2) and (3.3) by $e^{\sigma \phi(\sigma)}$ and proceeding to limit we get (3.1) in view of condition (d) of Definition.

Remark. This theorem generalizes and improves upon a result of Srivastava and Singh ([5], Lemma 2).

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ÖZET

Bu çalışmada, eksponentleri

$$\limsup_{n\to+\infty} \frac{\log n}{\lambda_n} = D \in R_+ \cup \{0\} \quad (R_+ \text{ pozitif reel sayılar cümlesi})$$

koşuluna uyan ve her yerde yakınsak bir Dirichlet serisi ile belirtilen

$$f(s) = \sum_{n\in N} a_n e^{s\lambda_n}$$

tam fonksiyonlarının bazı özellikleri incelenmektedir.